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Cycles in 2-Factors of Balanced Bipartite Graphs

Guantao Chen[1]*, Ralph J. Faudree[2†], Ronald J. Gould[3‡], Michael S. Jacobson[4§], and Linda Lesniak[5¶]

[1] Georgia State University, Atlanta, GA 30303, USA
[2] University of Memphis, Memphis, TN 38152, USA
[3] Emory University, Atlanta GA 30322, USA
[4] University of Louisville, Louisville, KY 40292, USA
[5] Drew University, Madison NJ 07940, USA

Abstract. In the study of hamiltonian graphs, many well known results use degree conditions to ensure sufficient edge density for the existence of a hamiltonian cycle. Recently it was shown that the classic degree conditions of Dirac and Ore actually imply far more than the existence of a hamiltonian cycle in a graph G, but also the existence of a 2-factor with exactly k cycles, where $1 \leq k \leq \frac{|V(G)|}{4}$. In this paper we continue to study the number of cycles in 2-factors. Here we consider the well-known result of Moon and Moser which implies the existence of a hamiltonian cycle in a balanced bipartite graph of order $2n$. We show that a related degree condition also implies the existence of a 2-factor with exactly k cycles in a balanced bipartite graph of order $2n$ with $n \geq max\left\{51, \frac{k^2}{2} + 1\right\}$.

1. Introduction

All graphs considered are simple, without loops or multiple edges. A 2-factor of a graph G is a 2-regular subgraph of G that spans the vertex set $V(G)$, that is, a 2-factor is a collection of vertex disjoint cycles that cover all vertices of G. For years mathematicians have investigated results ensuring the existence of 2-factors in graphs. Hundreds of results exist concerning the special case when the graph is hamiltonian, that is, the 2-factor is a single cycle. Recently, there have been efforts to determine more about the structure of general 2-factors. Questions about the number of cycles possible in a 2-factor or the lengths of the cycles forming the 2-factor have drawn interest.

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Such a question was considered in [1], where the following generalization of Ore's Theorem [6] was shown.

Theorem 1. *Let k be a positive integer and let G be a graph of order* $n \geq 4k$. *If* $deg\, u + deg\, v \geq n$ *for every pair of nonadjacent vertices u and v in* $V(G)$, *then G has a* 2*-factor with exactly k vertex disjoint cycles.*

An immediate Corollary to Theorem 1 generalizes the classic hamiltonian result of Dirac [3].

Corollary 2. *If G is a graph of order* $n \geq 4k$, *k a positive integer, and* $\delta(G) \geq \frac{n}{2}$, *then G contains a* 2*-factor with exactly k cycles.*

The complete bipartite graph $K_{n/2,n/2}$ shows that the conclusion of Theorem 1 and that of Corollary 2 are best possible in the sense that any 2-factor can contain at most $\left\lfloor \frac{n}{4} \right\rfloor$ cycles. Throughout this paper we let $G = (X \cup Y, E)$ be a balanced bipartite graph with vertex set $V = X \cup Y$, where $|X| = |Y|$, and edge set E which contains the edges with one vertex in X and the other one in Y. Corresponding to Dirac's Theorem, Moon and Moser [5] obtained the following result for balanced bipartite graphs.

Theorem 3. *If* $G = (X \cup Y, E)$ *is a balanced bipartite graph of order* $2n, (n \geq 2)$ *with* $deg\, u + deg\, v \geq n + 1$ *for each pair of nonadjacent vertices* $u \in X$ *and* $v \in Y$, *then G is hamiltonian.*

In this paper we show the following result, which generalizes Theorem 3 in a manner similar to the generalization of Ore's Theorem shown in Theorem 1.

Theorem 4. *Let k be a positive integer and let G be a balanced bipartite graph of order* $2n$ *where* $n \geq max\left\{51, \frac{k^2}{2} + 1\right\}$. *If* $deg\, u + deg\, v \geq n + 1$ *for every* $u \in V_1$ *and* $v \in V_2$, *then G contains a* 2*-factor with exactly k cycles.*

We will use the notation $P[u, v]$ to denote a path from u to v, while $C[u, v]$ shall mean the segment of the cycle C from vertex u to v (including u and v) under some orientation of C. We also let $\langle S \rangle$ denote the subgraph of G induced by the vertex set $S \subseteq V(G)$. We use the notation $deg\, v$ for the degree of the vertex v and $deg_S\, v$ for the degree of v relative to the subgraph S. Further, $N(x)$ represents the set of vertices adjacent to x and $N_C^-(x)$ and $N_C^+(x)$ represent the predecessors and successors of neighbors of x along some orientation of cycle C respectively.

Given a cycle C (or path P) with an orientation, we let v^+ denote the successor of vertex v along C and v^- the predecessor of v along C, according to this orientation. For terms not defined here, see [2].

We have recently learned of a related result due to Wang [7] that provides a minimum degree condition (namely $\delta(G) >= \lceil n/2 \rceil + 1$) for a balanced bipartite graph to have a 2-factor with exactly k cycles.

2. Preliminary Lemmas

In this section we provide some preliminary lemmas that will be useful in the proof of Theorem 4.

Lemma 1. *Let* $G = (X \cup Y, E)$ *be a bipartite graph and let* C *be a cycle of* G *and let* $P[u, v]$ *be a* $u - v$ *path in* $G - V(C)$ *such that* $u \in X$ *and* $v \in Y$. *If*

$$deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2},$$

then $\langle V(C) \cup V(P[u, v])\rangle$ *is hamiltonian, unless* $deg_C\, u = 0$ *or* $deg_C\, v = 0$. *If*

$$deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 1,$$

then $\langle V(C) \cup V(P[u, v])\rangle$ *is hamiltonian. Furthermore, if in this case* C *also contains a* 2*-factor with exactly two cycles, then so does* $\langle V(C) \cup V(P[u, v])\rangle$.

Proof. Since $deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2}$ and G is bipartite with $u \in X$ and $v \in Y$, either the cycle C has two consecutive vertices such that one is adjacent to u and the other is adjacent to v, and hence we obtain the desired hamiltonian cycle, or $deg_C\, u = 0$ or $deg_C\, v = 0$.

Now, if

$$deg_{C_1}\, u + deg_{C_1}\, v \geq \frac{|V(C)|}{2} + 1,$$

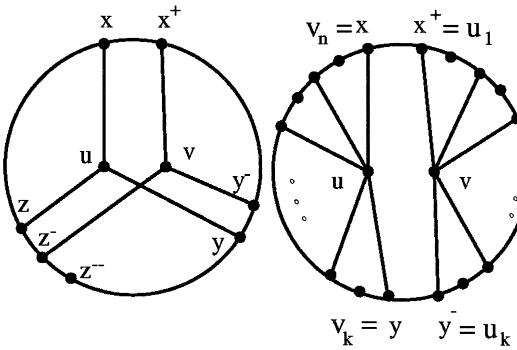
then we cannot have the situation that $deg_C\, u = 0$ or $deg_C\, v = 0$. Thus, again $\langle V(C) \cup V(P[u, v])\rangle$ is hamiltonian.

Now suppose that C also contains a 2-factor with exactly two cycles, say C_{11} and C_{12}. Then we have that either $deg_{C_{11}}\, u + deg_{C_{11}}\, v \geq \frac{|V(C)|}{2} + 1$ or $deg_{C_{12}}\, u + deg_{C_{12}}\, v \geq \frac{|V(C)|}{2} + 1$. Thus, either $\langle C_{11} \cup \{u, v\}\rangle$ or $\langle C_{12} \cup \{u, v\}\rangle$ is hamiltonian. In either case, we have the desired 2-factor of $\langle V(C) \cup V(P[u, v])\rangle$ with 2 cycles. □

Lemma 2. *Let* $G = (X \cup Y, E)$ *be a bipartite graph and let* $C = u_1 v_1 u_2 v_2 \dots u_n v_n u_1$ *be a cycle in* G. *If* $u \in X$ *and* $v \in Y$ *are two vertices of* $G - V(C)$ *and if*

$$deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 1,$$

then $\langle V(C) \cup \{u, v\}\rangle$ *is hamiltonian unless equality holds and, up to renumbering, we have that* v *is adjacent to* $u_1, \dots, u_k$ *and* u *is adjacent to* $v_k, \dots, v_n$, *for some* k.



(a) (b)

Fig. 1

Proof. Suppose, to the contrary, $\langle V(C) \cup \{u, v\} \rangle$ is not hamiltonian. Since $deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 1$, there are two consecutive vertices on C, say x and x^+, with $x \in N(u)$ and $x^+ \in N(v)$. Then, for any $w \neq x$, $w \in N(u)$ implies that $w^+ \notin N(v)$.

Now let y be the next neighbor of u along C from x following the orientation given to C. Because of the degree sum condition, $vy^- \in E(G)$ (note that y^- and x^+ may be the same vertex). Recall $u \in X$ and $v \in Y$. If there is a vertex $z \in C(y, x] \cap Y$ such that $z^{--} \notin N(u)$ and $z \in N(u)$, then $vz^- \in E(G)$, (or the degree condition would fail) which implies that $\langle V(C) \cup \{u, v\} \rangle$ is hamiltonian (see Figure 1a). Thus, $N(u) \cap V(C) = C[y, x] \cap Y$, which implies that $\langle V(C) \cup \{u, v\} \rangle$ is hamiltonian or $N(v) \cap C[y, x] = \varnothing$. Since

$$deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 1$$

we have that $N(v) \cap V(C) = C[x, y] \cap X$, that is, up to renumbering, v is adjacent to precisely $u_1, \ldots, u_k$ for some k and u is adjacent to precisely $v_k, \ldots, v_n$ (see Figure 1b), and hence equality holds in the degree sum. □

Lemma 3. *Let $G = (X \cup Y, E)$ be a bipartite graph and C a cycle in G with $|V(C)| \geq 6$. Let $u \in X$, $v \in Y$ and $u, v \in V(G) - V(C)$. If*

$$deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 2,$$

then $\langle V(C) \cup \{u, v\} \rangle$ has a 2-factor with exactly two cycles.

Proof. Since $deg_C\, u + deg_C\, v \geq \frac{|V(C)|}{2} + 2$, then $|N_C(u) \cap (N_C^-(v))| \geq 2$ and $|N_C(u) \cap (N_C^+(v))| \geq 2$. Thus, there are two distinct vertices $x, x_1 \in N_C(u)$ such that $x^+ \neq x_1^-$ and $\{x^+, x_1^-\} \subseteq N_C(v)$ (see Figure 2). A 2-factor is easily found. □

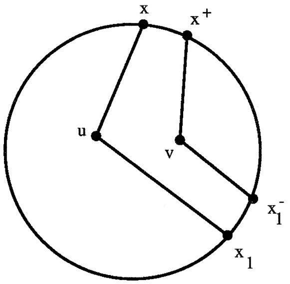


Fig. 2

3. Proof of Main Theorem

We now present the proof of our main result, Theorem 4.

Proof of Theorem 4. Assume that G does not contain a 2-factor with exactly k cycles. Since $deg\,u + deg\,v \geq n + 1$ for every $u \in X$ and $v \in Y$, we assume, without loss of generality, that $deg\,x \geq \frac{n+1}{2}$ for each $x \in X$.

We would fail to have a $K_{4,4}$ in G, if for each possible set of 4 vertices (in say X), there were at most 3 common neighbors (in Y). However, from our degree condition and since $n \geq 51$, we see that $\binom{\frac{n+1}{2}}{4} n > 3\binom{n}{4}$ and hence, that G contains a $K_{4,4}$.

Let C_1 be an 8-cycle in $K_{4,4}$. Clearly, $K_{4,4}$ also contains two vertex disjoint 4-cycles, call them C_{11} and C_{12}. Now we claim that in $G - V(C_1)$, there must exist at least $k - 2$ vertex disjoint 4-cycles. To see this, suppose that the claim fails to hold. Then there are at most $k - 3$ vertex disjoint 4-cycles in $G - V(C_1)$. Call a largest collection of 4-cycles F and say it contains s vertex disjoint 4-cycles. Let $X_R = X - V(C_1) - V(F)$ and $Y_R = Y - V(C_1) - V(F)$ and $t = |X_R| = |Y_R| = n - 2s - 4$. By our degree condition, we have $t \geq n - 2(k-3) - 4 \geq n - 2k + 2 > 0$. Since there are no 4-cycles in $\langle X_R \cup Y_R \rangle$, by counting the number of pairs of distinct vertices in Y_R which have the same neighbor in X_R, we see that

$$\binom{\frac{n+1}{2} - 2s - 4}{2} t \leq \binom{t}{2}.$$

Since $s \leq k - 3$, to reach a contradiction, we only need to show that

$$((n+1)/2 - 2k + 2)((n+1)/2 - 2k + 1) \geq n.$$

Note that $n \geq \max\{51, k^2/2 + 1\}$. Thus, if $51 \geq k^2/2 + 1$, then $k \leq 10$ and

$$\begin{aligned} ((n+1)/2 - 2k + 2)((n+1)/2 - 2k + 1) &\geq ((n+1)/2 - 8)((n+1)/2 - 9) \\ &\geq 7((n+1)/2 - 8) \geq n. \end{aligned}$$

Hence, we assume that $k^2/2 + 1 > 51$, and so, $k \geq 11$. Thus,

$$(n+1)/2 - 2k + 1 \geq k^2/4 - 2k + 2 \geq 10.$$

Hence,

$$\begin{aligned} ((n+1)/2 - 2k + 2)((n+1)/2 - 2k + 1) &\geq 10((n+1)/2 - 2k + 2) && (1) \\ &= n + 1 + 4(n+1) - 20(k+1) && (2) \\ &\geq n + 1 + 4(k^2/2 + 2) - 20(k+1) \\ &> n. && (3) \end{aligned}$$

Hence, we have shown what we needed and the inequality is established. In particular, we have shown the following:

Claim 1. *The bipartite graph G contains $k-1$ vertex disjoint cycles $C_1, C_2, C_3, \ldots, C_{k-1}$ such that there are two vertex disjoint cycles, C_{11} and C_{12}, with $V(C_1) = V(C_{11}) \cup V(C_{12})$.*

Now, among all collections of $k-1$ vertex-disjoint cycles in G, choose one that covers the largest possible number of vertices and in addition, has the property that $V(C_1)$ can be partitioned into two parts that each contain a spanning cycle. Since G does not contain a 2-factor with exactly k cycles, the graph $H = G - \bigcup_{i=1}^{k-1} V(C_i) \neq \emptyset$, in fact, H has at least 2 vertices since it has even order.

Claim 2. *The graph H does not contain two nontrivial components.*

Suppose that H does contain two nontrivial components, say H_1 and H_2. Without loss of generality suppose that $|V(H_1)| \geq |V(H_2)|$ and let $uv \in E(H_2)$. Note that

$$deg_H\, u + deg_H\, v \leq |V(H_2)| \leq \frac{|V(H)|}{2}.$$

Thus, there is a cycle C_i $(1 \leq i \leq k-1)$ such that

$$deg_{C_i}\, u + deg_{C_i}\, v \geq \frac{|V(C_i)|}{2} + 1$$

and hence, by Lemma 1, $\langle V(C_i) \cup \{u, v\} \rangle$ is hamiltonian. But this contradicts the maximality of the original collection of cycles, a contradiction to our assumptions. Thus, H_2 must be trivial if it exists. □

We now note that if B is a connected bipartite graph with partite sets W_1 and W_2, where $|W_1| \leq |W_2|$, then B has a balanced connected subgraph.

If H has a nontrivial connected component H_1, let F_1 be a balanced connected subgraph of H_1. Further, we select F_1 such that $|V(F_1)|$ is maximum under the above restrictions. Then as before, all other components are trivial.

Claim 3. *The graph $F_1 \neq K_2$.*

Suppose to the contrary that $F_1 = K_2$. Let $V(F_1) = \{u, v\}$ where $uv \in E(G)$. Then,

$$deg_H\, u + deg_H\, v \leq \frac{|V(H)|}{2} + 1. \tag{4}$$

Note that equality holds in equation (4) if, and only if, H_1 is a star centered either at u or v. Without loss of generality, we assume that H_1 is a star centered at v.

By Lemma 1, we have that

$$deg_{C_i}\, u + deg_{C_i}\, v \leq \frac{|V(C_i)|}{2}$$

for each $i = 1, 2, \ldots, k-1$ or our cycle system could be enlarged, a contradiction. Since $deg\, u + deg\, v \geq n + 1$, we have that

$$deg_{C_i}\, u + deg_{C_i}\, v = \frac{|V(C_i)|}{2}$$

for each i. Then, again by Lemma 1, we have that either $deg_{C_i}\, u = \frac{|V(C_i)|}{2}$ and $deg_{C_i}\, v = 0$ or $deg_{C_i}\, v = \frac{|V(C_i)|}{2}$ and $deg_{C_i}\, u = 0$, for each $i = 2, \ldots, k-1$.

We shall show that $H = F_1 = K_2$. Suppose, to the contrary, $H - F_1 \neq \varnothing$. Now suppose there is a cycle C_i $(i \geq 2)$ such that $deg_{C_i}\, u = \frac{|V(C_i)|}{2}$. Let $u^* \in V(C_i) \cap X$. We interchange u and u^* to get a new cycle C_i^*. Then replacing C_i by C_i^* in our cycle system (and renaming C_i^* to C_i) preserves the properties of the system. Now let $H^* = \langle H - u + u^* \rangle$ and select a vertex $u_1 \neq u^*$ with $u_1 \in V(H) \cap X$. Note here that u_1 is adjacent to v. Then we have

$$deg_{H^*}\, u_1 + deg_{H^*}\, v \leq \frac{|V(H)|}{2}.$$

But then there is a cycle C_j such that

$$deg_{C_j}\, u_1 + deg_{C_j}\, v \geq \frac{|V(C_j)|}{2} + 1.$$

Thus, by Lemma 1, $\langle C_j^* \cup \{u_1, v\} \rangle$ has a hamiltonian cycle C_j^{**} which preserves the properties of C_j. But then replacing C_j by C_j^{**} contradicts the maximality of our cycle system. Thus, $deg_{C_i}\, u = 0$ for each $i \geq 2$. Since $deg\, u \geq 2$, then $deg_{C_1}\, u \neq 0$. If $deg_{C_1}\, v = 0$, then $deg_{C_1}\, u = \frac{|V(C_1)|}{2}$. Therefore,

$$deg_{C_{11}}\, u = |V(C_{11})|/2 \quad \text{and} \quad deg_{C_{12}}\, u = |V(C_{12})|/2,$$

since $V(C_1) = V(C_{11}) \cup V(C_{12})$. Let $u^* \in V(C_{11}) \cap X$. Since both the successor (on C_{11}) and the predecessor of u^* on C_{11} are neighbors of u, $\langle V(C_{11}) \cup \{u\} - \{u^*\} \rangle$ has a hamiltonian cycle C_{11}^*. For the same reason, $\langle V(C_1) \cup \{u\} - \{u^*\} \rangle$ has a hamiltonian cycle C_1^*. Then, replacing C_1 by C_1^* in our cycle system preserves the properties of the system. Let $H^* = \langle H \cup \{u\} - \{u^*\} \rangle$ and select a vertex $u_1 \neq u^*$ in $V(H) \cap X$. Then, again

$$deg_{H^*} u_1 + deg_{H^*} v \leq \frac{|V(H)|}{2}.$$

Then, there is a cycle C_j such that

$$deg_{C_j} u_1 + deg_{C_j} v \geq \frac{|V(C_j)|}{2} + 1$$

which, by Lemma 1, yields a contradiction.

Thus, $deg_{C_1} v \neq 0$. If for some $j = 1, 2$, we have that $deg_{C_{1j}} u \neq 0$ and $deg_{C_{1j}} v \neq 0$, then by Lemma 1, $\langle V(C_{1j}) \cup \{u, v\} \rangle$ is hamiltonian, and $\langle V(C_1) \cup \{u, v\} \rangle$ is hamiltonian, a contradiction. Therefore, since $deg_{C_1} u + deg_{C_1} v = \frac{|V(C_1)|}{2}$, we may assume without loss of generality that

$$deg_{C_{11}} u = |V(C_{11})|/2 \quad \text{and} \quad deg_{C_{12}} v = |V(C_{12})|/2,$$

that is, $N(u) \supseteq V(C_{11}) \cap Y$ and $N(v) \supseteq V(C_{12}) \cap X$. For each $u^* \in V(C_{11}) \cap X$, if its successor and predecessor on C_1 are both in $V(C_{11}) \cap Y$, we interchange u and u^*. In the same manner as above, we again obtain a contradiction. Thus, u^* must have a neighbor in $V(C_{12}) \cap Y$ for each $u^* \in V(C_{11}) \cap X$. It is readily seen that $V(C_1) \cup \{u, v\}$ is hamiltonian and has a 2-factor with exactly two cycles (see Figure 3), unless $|V(C_{11})| = |V(C_{12})| = 4$. However, the later case can happen only when $\langle V(C_1) \rangle$ is a $K_{4,4}$ by our choice of C_1. Clearly, in this case, we can enlarge the cycle system by inserting u and v to C_1, a contradiction. Therefore, we can conclude that $H - F_1 = \varnothing$ and that $H = F_1 = K_2$.

We now relabel the cycles $C_{11}, C_{12}, C_2, \ldots, C_{k-1}$ as $C_1^*, \ldots, C_k^*$. The cycle C_i^* is called a u-type cycle if $deg_{C_i^*} u = \frac{|V(C_i^*)|}{2}$ and C_i^* is called a v-type cycle if $deg_{C_i^*} v = \frac{|V(C_i^*)|}{2}$. Note that each C_i^* is either a v-type or u-type cycle and the degree sum condition implies there are both types of cycles. Assume without loss of generality that $C_1^*, \ldots, C_m^*$ are u-type cycles and $C_{m+1}^*, \ldots, C_k^*$ are v-type cycles.

If $\delta(G) \geq \frac{n+1}{2}$ and $deg\, u + deg\, v = n + 1$, we have that $deg\, u = deg\, v = \frac{n+1}{2}$. Thus, the total number of vertices in u-type cycles is $n - 1$ and the total number of vertices in v-type cycles is $n - 1$. Since $n \geq \frac{k^2}{2} + 1 \geq 2m(k - m) + 1$. Note that equality holds throughout if and only if $m = k/2$ and $n = k^2/2 + 1$. Now $\frac{n-1}{m} \geq 2(k - m)$. Let C_r^* be the longest cycle among the u-type cycles. Thus, $|V(C_r^*)| \geq 2(k - m)$. Note that if equality holds above, each u-type cycle has the same length, k. Since $\sum_{i=1}^{m} |V(C_i^*)| = n - 1$, each $u^* \in X \cap (\bigcup_{i=1}^{m} V(C_i^*))$ must have a neighbor in $\bigcup_{i=m+1}^{k} V(C_i^*)$. If either $|V(C_r^*)| > 2(k - m)$ or there is a vertex of C_r^* with at least two neighbors in $\bigcup_{m+1}^{k} V(C_i^*)$, then, by the pigeon hole principle, there are two vertices $u^*, u^{**} \in X \cap V(C_r^*)$ so that both u^* and u^{**} have a

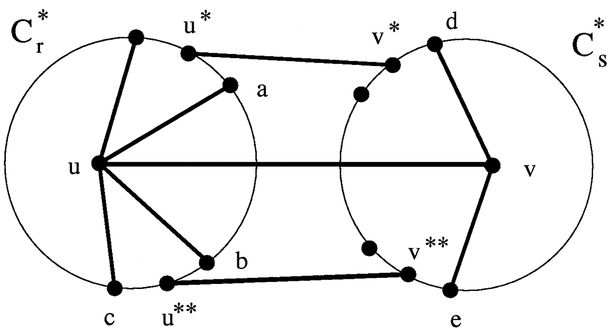


Fig. 3

neighbor in some cycle C_s^*, $(s > m)$. Then the configuration of Figure 3 shows that $\langle C_1^* \cup C_s^* \cup \{u, v\} \rangle$ has a 2-factor with exactly 2 cycles, namely

$$u^*, v^*, \ldots, v^{**}, u^{**}, b, \ldots, a, u, c, \ldots, u^*$$

and

$$v, d, \ldots, e, v.$$

Thus, the longest u-type cycle has length exactly $2(k - m)$ (which implies each u-type cycle is a longest such cycle) and has exactly one neighbor in $\bigcup_{m+1}^{k} V(C_i^*)$. Thus, the subgraph induced by the u-type (or v-type) cycles are complete bipartite graphs. Further, there is a perfect matching between the vertices in the u-type cycles and the vertices in the v-type cycles. It is easy then to construct a 2-factor with exactly k cycles in this graph. Thus G has a 2-factor with exactly k cycles.

Now if $deg\, u \geq \frac{n+1}{2}$ and $deg\, v < \frac{n+1}{2}$ (a similar argument applies if these conditions are reversed), then as before, there is a u-type cycle, say C_d^*, of length greater than $2(k - m)$. Since $deg\, v < \frac{n+1}{2}$, we see that for any $u^* \in V(C_d^*) \cap X$, $deg\, u^* \geq deg\, u \geq \frac{n+1}{2}$. Further, u^* is not adjacent to v or we could extend our cycle system. Thus, each $u^* \in V(C_d^*) \cap X$ must have at least one adjacency to the v-type cycles $C_{m+1}^*, \ldots, C_k^*$. We now proceed as before to obtain a contradiction. Hence, we conclude that $F_1 \neq K_2$. □

Claim 4. *If $E(F_1) \neq \emptyset$, then F_1 is hamiltonian.*

By Claim 3, if $E(F_1) \neq \emptyset$, then $|V(F_1)| \geq 4$. If F_1 is not hamiltonian, then there are two nonadjacent vertices $u, v \in V(F_1)$ such that $u \in X$ and $v \in Y$ and

$$deg_{F_1} u + deg_{F_1} v \leq \frac{|V(F_1)|}{2}$$

and so, by our choice of F_1,

$$deg_H u + deg_H v \leq \frac{|V(H)|}{2}.$$

Let $P[u,v]$ be a path in F_1 from u to v. Then from the above inequality we know that there is some $C_i, i \geq 1$, such that

$$deg_{C_i} u + deg_{C_i} v \leq \frac{|V(C_i)|}{2} + 1.$$

Thus, by Lemma 1, $\langle V(C_i) \cup V(P[u,v]) \rangle$ has a hamiltonian cycle C_i^* and as before, C_i^* preserves the properties of C_i. But then the cycles $C_1, \dots, C_{i-1}, C_i^*, C_{i+1}, \dots, C_{k-1}$ contradict the maximality of $\sum_{i=1}^{k-1} |V(C_i)|$. Thus, F_1 must contain a hamiltonian cycle. □

Since G does not contain a 2-factor with k cycles, it must be the case that $H - F_1 \neq \emptyset$, or we could add the cycle in F_1 to our cycle system and obtain a 2-factor with exactly k cycles, contradicting our assumptions.

Claim 5. $E(F_1) = \emptyset$.

Assume that $E(F_1) \neq \emptyset$, then by Claim 4, F_1 is hamiltonian. Let C be a hamiltonian cycle of F_1 and let $u \in X \cap V(H - F_1)$ and $v \in Y \cap V(H - F_1)$. Then, by our choice of F_1,

$$deg_H u + deg_H v \leq \frac{|V(F_1)|}{2} \leq \frac{|V(H)|}{2} - 1.$$

Thus,

$$\sum_{i=1}^{k-1} (deg_{C_i} u + deg_{C_i} v) \geq \sum_{i=1}^{k-1} \frac{|V(C_i)|}{2} + 2.$$

Thus, by Lemma 2 and Lemma 3, there is some $i \geq 2$ such that

$$deg_{C_i} u + deg_{C_i} v \geq \frac{|V(C_i)|}{2} + 1$$

Without loss of generality, we assume that $i = k-1$. Since $\langle V(C_{k-1}) \cup \{u, v\} \rangle$ is not hamiltonian, we have, by Lemma 2, the configuration with adjacencies up to renumbering, as shown in Figure 1b.

If $x = y$, replace C_{k-1} by the cycle $vC_{k-1}[x^+, y^-]v$. Then, note that $H^* = \langle (H - v) \cup \{x\} \rangle$. Let F_1^* be the largest component in H^*. Then, F_1^* is the only possible nontrivial component in H^* as we have shown before. Since $ux \in E(G)$, then $V(F_1^*) \supseteq V(F_1) \cup \{u, x\}$, a contradiction to the maximality of F_1.

Thus, $x \neq y$ and similarly, $x^+ \neq y^-$. Now select y^+ and $w = y^{--}$ and form two paths $P[u,v] = uC_{k-1}[y^{++}, w^-]v$ and $P^*[w, y^+] = wy^-yy^+$. Since $N(u) \cap C_{k-1}[x^+, w^-] = \emptyset$ and $N(v) \cap C_{k-1}[(y)^{++}, x] = \emptyset$, we have that

$$deg_P u + deg_P v \leq \frac{|V(P)|}{2}$$

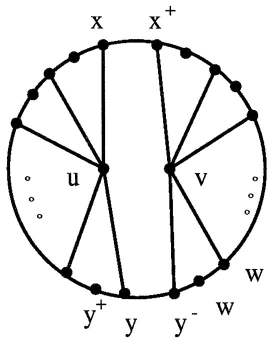


Fig. 4

and similarly,

$$deg_{P^*}\, y^+ + deg_{P^*}\, w \leq \frac{|V(P^*)|}{2}.$$

Also note that either $N(y^+) \cap V(H) = \emptyset$ or $N(w) \cap V(H) = \emptyset$. Otherwise, swapping $\{y^+, w\}$ and $\{u, v\}$, we obtain a set of $k-1$ cycles preserving the properties of $C_1, \dots, C_{k-1}$ and the remaining graph H^* obtained by deleting these cycles either contains two nontrivial components or the balanced component in H^* is larger than that in H, in either case a contradiction.

Hence, there is a cycle C_t $(t \neq i-1)$ such that

$$deg_{C_t}\, u + deg_{C_t}\, v \geq \frac{|V(C_t)|}{2} + 1$$

which, by Lemma 1, implies that $\langle V(C_t) \cup P[u, v] \rangle$ has a hamiltonian cycle C_t^* and (again by Lemma 1) it preserves the properties of $C_1, C_2, \dots, C_{k-1}$.

Let $C_1^* = C_1, C_2^* = C_2, \dots, C_t^*, \dots, C_{k-2}^* = C_{k-2}$. Since $deg\, y^+ + deg\, w \geq n+1$, there is a cycle C_j^* such that

$$deg_{C_j^*}\, y^+ + deg_{C_j^*}\, w \geq \frac{|V(C_j^*)|}{2} + 1.$$

Then, by Lemma 1, $\langle C_j^* \cup P^*[y^+, w] \rangle$ has a hamiltonian cycle, say C_j^{**}. Replacing C_j^* by C_j^{**} produces a collection of $k-2$ cycles, which, along with the hamiltonian cycle C in F_1, provides a collection of $k-1$ cycles which contradicts the maximality of $\sum_{i=1}^{k-1} |V(C_i)|$. Thus, we conclude that $F_1 = \emptyset$. □

We now note that since $E(F_1) = \emptyset, H$ is an empty graph.

Claim 6. *The graph H has order two.*

Suppose to the contrary that $|V(H)| \geq 4$ (recall H has even order), and say $u_1, u_2 \in V(H) \cap X$ and $v_1, v_2 \in V(H) \cap Y$. Since $deg\, u_1 + deg\, v_1 \geq n+1$ and by Lemma 2 , $deg_{C_i}\, u_1 + deg_{C_i}\, v_1 \leq \dfrac{|V(C_i)|}{2} + 1$, a direct count shows us that there

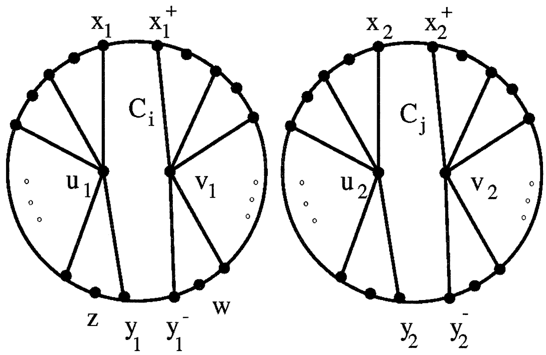


Fig. 5

are at least three cycles $C_{i_1}, C_{i_2}, C_{i_3}$ such that

$$deg_{C_{i_s}} u_1 + deg_{C_{i_s}} v_1 = \frac{|V(C_{i_s})|}{2} + 1,$$

$(s = 1, 2, 3)$. Similarly, there are three cycles $C_{j_1}, C_{j_2}, C_{j_3}$ such that

$$deg_{C_{j_t}} u_2 + deg_{C_{j_t}} v_2 = \frac{|V(C_{j_t})|}{2} + 1,$$

$(t = 1, 2, 3)$. Without loss of generality, assume $i_1 \neq j_1$ and $i_1 \geq 2, j_1 \geq 2$. Let $i = i_1$ and $j = j_1$.

By Lemma 3 we have the following two configurations of Figure 5.

If $x_1 = y_1$, then operating as before, we exchange v_1 with x_1 and obtain $k-1$ cycles $C_1^*, \ldots, C_{k-1}^*$ and $H = G - \bigcup_{i=1}^{k-1} V(C_i^*)$ where H now contains an edge, contradicting our previous claim. Similarly, $x_1^+ = y_1^-, x_2 = y_2$ and $x_2^+ = y_2^-$ all lead to contradictions.

But now, $u_2 C_j[y_2, x_2] u_2$ and $v_2 C_j[x_2^+, y_2^-] v_2$ provide a 2-factor of $\langle C_j \cup \{u_2, v_2\} \rangle$.

Assign one of these two cycles to C_i^* and the other one to C_j^*. These two cycles along with all other cycles $C_l, l \neq i, j$ gives a collection of $k-1$ cycles $C_1^*, \ldots, C_{k-1}^*$ with $C_1^* = C_1$.

Let $y_1^+ = z$ and $y_1^{--} = w$. Also let

$$P[u_1, v_1] = u_1 C[z^+, w^-] v_1$$

and

$$P^*[w, z] = w y_1^- y_1 z.$$

Clearly, $N(w) \cap V(H) = \emptyset$ and $N(z) \cap V(H) = \emptyset$. Otherwise, we may exchange u and z or v and w and then H^* will have at least one edge, contradicting our earlier claims.

Note that $deg_P u_1 + deg_P v_1 \le \frac{|V(P)|}{2}$ and $deg_{P^*} z + deg_{P^*} w \le \frac{|V(P^*)|}{2}$. Since $deg\, u_1 + deg\, v_1 \ge n+1$, there is a cycle C_s^* such that $deg_{C_s^*} u_1 + deg_{C_s^*} v_1 \ge \frac{|V(C_s^*)|}{2} + 1$.

Then $\langle V(C_s^*) \cup V(P[u_1, v_1])\rangle$ has a hamiltonian cycle, say C_s^{**} and by Lemma 1 it preserves the properties of C_s^*. Let $C_1^{**} = C_1^*, \ldots, C_s^{**} = C_s^{**}, \ldots, C_{k-1}^{**} = C_{k-1}^*$. Since $deg\, z + deg\, w \ge n+1$ and $deg_{P^*} z + deg_{P^*} w \le \frac{|V(P^*)|}{2}$, and $N(z) \cap V(H) = \varnothing$ and $N(w) \cap V(H) = \varnothing$, there is a cycle C_t^{**} such that

$$deg_{C_t^{**}} z + deg_{C_t^{**}} w \ge \frac{|V(C_t^{**})|}{2} + 1.$$

By Lemma 1, $\langle V(C_t^{**}) \cup V(P[w, z])\rangle$ is hamiltonian and the cycle preserves the properties of C_t^{**}, which again allows us to contradict the maximality of $\sum |V(C_i)|$, completing the proof of the claim. □

Thus, $|V(H)| = 2$, say $V(H) = \{u, v\}$. Since, by Lemma 2,

$$deg_{C_1} u + deg_{C_1} v = \frac{|V(C_1)|}{2} + 1$$

and $deg\, u + deg\, v \ge n+1$, there is an $i \ge 2$ such that

$$deg_{C_i} u + deg_{C_i} v = \frac{|V(C_i)|}{2} + 1.$$

By Lemma 2, $\langle V(C_i) \cup \{u, v\}\rangle$ has the subgraph of Figure 1b, or we would be able to again contradict the maximality of our collection of cycles.

Note that if $x = y$, we could swap v with x to obtain the cycles

$$C_1^* = C_1, \quad C_2^* = C_2, \ldots, C_i^* = vC[x^+, y^-]v, \quad C_{i+1}^*, \ldots, C_{k-1}^*$$

But these $k-1$ cycles preserve the properties of $C_1, \ldots, C_{k-1}$. However, then $G - \bigcup_{i=1}^{k-1} V(C_i^*) = K_2$, a contradiction to Claim 4. Similarly, we have $x^+ \neq y^-$. Thus, the graph $\langle V(C_i) \cup \{u, v\}\rangle$ has two cycles,

$$C_{i_1} = uC[y, x]u$$

and

$$C_{i_2} = vC[x^+, y^-]v.$$

Now, $C_1, \ldots, C_{i_1}, C_{i_2}, \ldots, C_{k-1}$ forms a 2-factor of G with exactly k cycles, a contradiction.

This contradiction completes the proof of the theorem. □

The following Corollary is immediate.

Corollary 5. *If G is a balanced bipartite graph of order $2n$ with $n \geq \max\left\{51, \frac{k^2}{2}+1\right\}$ and $\delta(G) \geq \frac{n+1}{2}$, then G contains a 2-factor with exactly k cycles.*

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