Graphs and Combinatorics (2024) 40:32
https://doi.org/10.1007/s00373-024-02761-0

ORIGINAL PAPER

The Spectral Radius, Maximum Average Degree and Cycles of Consecutive Lengths of Graphs

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Received: 17 May 2023 / Revised: 12 December 2023 / Accepted: 7 February 2024 /
Published online: 5 March 2024

Abstract
In this paper, we study the relationship between spectral radius and maximum average degree of graphs. By using this relationship and the previous technique of Li and Ning in (J Graph Theory 103:486–492, 2023), we prove that, for any given positive number $\varepsilon < \frac{1}{3}$, if n is a sufficiently large integer, then any graph G of order n with $\rho(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor}$ contains a cycle of length t for all integers $t \in [3, (\frac{1}{3} - \varepsilon)n]$, where $\rho(G)$ is the spectral radius of G. This improves the result of Li and Ning (2023).

Keywords Spectral radius · Cycles of consecutive lengths · Spectral extrema

Mathematics Subject Classification 05C50

1 Introduction

All graphs considered in this paper are finite, undirected and simple. For a graph G, let $\overline{G}$ denote the complement of G. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For a subset B of $V(G)$, let $G[B]$ be the subgraph of G induced by B, and let $G - B$ be the graph $G[V(G) - B]$. For a vertex u of G, let $G - u = G - \{u\}$, and let $d_G(u)$ be the *degree* of u in G. The vertex u is called a *cut vertex* of G if $G - u$ has more components than G. Let $\delta(G)$ denote the *minimum degree* of G. The *spectral radius* of G, denoted by $\rho(G)$, is the largest eigenvalue of its adjacency matrix. By Perron–Frobenius Theorem, $\rho(G)$ has a non-negative eigenvector. A non-negative eigenvector corresponding to $\rho(G)$ is called a *Perron vector* of G. If G is connected, then any Perron vector of G has

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positive entries. For any terminology used but not defined here, one may refer to [1, 5].

For a graph G, the *maximum average degree* of G, denoted by $\mathrm{mad}(G)$, is defined as

$$\max_{\emptyset \neq S \subseteq V(G)} \frac{2|E(G[S])|}{|S|}.$$

Clearly, $\mathrm{mad}(G) \leq \rho(G)$ (see [1]). A subset $B \neq \emptyset$ of $V(G)$ is called *critical*, if the average degree of $G[B]$ equals $\mathrm{mad}(G)$, i.e., $2|E(G[B])| = \mathrm{mad}(G) \cdot |B|$. A *pseudo-forest* is a graph of which each component contains at most one cycle. As is well known (see [10]), a graph can be decomposed into k pseudoforests if and only if its maximum average degree is at most $2k$. For more study on the relationship between decomposition into pseudoforests of a graph and its maximum average degree, one may refer to [6, 7].

For a certain integer n, let K_n and C_n be the complete graph and the cycle on n vertices, respectively. For two vertex-disjoint graphs G_1 and G_2, let $G_1 \cup G_2$ be the disjoint union of them, and let $G_1 \vee G_2$ be the graph obtained from G_1 and G_2 by adding all the edges between $V(G_1)$ and $V(G_2)$.

In 2008, Nikiforov [12] studied spectral radius condition for cycles of consecutive lengths in graphs, and proposed the following open problem.

Problem 1 (Nikiforov [12]). *What is the maximum positive number c_0 such that for any given real number $0 < \varepsilon < c_0$ and sufficiently large n, every graph G of order n with $\rho(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor}$ contains a cycle of length t for every integer $3 \leq t \leq (c_0 - \varepsilon)n$.*

Considering the graph $K_s \vee \overline{K_{n-s}}$ (see [12]), where $s = \left\lfloor \frac{(3-\sqrt{5})n}{4} \right\rfloor$, we see that $c_0 \leq \frac{3-\sqrt{5}}{2} = 0.38196\cdots$. Nikiforov [12] proved that $c_0 \geq \frac{1}{320}$. Ning and Peng [14] slightly refined this result by showing $c_0 \geq \frac{1}{160}$. Very recently, Zhai and Lin [18] proved that $c_0 \geq \frac{1}{7}$; Li and Ning [11] improved these results to $c_0 \geq \frac{1}{4}$. For other related results, one may refer to [19].

As one main result of this paper, we prove that $c_0 \geq \frac{1}{3}$. The rest of this paper is organized as follows. In Sect. 2, we study the relationship between spectral radius and maximum average degree of graphs. In Sect. 3, we give a proof of $c_0 \geq \frac{1}{3}$.

2 Spectral Radius and Maximum Average Degree

In this section, we study the relationship between spectral radius and maximum average degree of graphs. To prove the main result of this section, we need several lemmas. The first one is from [1], and the second one is the Theorem 8.1.3 of [5].

Lemma 2.1 ([1]) *If H is a subgraph of a connected graph G, then $\rho(H) \leq \rho(G)$, with equality if and only if $H = G$.*

Lemma 2.2 ([5]) *Let G be a connected graph with a Perron vector $\mathbf{x} = (x_w)_{w \in V(G)}$. For an edge uv_1 and a non-edge uv_2 of G, let G' be the graph obtained from G by deleting the edge uv_1, and adding the edge uv_2. If $x_{v_1} \leq x_{v_2}$, then $\rho(G') > \rho(G)$.*

The following lemma can be deduced from Proposition 2.2 and Theorem 2.3 of [9].

Lemma 2.3 ([9]) *Let G be a connected graph on n vertices and m edges with $\delta(G) \geq k$. Then $\rho(G) \leq \frac{k-1}{2} + \sqrt{2m - kn + \frac{(k+1)^2}{4}}$. Equality holds if and only if G is either a k-regular graph or a bidegreed graph in which each vertex is of degree either k or $n-1$.*

We firstly study critical subsets in graphs.

Lemma 2.4 *Let G be a graph, and let B and C be two critical subsets of $V(G)$. If $B \cap C \neq \emptyset$, then $B \cap C$ is a critical subset of $V(G)$.*

Proof Set $\mathrm{mad}(G) = k$, where k is a rational number. Clearly, a subset $S \neq \emptyset$ of $V(G)$ is critical if and only if $2|E(G[S])| = k|S|$. Note that $|B| + |C| = |B \cup C| + |B \cap C|$. Since each edge e of $G[B \cup C]$ is counted at most once in $|E(G[B])| + |E(G[C])|$ except that e is counted precisely twice for $e \in E(G[B \cap C])$, we have

$$|E(G[B])| + |E(G[C])| \leq |E(G[B \cup C])| + |E(G[B \cap C])| \quad (*).$$

Since B and C are two critical subsets of $V(G)$, we have $2|E(G[B])| = k|B|$ and $2|E(G[C])| = k|C|$. By $(*)$ we have

$$2|E(G[B \cup C])| + 2|E(G[B \cap C])| \geq k|B| + k|C| = k|B \cup C| + k|B \cap C|.$$

Since $2|E(G[B \cup C])| \leq k|B \cup C|$ and $2|E(G[B \cap C])| \leq k|B \cap C|$ by definition of k, we have $2|E(G[B \cup C])| = k|B \cup C|$ and $2|E(G[B \cap C])| = k|B \cap C|$. Thus $B \cap C$ is a critical subset of $V(G)$. This completes the proof. □

For two integers $k \geq 1$ and $n \geq 2k+1$, let $\mathcal{G}_n^k$ be the set of graphs on n vertices with maximum average degree at most $2k$.

Theorem 2.5 *For two integers $k \geq 1$ and $n \geq 2k+1$, let G be an extremal graph with maximum spectral radius in $\mathcal{G}_n^k$. Then $G = K_k \vee H$, where H is a graph on $n-k$ vertices with $\binom{k+1}{2}$ edges.*

Proof Note that $\mathrm{mad}(K_{2k+1}) = 2k$ and $K_k \vee (K_{k+1} \cup \overline{K_{n-1-2k}}) \in \mathcal{G}_n^k$. Thus, for any graph H' on $n-k$ vertices with $\binom{k+1}{2}$ edges, the graph $K_k \vee H'$ is in $\mathcal{G}_n^k$.

Now let G be an extremal graph with maximum spectral radius in $\mathcal{G}_n^k$. Then $\rho(G) \geq \rho(K_k \vee (K_{k+1} \cup \overline{K_{n-1-2k}})) \geq 2k$ as $K_k \vee (K_{k+1} \cup \overline{K_{n-1-2k}}) \in \mathcal{G}_n^k$.

Case 1. G is connected. Let $\mathbf{x} = (x_w)_{w \in V(G)}$ be a Perron vector of G. Without loss of generality, assume that $x_{u_1} \geq x_{u_2} \geq \cdots \geq x_{u_n}$, where u_i for $1 \leq i \leq n$ are the vertices of G.

Now we shall prove that $d_G(u_i) = n-1$ for any $1 \leq i \leq k$. If this is not true, then there is an integer $1 \leq i_0 \leq k$ such that $d_G(u_{i_0}) < n-1$. Let w be a vertex of G not adjacent to u_{i_0}. Let G_1 be the graph obtained from G by adding the edge wu_{i_0}. By Lemma 2.1 we have $\rho(G_1) > \rho(G)$. Thus $\mathrm{mad}(G_1) > 2k$ by the choice of G. Let B be a critical subset of $V(G_1)$. Clearly, $w, u_{i_0} \in B$. Then $2|E(G_1[B])| = \mathrm{mad}(G_1) \cdot |B| > 2k|B|$, implying that $2|E(G_1[B])| \geq 2k|B| + 2$ by parity. Thus $2|E(G[B])| = 2|E(G_1[B])| - 2 \geq 2k|B|$. Hence $2|E(G[B])| = 2k|B|$ as $\mathrm{mad}(G) \leq 2k$. So, B is also a critical subset of $V(G)$ and $\mathrm{mad}(G) = 2k$.

Let B_j for $1 \leq j \leq \ell$ be all the critical subsets of $V(G)$ containing vertices w and u_{i_0}, where $\ell \geq 1$. Let $S = \cap_{1 \leq j \leq \ell} B_j$. Note that $w, u_{i_0} \in S$. By Lemma 2.4 we have that S is a critical subset of $V(G)$. Let $S_0 = S - \{w\}$. Since $|E(G[S])| = |E(G[S_0])| + d_{G[S]}(w)$, $2|E(G[S])| = 2k|S|$ and $2|E(G[S_0])| \leq 2k|S_0|$, we have

$$2d_{G[S]}(w) = 2|E(G[S])| - 2|E(G[S_0])| \geq 2k|S| - 2k|S_0| = 2k,$$

implying that $d_{G[S]}(w) \geq k$. Since w is not adjacent to u_{i_0}, there is a vertex in $S - \{u_1, u_2, ..., u_k\}$, say u_t with $t \geq k+1$, such that w is adjacent to u_t. Let G_2 be the graph obtained from G by adding the edge wu_{i_0} and deleting the edge wu_t. By Lemma 2.1 we have $\rho(G_2) > \rho(G)$ as $x_{u_{i_0}} \geq x_{u_t}$. Since G is an extremal graph with maximum spectral radius in $\mathcal{G}_n^k$, we have $\mathrm{mad}(G_2) > 2k$. Let C be a critical subset of $V(G_2)$. Then $2|E(G_2[C])| > 2k|C|$, implying that $w, u_{i_0} \in C$ and $u_t \notin C$. Using a similar discussion as on B, we can show that C is also a critical subset of $V(G)$ containing vertices u_{i_0} and w. By the definition of S, we have $S \subseteq C$. However, this is a contradiction, since $u_t \in S$ and $u_t \notin C$. Thus we obtain that $d_G(u_i) = n-1$ for any $1 \leq i \leq k$.

Consequently, $G = K_k \vee H$, where H is a graph on $n-k$ vertices. From $2|E(G)| \leq 2k|G| = 2kn$ we obtain $|E(H)| \leq \binom{k+1}{2}$. Since G is an extremal graph with maximum spectral radius in $\mathcal{G}_n^k$, we have $|E(H)| = \binom{k+1}{2}$ by Lemma 2.1.

Case 2. G is not connected. Let Q be a component of G with $\rho(Q) = \rho(G)$. Let $|Q| = n_1 < n$. If $n_1 \leq 2k$, then $\rho(G) = \rho(Q) \leq n_1 - 1 \leq 2k - 1$, a contradiction. Thus $n_1 \geq 2k+1$. So, Q is a connected extremal graph with maximum spectral raidus in $\mathcal{G}_{n_1}^k$. Similar to Case 1, we can show that $Q = K_k \vee H_0$, where H_0 is a graph on $n_1 - k$ vertices with $\binom{k+1}{2}$ edges. Let $H_1 = H_0 \cup \overline{K_{n-n_1}}$. Then $K_k \vee H_1 \in \mathcal{G}_n^k$. Since Q is a proper subgraph of $K_k \vee H_1$, we have $\rho(G) = \rho(Q) < \rho(K_k \vee H_1)$ by Lemma 2.1. This contradicts the choice of G.

By the above discussion, we have that $G = K_k \vee H$, where H is a graph on $n-k$ vertices with $\binom{k+1}{2}$ edges. This completes the proof. $\square$

As in [15], a matrix $A = (a_{ij})_{n \times n}$ is called *a stepwise matrix*, if it is deduced that $a_{hk} = 1$ from $a_{ij} = 1$ with $i < j$, whenever $h < k \leq j$ and $h \leq i$. Following [2], we can (easily) show that the adjacency matrix of the graph H in Theorem 2.5 is a stepwise matrix. However, it seems hard to characterize the structure of H completely for general values of n and k. For a similar problem on determining the extremal graphs with maximum spectral radius among all connected graphs of order n and size m, it is only known for special values of n and m (see [2–4]). In particular, when k is given and n is sufficiently large, by a very similar proof as Lemma 2.8 of [20], we can obatin that the extremal graph $G = K_k \vee H$ in Theorem 2.5 has a *local degree sequence majorization* (see Lemma 2.8 of [20] for the definition). Hence the graph H in Theorem 2.5 must be the disjoint union of the star graph of order $\binom{k+1}{2} + 1$ and $n - k - 1 - \binom{k+1}{2}$ isolated vertices when n is large enough.

Corollary 2.6 *For two integers $k \geq 1$ and $n \geq 2k + 1$, let G be a graph in $\mathcal{G}_n^k$. Then $\rho(G) < \frac{k-1}{2} + \sqrt{kn + \frac{(k+1)^2}{4}}$.*

Proof By Theorem 2.5 we have $\rho(G) \leq \rho(K_k \vee H)$, where H is a graph on $n - k$ vertices with $\binom{k+1}{2}$ edges. Clearly, $K_k \vee H$ is a connected graph with kn edges and $\delta(K_k \vee H) \geq k$. By Lemma 2.3, we have that $\rho(K_k \vee H) \leq \frac{k-1}{2} + \sqrt{kn + \frac{(k+1)^2}{4}}$. Note that equality can not hold, since $K_k \vee H$ is neither a k-regular graph nor a bidegreed graph in which each vertex is of degree either k or $n-1$. Thus $\rho(G) \leq \rho(K_k \vee H) < \frac{k-1}{2} + \sqrt{kn + \frac{(k+1)^2}{4}}$. This completes the proof. □

3 Spectral Radius and Cycles of Consecutive Lengths

As pointed out in [11], to attack Nikiforov's problem, one main technique is to use both Gould–Haxell–Scott Theorem [8] and Voss and Zuluaga's theorem [17], together with Sun–Das inequality [16], to find a subgraph with large connectivity and average degree, which also contains cycles of consecutive lengths. We will use Corollary 2.6 and the method used in Li and Ning [11] to find a required one.

For a graph G, let $ec(G)$ and $oc(G)$ denote the length of a longest even cycle and the length of a longest odd cycle of G, respectively. To prove the main result of this section, we need some preliminaries. The whole machine from [11] due to Li and Ning includes the following four results. The first one is from [17], and the second one is from [8].

Theorem 3.1 ([17]) *Let G be a 2-connected graph with $\delta(G) \geq k \geq 3$ having at least $2k + 1$ vertices. Then $ec(G) \geq 2k$, and $oc(G) \geq 2k - 1$ if G is non-bipartite.*

Theorem 3.2 ([8]) *For any real number $c > 0$, there exits a constant $K := K(c) = \frac{7.5\times 10^5}{c^5}$ such that the following holds. Let G be a graph on $n \geq \frac{45K}{c^4}$ vertices with $\delta(G) \geq cn$. Then G contains a cycle of length t for every even $t \in [4, ec(G) - K]$ and every odd $t \in [K, oc(G) - K]$.*

The following two lemmas are from [11].

Lemma 3.3 ([11]) *Let G be a graph on n vertices. If $ec(G) \leq 2k$ where $k \geq 1$ is an integer, then* $|E(G)| \leq \frac{(2k+1)(n-1)}{2}$.

Lemma 3.4 ([11]) *Let G be a graph. For any $v \in V(G)$, we have* $\rho^2(G) \leq \rho^2(G-v) + 2d_G(v)$.

The following fact (from Theorem 1 of [13]) will be used in the proof of Theorem 3.5.

Fact 1. For a graph G on n vertices, if $\rho(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor}$, then G contains a triangle.

Theorem 3.5 *For any real number $0 < \varepsilon < \frac{1}{3}$, there is a positive integer $N := N(\varepsilon)$ satisfying: if G is a graph on $n \geq N$ vertices satisfying $\rho(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor}$, then G contains the cycle C_r for all integers $r \in [3, (\frac{1}{3} - \varepsilon)n]$.*

Proof Let $c = \frac{1}{7}$ and $K = K(\frac{1}{7})$ be given as in Theorem 3.2. Let $0 < \varepsilon < \frac{1}{3}$ be fixed and let $N(\varepsilon)$ be sufficiently large such that all the inequalities appeared in the following hold when $n \geq N(\varepsilon)$. (A specified value of $N(\varepsilon)$ can be given in our proof, though we will not do this.) Thus we can assume that n is sufficiently large in the following discussion.

Let k be the least integer such that $k \geq \frac{1}{2} \cdot \text{mad}(G)$. Then $\text{mad}(G) \leq 2k$. We will prove $k > \frac{n}{6}$. Suppose $k \leq \frac{n}{6}$. Note that G is in $\mathcal{G}_n^k$ defined in Sect. 3. By Corollary 2.6, we have that

$$\rho(G) < \frac{k-1}{2} + \sqrt{kn + \frac{(k+1)^2}{4}}.$$

Since $\rho(G) \geq \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \geq \sqrt{\frac{n^2-1}{4}}$, we obtain that

$$\frac{k-1}{2} + \sqrt{kn + \frac{(k+1)^2}{4}} > \sqrt{\frac{n^2-1}{4}},$$

implying that

$$k > \frac{\frac{n^2-1}{4} + \sqrt{\frac{n^2-1}{4}}}{n + 1 + \sqrt{\frac{n^2-1}{4}}}.$$

For large n, it is easy to check that

$$\frac{\frac{n^2-1}{4}+\sqrt{\frac{n^2-1}{4}}}{n+1+\sqrt{\frac{n^2-1}{4}}} > \frac{n}{6}.$$

Hence $k > \frac{n}{6}$, implying that $\frac{1}{2}\mathrm{mad}(G) > \left\lfloor \frac{n}{6} \right\rfloor$, i.e., $\mathrm{mad}(G) > 2\left\lfloor \frac{n}{6} \right\rfloor$.

Let B be a critical subset of $V(G)$. Then $2|E(G[B])| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |B|$. Let G_0 be an induced subgraph of G with maximum $|V(G_0)|$ such that $2|E(G_0)| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(G_0)|$. (Such G_0 exists as $G[B]$ exists.)

If $G_0 \neq G$, let $V(G) - V(G_0) = \left\{u_1, u_2, ..., u_\ell\right\}$, where $\ell = |V(G)| - |V(G_0)| \geq 1$. For any $1 \leq i \leq \ell$, let G_i be the subgraph of G induced by $V(G_0) \cup \left\{u_1, u_2, ..., u_i\right\}$. If

$$\sum_{1\leq i\leq \ell} d_{G_i}(u_i) > \left\lfloor \frac{n}{6} \right\rfloor \cdot \ell,$$

then

$$2|E(G)| = 2|E(G_0)| + 2\sum_{1\leq i\leq \ell} d_{G_i}(u_i) > 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(G_0)| + 2\left\lfloor \frac{n}{6} \right\rfloor \cdot \ell = 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(G)|,$$

implying that $G_0 = G$ by the choice of G_0, a contradiction. Thus

$$\sum_{1\leq i\leq \ell} d_{G_i}(u_i) \leq \left\lfloor \frac{n}{6} \right\rfloor \cdot \ell.$$

If $G_0 = G$, the above equality holds trivially, if we assume that $\ell = 0$ and $\sum_{1\leq i\leq \ell} d_{G_i}(u_i) = 0$ (this assumption does not affect the following discussion).

Let $H_0 = G_0$. If there is some vertex $v_1 \in V(H_0)$ with $d_{H_0}(v_1) < \left\lfloor \frac{n}{6} \right\rfloor$, let $H_1 = H_0 - v_1$. Repeating this process as far as possible, we obtain some vertices $v_1, v_2, ..., v_t$ in $V(H_0)$ with $t \geq 1$, such that $d_{H_{i-1}}(v_i) < \left\lfloor \frac{n}{6} \right\rfloor$ for any $1 \leq i \leq t$, where H_i is the subgraph of H_0 induced by $V(H_0) - \left\{v_1, v_2, ..., v_i\right\}$. Let $H = H_t$ (and let $H = H_0$ if $\delta(H_0) \geq \left\lfloor \frac{n}{6} \right\rfloor$, i.e. $t = 0$). Since

$$2|E(H_0)| = 2|E(G_0)| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(G_0)| = 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(H_0)|$$

and

$$2|E(H_0)| = 2|E(H)| + 2\sum_{1\leq i\leq t} d_{H_{i-1}}(v_i) < 2|E(H)| + 2t\left\lfloor \frac{n}{6} \right\rfloor,$$

we have

$$2|E(H)| > 2\left\lfloor \frac{n}{6} \right\rfloor \cdot (|V(H_0)| - t) = 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(H)|.$$

This implies that $E(H)$ is not empty and thus $t < |V(H_0)|$. Note that $\delta(H) \geq \left\lfloor \frac{n}{6} \right\rfloor$, since the process stops at step t. Recall that $|E(H)| \geq \left\lfloor \frac{n}{6} \right\rfloor \cdot |V(H)|$.

(i) Even cycle.

Since

$$|E(H)| \geq \left\lfloor \frac{n}{6} \right\rfloor \cdot |V(H)| > \frac{2\left\lfloor \frac{n}{6} \right\rfloor \cdot (|V(H)| - 1)}{2},$$

we have $ec(H) \geq 2\left\lfloor \frac{n}{6} \right\rfloor \geq \frac{n}{3} - 2$ by Lemma 3.3. Recall that $\delta(H) \geq \left\lfloor \frac{n}{6} \right\rfloor \geq \frac{|V(H)|}{7}$. By Theorem 3.2, H contains a cycle C_r with each integer $r \in [4, ec(H) - K]$ for large n. Thus G contains a cycle C_r with each integer $r \in [4, (\frac{1}{3} - \varepsilon)n]$ for large n.

(ii) Odd cycle.

Let $h = |V(H)|$. Then $n = h + \ell + t$. Note that $h \geq \frac{n}{3} - 1$ as $2|E(H)| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot |V(H)|$. By Lemma 3.4 we have that

$$\rho^2(G) - \rho^2(H) \leq 2 \sum_{1 \leq i \leq \ell} d_{G_i}(u_i) + 2 \sum_{1 \leq i \leq t} d_{H_{i-1}}(v_i) \leq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot (\ell + t) \leq \frac{n}{3}(n - h),$$

implying that

$$\rho^2(H) \geq \frac{n^2 - 1}{4} - \frac{n}{3}(n - h) = \frac{hn}{3} - \frac{n^2}{12} - \frac{1}{4}.$$

We shall prove that H is non-bipartite. Note that

$$\rho(H) \geq \frac{2|E(H)|}{|V(H)|} \geq 2\left\lfloor \frac{n}{6} \right\rfloor \geq \frac{n}{3} - 2.$$

By Fact 1, G is non-bipartite, since $\rho^2(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$. If H is bipartite, then $n - h \geq 1$ and $\rho(H) \leq \frac{h}{2}$ by Fact 1. Thus $\frac{n}{3} - 2 \leq \frac{h}{2}$, implying that $h \geq \frac{2}{3}n - 4$. But then

$$\rho^2(H) \geq \frac{hn}{3} - \frac{n^2}{12} - \frac{1}{4} > \frac{h^2}{4}$$

as $\frac{2}{3}n - 4 \leq h \leq n - 1$, implying that H contains a triangle by Fact 1. Thus H is non-bipartite.

Let $F_0 = H$. If F_0 has a cut vertex, say p_1, let $F_1 = F_0 - p_1$. If F_1 has a cut vertex, say p_2, let $F_2 = F_1 - p_2$. Repeating this process as far as possible, we obtain some vertices $p_1, p_2, ..., p_s$ of F_0 such that p_i is a cut vertex of F_{i-1} for each $1 \leq i \leq s$, where F_i is the subgraph of F_0 induced by $V(F_0) - \{p_1, p_2, ..., p_i\}$. Let $F = F_s$. Then F has no cut vertices as the process stops at step s.

We claim that $s \leq 5$. Otherwise the graph F_6 exists. Clearly, F_6 has at least 7 components. Thus F_6 has a vertex with degree at most $\frac{|V(F_6)|}{7} < \frac{n}{7}$. Note that

$\delta(F_6) \geq \delta(F_0) - 6 = \delta(H) - 6 \geq \frac{n}{6} - 7$. Then $\frac{n}{6} - 7 < \frac{n}{7}$, a contradiction. So we obtain that $s \leq 5$. Then $|V(F)| \geq h - 5 \geq \frac{n}{3} - 6$ and $\delta(F) \geq \delta(F_0) - 5 \geq \frac{n}{6} - 6$.

By Lemma 3.4, we have that

$$\rho^2(H) - \rho^2(F) \leq 2 \sum_{1\leq i\leq s} d_{F_{i-1}}(p_i) \leq 2s(h-1) \leq 10(h-1),$$

implying that

$$\rho^2(F) \geq \rho^2(H) - 10(h-1) \geq \frac{hn}{3} - \frac{n^2}{12} - \frac{1}{4} - 10(h-1).$$

Note that

$$2|E(F)| \geq 2|E(H)| - 2\sum_{1\leq i\leq s} d_H(p_i) \geq 2|E(H)| - 2s(h-1).$$

Recall that $2|E(H)| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot h$. Then $2|E(F)| \geq 2\left\lfloor \frac{n}{6} \right\rfloor \cdot h - 10h$ as $s \leq 5$. Thus

$$\frac{2|E(F)|}{|V(F)|} \geq \frac{2|E(F)|}{h} \geq \frac{n}{3} - 12.$$

Hence $\rho(F) \geq \frac{2|E(F)|}{|V(F)|} \geq \frac{n}{3} - 12$.

Since F has no cut vertices and $\delta(F) \geq \frac{n}{6} - 6 \geq 2$, each component of F is 2-connected. Let Q be a component of F such that $\rho(Q) = \rho(F)$. Then $\rho(Q) \geq \frac{n}{3} - 12$ and thus $|V(Q)| \geq \rho(Q) + 1 \geq \frac{n}{3} - 11$. Note that Q is 2-connected and $\delta(Q) \geq \frac{n}{6} - 6 \geq \frac{|V(Q)|}{7}$.

Now we shall prove that Q is non-bipartite. Recall that H is non-bipartite. We can assume that $Q \neq H$. Then F has another component, say Q_1. Since $\delta(F) \geq \left\lfloor \frac{n}{6} \right\rfloor - s$, we have $|V(Q_1)| \geq \left\lfloor \frac{n}{6} \right\rfloor - s + 1$. So $|V(Q)| \leq h - s - (\left\lfloor \frac{n}{6} \right\rfloor - s + 1) \leq h - \frac{n}{6}$. If $\frac{n}{3} - 12 > \frac{h-\frac{n}{6}}{2}$, then $\rho(Q) \geq \frac{n}{3} - 12 > \frac{|V(Q)|}{2}$, implying that Q contains a triangle. Thus we can assume that $\frac{n}{3} - 12 \leq \frac{h-\frac{n}{6}}{2}$, i.e., $h \geq \frac{5}{6}n - 24$. When $\frac{5}{6}n - 24 \leq h \leq n$ and n is sufficiently large, it is easy to show that

$$\frac{hn}{3} - \frac{n^2}{12} - \frac{1}{4} - 10(h-1) > \frac{(h-\frac{n}{6})^2}{4}$$

(which is equivalent to $\frac{5}{3}\frac{h}{n} - (\frac{h}{n})^2 - \frac{13}{36} > \frac{40h-39}{n^2}$, where $\frac{5}{6} - \frac{24}{n} \leq \frac{h}{n} \leq 1$). Thus

$$\rho^2(Q) = \rho^2(F) \geq \frac{hn}{3} - \frac{n^2}{12} - \frac{1}{4} - 10(h-1) > \frac{(h-\frac{n}{6})^2}{4} \geq \frac{|V(Q)|^2}{4}.$$

Then Q contains a triangle by Fact 1. Hence Q is non-bipartite.

Recall that $|V(Q)| \geq \frac{n}{3} - 11$ and $\delta(Q) \geq \frac{n}{6} - 6$. By Theorem 3.1 we have that

$$oc(Q) \geq \min\{2\delta(Q) - 1, |V(Q)|\} \geq \frac{n}{3} - 13.$$

By Theorem 3.2, Q contains a cycle C_r for all odd integers $r \in [K, \frac{n}{3} - 13 - K]$ for $n \geq N(\varepsilon)$. By Theorem 1 of [12], there is an integer $N_0(:= 10^4$, see [11]) such that the graph G on $n \geq N(\varepsilon) \geq N_0$ vertices contains a cycle C_r for all integers $r \in [3, \frac{n}{320}]$ as $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$. Then G contains a cycle C_r for all odd integers $r \in [3, (\frac{1}{3} - \varepsilon)n]$ for sufficiently large n.

By (i) and (ii), we have that G contains a cycle C_r for all integers $r \in [3, (\frac{1}{3} - \varepsilon)n]$ for sufficiently large n. This completes the proof. □

Using Corollary 2.6, we can only obtain $\text{mad}(G) \geq 2\lfloor \frac{n}{6} \rfloor$ if $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$. To improve the current constant $\frac{1}{3}$ in Theorem 3.5, one needs to obtain a larger constant $C > \frac{1}{3}$, such that $\text{mad}(G) \geq (C - \varepsilon)n$ if $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$. To do this, we need to strengthen Corollary 2.6 by using Theorem 2.5 when $n \leq 6k$ (i.e., we need to reduce the upper bound in Corollary 2.6 by $\Theta(k)$).

Acknowledgements The author would like to thank the editors and the referees for their valuable comments.

Funding This work is supported by NSFC (No. 12301448) and NSF of Shandong Province (No. ZR2022QA045).

Data Availability There is no associated data.

Declaration

Conflict of Interest There is no conflict of interest.

References

1. Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Springer (2012)
2. Brualdi, R.A., Solheid, E.S.: On the spectral radius of connected graphs. Publ. Inst. Math. **39**, 45–54 (1986)
3. Bell, F.K.: On the maximal index of connected graphs. Linear Algebra Appl. **144**, 135–151 (1991)
4. Cvetković, D., Rowlinson, P.: On connected graphs with maximal index. Publ. Inst. Math. **44**, 29–34 (1988)
5. Cvetković, D., Rowlinson, P., Simić, S.: An Introduction to the Theory of Graph Spectra. Cambridge University Press, Cambridge (2010)
6. Fan, G., Li, Y., Song, N., Yang, D.: Decomposing a graph into pseudoforests with one having bounded degree, Theory. J. Comb. Ser. B **115**, 72–95 (2015)
7. Grout, L., Moore, B.: The pseudoforest analogue for the Strong Nine Dragon Tree Conjecture is true. J. Comb. Theory Ser. B **145**, 433–449 (2020)
8. Gould, R.J., Haxell, P.E., Scott, A.D.: A note on cycle lengths in graphs. Graphs Combin. **18**, 491–498 (2002)

9. Hong, Y., Shu, J.L., Fang, K.F.: A sharp upper bound of the spectral radius of graphs, Theory. J. Comb. Ser. B **81**, 177–183 (2001)
10. Hakimi, S.L.: On the degrees of the vertices of a directed graph. J. Frankl. Inst. **279**(4), 290–308 (1965)
11. Li, B., Ning, B.: Eigenvalues and cycles of consecutive lengths. J. Graph Theory. **103**, 486–492 (2023)
12. Nikiforov, V.: A spectral condition for odd cycles in graphs. Linear Algebra Appl. **428**, 1492–1498 (2008)
13. Nikiforov, V.: Bounds on graph eigenvalues II. Linear Algebra Appl. **427**, 183–189 (2007)
14. Ning, B., Peng, X.: Extensions of the Erdős-Gallai theorem and Luo's theorem. Combin. Probab. Comput. **29**, 128–136 (2020)
15. Rowlinson, P.: On the maximal index of graphs with a prescribed number of edges. Linear Algebra Appl. **110**, 43–53 (1988)
16. Sun, S.W., Das, K.: A conjecture on the spectral radius of graphs. Linear Algebra Appl. **588**, 74–80 (2020)
17. Voss, H., Zuluaga, C.: Maximale gerade und ungerade Kreise in Graphen I (German). Wiss. Z. Tech. Hochsch. Ilmenau. **23**, 57–70 (1977)
18. Zhai, M., Lin, H.: A strengthening of the spectral chromatic critical edge theorem: books and theta graphs. J. Graph Theory. **102**, 502–520 (2023)
19. Zhang, Z., Zhao, Y.: A spectral condition for the existence of cycles with consecutive odd lengths in non-bipartite graphs. Discrete Math. **346**, 113365 (2023)
20. Zhai, M., Lin, H.: Spectral extrema of $K_{s,t}$-minor free graphs—on a conjecture of M. Tait, J. Comb. Theory Ser. B **157**, 184–215 (2022)