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ORIGINAL PAPER

The Existence of a Path with Two Blocks in Digraphs

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Abstract
We give a new elementary proof of El Sahili conjecture El Sahili (Discrete Math 287:151–153, 2004) stating that any n-chromatic digraph D, with $n \geq 4$, contains a path with two blocks of order n.

Keywords Path with two blocks · Chromatic number · Maximal forest · Final forest

Mathematics Subject Classification 05C05 · 05C15 · 05C20 · 05C38

1 Introduction and Elementary Definitions

A **graph** G is a couple $G = (V(G), E(G))$ where $V(G)$ is the set whose elements are the **vertices** of G, and $E(G)$ is the set whose elements are called the **edges** of G. The **neighborhood** of a vertex v in a graph G is denoted by $N_G(v)$ (or simply $N(v)$), and it is defined by $N(v) = \{u \in V(G);\ uv \in E(G)\}$.
A **proper coloring** of a graph is an assignment of colors to its vertex set so that no two adjacent vertices have the same color. A ***k*-coloring** of a graph is a proper coloring of the graph using k colors. For any graph G, we denote by $\chi(G)$ the smallest integer k such that G has a proper k-coloring.

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A **digraph** D is a couple $D = (V(D), A(D))$, where $V(D)$ is the set of **vertices** of D and $A(D)$ is its set of **arcs**. Consider a digraph D. The **out-neighborhood** of a vertex v of D is denoted by $N_D^+(v)$ (or simply $N^+(v)$), and it is defined by $N^+(v) = \{u \in V(D);\ (v, u) \in A(D)\}$. Similarly, the **in-neighborhood** of a vertex v of D is denoted by $N_D^-(v)$ (or simply $N^-(v)$), and it is defined by $N^-(v) = \{u \in V(D);\ (u, v) \in A(D)\}$. The **out-degree** (resp. **in-degree**) of v in D is $d_D^+(v) = |N_D^+(v)|$ (resp. $d_D^-(v) = |N_D^-(v)|$), and for simplicity we say $d^+(v)$ (resp. $d^-(v)$). A digraph H is said to be an **induced subdigraph** of D if $V(H) \subseteq V(D)$ and for all vertices u and v of H, (u, v) is an arc of H if and only if (u, v) is an arc of D. Let $S \subseteq V(D)$. If H is an induced subdigraph of D and $V(H) = S$, then we write $H = D[S]$. Let a be an arc. We define $D + a$ (resp. $D - a$) to be the digraph obtained after adding the arc a to D (resp. removing the arc a from D if $a \in A(D)$).
The **underlying graph** of a digraph D, denoted by $G[D]$, is obtained upon removing all the orientations of the arcs of D. The **chromatic number** of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph $G[D]$. A digraph D is said to be ***n*-chromatic** whenever $\chi(G[D]) = n$. An arc a of D is said to be **monochromatic** if its ends have the same color.

In our work, the digraphs considered are finite having neither loops nor multiple edges.

For a directed path $P = v_1 v_2 \cdots v_n$ (also denoted by a $v_1 v_n$-path) in a digraph, v_1 is said to be the origin of P, v_n is its end and v_{i+1} is the successor of v_i on P for all $i \in \{1, \ldots, n-1\}$. A subpath of P, denoted by $P_{[v_i, v_j]}$ with $1 \leq i < j \leq n$, is the directed path contained in P of origin v_i and end v_j. Similarly, a subpath of P, denoted by $P_{]v_i, v_j]}$ (resp. $P_{[v_i, v_j[}$) with $1 \leq i < j \leq n$, is the directed path contained in P of origin v_{i+1} (resp. v_i) and end v_j (resp. v_{j-1}).

A **block** of an oriented path P in a digraph is a maximal directed subpath of P. A $\boldsymbol{P(k, l)}$ is defined to be a path with two blocks, where the first block consists of k forward arcs followed by a second block consisting of l backward arcs.
If D is an n-tournament, Thomason in [8] proves that D contains every oriented path of length $n - 1$ if n is large enough. Havet and Thomassé give in [6] a refinement of Thomason's result, proving that this is valid for every tournament except for three cases: the directed 3-cycle, the regular tournament on 5 vertices and the Paley tournament on 7 vertices; where in these cases D contains no antidirected path of length $n - 1$.

An **outbranching** is a connected digraph containing a vertex of in-degree zero which is called the **source**, and all other vertices are of in-degree one. An **outforest** F is a digraph such that each connected component is an outbranching. For every u of $V(F)$, $\boldsymbol{P_u(F)}$ denotes the unique directed path in F starting from a source and reaching u. The **level** of u in F, denoted by $\ell_F(u)$, is the order of $P_u(F)$. Define for each $i \geq 1$, $\boldsymbol{L_i(F) = \{u \in V(F);\ \ell_F(u) = i\}}$. Let $y \in V(F)$, we denote by $\boldsymbol{T_F(y)}$ the sub-outbranching in F of source y. Note that any digraph contains a spanning outforest. Let F be a spanning outforest of a digraph D. An arc $(u, v) \in A(D)$ is said to be a **backward arc** with respect to F whenever $\ell_F(u) \geq \ell_F(v)$, else it is called a **forward arc**.

A spanning outforest of a digraph D is said to be a **maximal forest** if $\sum_{v \in V(D)} \ell_F(v)$ is maximal. El Sahili and Kouider [4], after introducing the concept of a maximal forest, proved that it verifies this crucial property: For any backward arc (u, v) with respect to a maximal forest F of a digraph D, F contains a vu-directed path, and we will denote it by $\boldsymbol{P_{[v,u]}(F)}$.

Addario et al. [1] defined a **final forest** of a digraph D to be each spanning outforest of D verifying the previous property.

For any final forest F of a digraph D, it can be easily proved that $L_i(F)$ is stable in D for every $i \geq 1$, and consequently $\ell(F) \geq \chi(D)$, where $\ell(F)$ is the maximum integer i such that $L_i(F)$ is non empty, and this gives a very simple proof of Gallai-Roy Theorem [5, 7]:

Theorem 1.1 *Every n-chromatic digraph contains a directed path of length $n - 1$.*

Starting from any spanning outforest, an algorithmic way to obtain a final forest is described in [1]. Let F be a spanning outforest of a digraph D. The process is done by defining a sequence of spanning outforests, say $F_1, F_2, \ldots, F_n$ such that $F_1 = F$, F_n is final and F_{i+1} is obtained from F_i by adding a backward arc (x, y) (relative to F_i) such that F_i contains no directed yx-path, and by deleting the arc of F_i with head y (if any). F_{i+1} is said to be a rectification of F_i, and F_n is said to be a closure of F. Note that due to this algorithm, at least one vertex has a strictly higher level afterwards, and so a final forest is reachable since the process of rectification tends to maximize the levels of the vertices of the finite digraph we are dealing with.

El Sahili in [4] introduced the function $f(n)$ defined to be the smallest integer such that any $f(n)$-chromatic digraph contains all paths $P(k, j)$ with $k + j = n - 1$, and he conjectured that $f(n) = n$. Using maximal forests, El Sahili and Kouider [4] proved that $f(n) \leq n + 1$. Then, Addario et al. [1] proved the conjecture using the notion of final forests and a generalization of Bondy's Theorem [2] about strongly connected digraphs.

In this paper, we give a new elementary proof of El Sahili conjecture without using strongly connected digraphs.

2 The Elementary Proof

Theorem 2.1 *Every n-chromatic digraph with $n \geq 4$ contains any $P(k, l)$ such that $k + l = n - 1;\ k,\ l \in \mathbb{N}^*$.*

Proof Due to symmetry, we may assume that $k \leq l$. Let D be an n-chromatic digraph. Suppose that D contains no $P(k, l)$ for some $k,\ l \in \mathbb{N}^*$. Let F be a final forest of D, set $U_i(F) = L_i(F)$ for all $i \in \{1, \ldots, k-1\}$, $U_i(F) = \bigcup_{r \geq 0} L_{i+r(l+1)}(F)$ and denote by $D_i = D[U_i(F)]$ for all $i \in \{k, \ldots, k+l\}$. Color the vertices in $U_i(F)$ by the color i for all $i \in \{1, \ldots, n-1\}$. Note that if $(u, v) \in A(D_i)$ for some $i \in \{k, \ldots, k+l\}$, then $i = k$ and $v \in L_k(F)$; otherwise $P_u(F) \cup P_v(F) \cup (u, v)$ contains a $P(k, l)$.

Thus, $U_i(F)$ is stable for every $i \neq k$ and $U_k(F)$ is not stable since $\chi(D) = n$ and $V(D) = \bigcup_{i=1}^{n-1} U_i(F)$.

From now on, v is said to be a **bad vertex** for every $(u, v) \in A(D_k)$. Moreover, $l(P_{[v,u]}(F)) \geq l+1$ which implies that every vertex $z \in T_F(v)$ is the end of a directed path of length $l+1$, denoted by Q_z, and it is contained in $(u, v) \cup P_{[v,u]}(F) \cup P_{[v,z]}(F)$.
Claim 1. For every arc $(u, v) \in A(D_k)$, $zz' \notin E(G[D])$ for every $z \in T_F(v)$ and for every $z' \notin T_F(v)$ with $\ell_F(z') \geq k$. □

Proof If $(z, z') \in A(D)$, then (z, z') is a forward arc as F is a final forest, and so $\ell_F(z') > \ell_F(z) \geq k$. Knowing that $l(P_{z'}(F)) \geq k$, $P_{z'}(F) \cap T_F(v) = \phi$ and $Q_z \subseteq T_F(v)$, so $Q_z \cup (z, z') \cup P_{z'}(F)$ contains a $P(k, l)$; which gives a contradiction. Similarly, one can observe that a $P(k, l)$ will appear in $Q_z \cup (z', z) \cup P_{z'}(F)$ if $(z', z) \in A(D)$. □

It is now clear that the only monochromatic arcs appear in D_k, which implies that a proper coloring may be established if we recolor properly $T_F(v)$ for every bad vertex v.

Let $(u, v) \in A(D_k)$, and set $C_v^F = P_{[v,u]}(F) \cup (u, v)$, $P_{[v,u]}(F) = v_k v_{k+1} \cdots v_p$ where $v_k = v$, $v_p = u$ and $p = \ell_F(u)$.
A vertex x in D is said to be **rich** in F if $\ell_F(x) \geq k$ and $N(x) \cap L_i(F) \neq \phi$ for all $i \in \{1, \ldots, k-1\}$.

From now on, we may consider the following recoloring. For every bad vertex v, if v is not rich, then recolor it by some color i whenever $N(v) \cap L_i(F) = \phi$ for some $i \in \{1, \ldots, k-1\}$. Otherwise, if v is bad rich but has no rich neighbor in $U_j(F)$ for some $j \in \{k, \cdots, k+l\}$, then recolor its neighbors in $U_j(F)$, if any,by the suitable colors from $\{1, \ldots, k-1\}$, and finally color v by j. In both cases, $T_F(v)$ is colored properly by the colors $\{1, \ldots, n-1\}$.

A direct consequence of the previous paragraph states that there exists a rich bad vertex v such that v has a rich neighbor in $U_i(F)$ for every $i \geq k$, which implies that we are concerned now with recoloring $T_F(v)$ properly in this case.
Claim 2. If u is a rich in-neighbor of v in $U_k(F)$, then u is the unique in-neighbor of v in $U_k(F)$.

Proof Suppose that there exists $u' \in U_k(F)$ such that $u' \neq u$ and $(u', v) \in A(D)$. Let $j = min\{i : 1 \leq i \leq k$ and $N^+(u) \cap L_i(F) \neq \phi\}$, $x \in N^+(u) \cap L_j(F)$ and $y \in N^-(u) \cap L_{j-1}(F)$ if $j > 1$. Clearly, $x \in P_v(F)$. If $j = 1$, then the path $(u, x) \cup P_v(F) \cup P_{]v,u']}(F) \cup (u', v)$ contains a $P(k, l)$, else the path $P_y(F) \cup (y, u) \cup (u, x) \cup P_{[x,v]}(F) \cup P_{]v,u']}(F) \cup (u', v)$ contains a $P(k, l)$; which gives a contradiction. □

Taking into consideration the proof of Claim 2, P'_z denotes the path $(z, x) \cup P_v(F)$ if $j = 1$ and the path $P_y(F) \cup (y, z) \cup (z, x) \cup P_{[x,v]}(F)$ if $j > 1$ whenever z is a rich in-neighbor of a bad vertex v.
Set $B = \{v : v$ is a rich bad vertex and has a rich in-neighbor in $U_k(F)\}$. Let $v \in B$ and let u be the rich in-neighbor of v in $U_k(F)$. Recall that $C_v^F = v_k v_{k+1} \cdots v_p$ where $v_k = v$, $v_p = u$ and $p = \ell_F(u)$.

Claim 3. For every vertex $v \in B$, the rich neighbors of v belong to $L_i(F)$ for every $i \in \{k+1, \ldots, k+l\}$.

Proof Suppose that there exists a rich neighbor w of v such that $w \in U_i(F) - L_i(F)$ for some $i \in \{k+1, \ldots, k+l\}$. If $w \notin T_F(u)$, then $P'_u \cup P_{[v,w]}(F) \cup vw$ contains a $P(k,l)$, which gives a contradiction. If $w \in T_F(u)$, then consider the path $P'_w \cup P_{[v,u]}(F) \cup (u, v)$ if $(w, v) \in A(D)$ and the path $P_v(F) \cup (v, w) \cup P_{[v,w]}(F)$ otherwise. Note that both paths contain a $P(k,l)$, which gives a contradiction. Let $r \in \{1, \ldots, l\}$. We will consider a recoloring $T_F(v)$ for every $v \in B$ having a non rich neighbor v_{k+r}. If v has a rich neighbor x in $U_{k+r}(F)$, then $\ell_F(x) = k + r$ by Claim 3. Since $\ell_F(x) = k + r$ and $x \neq v_{k+r}$, then $x \notin C^F_v$, and so $N(x) \cap L_i(F) = N^-(x) \cap L_i(F)$ for every $i \in \{1, \ldots, k\}$, since otherwise $P'_x \cup C^F_v$ contains a $P(k,l)$ which gives a contradiction.

Let $wz \in E(G[D])$, where $w \in T_F(v) - T_F(x)$ and $z \in T_F(x)$, then $(w, z) \in A(D)$ such that $z = x$ and $\ell_F(w) < \ell_F(x)$. Otherwise, since $w \notin T_F(x)$, then $Q_w \cap T_F(x) = \phi$. If $(z, w) \in A(D)$, then a $P(k,l)$ appears in $P_y(F) \cup (y, x) \cup P_{[x,z]}(F) \cup (z, w) \cup Q_w$ where $y \in N^-(x) \cap L_{k-1}(F)$, which gives a contradiction. Hence, $(w, z) \in A(D)$, and so $\ell_F(w) < \ell_F(z)$ as $w \notin T_F(x)$. If $z \neq x$, a $P(k,l)$ appears in $P_y(F) \cup (y, x) \cup P_{[x,z]}(F) \cup (w, z) \cup Q_w$ where $y \in N^-(x) \cap L_{k-1}(F)$.

So for any rich neighbor x of v in $U_{k+r}(F)$, recolor z by $i + 1$ for every $z \in T_F(x) \cap U_i(F)$ with $i \in \{k+1, \ldots, k+l-1\}$, recolor z by k for every $z \in T_F(x) \cap U_{k+l}(F)$ and finally recolor the remaining neighbors of v in $U_{k+r}(F)$ by the suitable color from $\{1, \ldots, k-1\}$. Now, $T_F(v)$ will be colored properly if we give v the color $k + r$.

Claim 4. There exists no vertex $v \in B$ such that v_{k+r} is a rich neighbor of v for every $r \in \{1, \ldots, l\}$. □

Proof Let $F_1 = F + (u, v) - (v, v_{k+1})$ if $k = 1$ and $F_1 = F + (y, v_{k+1}) + (u, v) - (v, v_{k+1}) - (x, v)$ if $k \geq 2$ where $x \in N^-(v) \cap L_{k-1}(F)$ and $y \in N^-(v_{k+1}) \cap L_{k-1}(F)$. Let F_c be a closure of F_1. Since $u(F)$ is minimal, then $\ell_{F_c}(z) = \ell_F(z)$ for all $z \in L_i(F)$, $i \in \{1, \ldots, k-1\}$, if any. Thus, $\ell_{F_c}(v_{k+1}) = k$ and v is still rich in F_c. Due to the minimality of $|B|$ and maximality of $\sum_{w \in B} h_F(w)$, v_{k+1} is bad in F_c and $h_{F_c}(v_{k+1}) = h_F(v)$ and so $\ell_{F_c}(v) = \ell_F(u)$. Moreover, it can be easily proved that $\ell_{F_c}(v_i) = \ell_F(v_i) - 1$ for every $i \in \{k+1, \ldots, p\}$. Therefore, $C^{F_c}_{v_{k+1}} = C^F_v$.

Repeating the same reasoning, we can show that v_i plays the same role, that is the l successive vertices of v_i on C^F_v are rich neighbors of v_i and $N(v_i) \cap L_{k-1}(F) = N^-(v_i) \cap L_{k-1}(F)$ for every $i \in \{k+1, \ldots, p\}$. Using the fact that $k \leq l$ and $n \geq 4$, we get that $l \geq 2$ and so $l(C^F_v) \geq 4$. Moreover, $(v_{k+2}, v_k) \in A(D)$, otherwise a $P(k,l)$ appears in $P_{[v_{k+3}, v_p]}(F) \cup (v_p, v_k) \cup (v_k, v_{k+2}) \cup P_z(F) \cup (z, v_{k+1}) \cup (v_{k+1}, v_{k+2})$, where $z \in N^-(v_{k+1}) \cap L_{k-1}(F)$. By symmetry, $(v_{k+1}, v_p) \in A(D)$. Finally, $(v_k, v_{k+l}) \in A(D)$, otherwise $P_{[v_{k+1}, v_{k+l}]}(F) \cup (v_{k+l}, v_k) \cup P'_{v_p}$ contains a $P(k,l)$. Thus, there exists $i \in \{k+2, \ldots, k+l-1\}$ such that $(v_i, v_k) \in A(D)$ and $(v_k, v_{i+1}) \in A(D)$. Hence, $P_{[v_{k+2}, v_i]}(F) \cup (v_i, v_k) \cup (v_k, v_{i+1}) \cup P_{[v_{i+1}, v_p]}(F) \cup P_z(F) \cup (z, v_{k+1}) \cup (v_{k+1}, v_p)$, where $z \in N^-(v_{k+1}) \cap L_{k-1}(F)$, contains a $P(k,l)$; which gives a contradiction. Thus, we can say now that D is colored properly using $(n-1)$-colors due to Claim 1, which gives a contradiction. □

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