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ORIGINAL PAPER

Multipermutations and Stirling Multipermutations

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Abstract
We consider multipermutations and a certain partial order, the weak Bruhat order, on this set. This generalizes the Bruhat order for permutations, and is defined in terms of containment of inversions. Different characterizations of this order are given. We also study special multipermutations called Stirling multipermutations and their properties.

Keywords Permutation · Multipermutation · Stirling permutation · Weak Bruhat order

Mathematics Subject Classification 05A05 · 06A07

1 Introduction

Let n and $m_1, m_2, \ldots, m_n$ be positive integers and let $\{1, 2, \ldots, n\}^{\times(m_1,m_2,\ldots,m_n)}$ denote the multiset $\{m_1 \cdot 1, m_2 \cdot 2, \ldots, m_n \cdot n\}$ consisting of m_i integers equal to i $(1 \le i \le n)$. If $m_1 = m_2 = \cdots = m_n = k$ for some integer k, then we abbreviate this to $\{1, 2, \ldots, n\}^{\times k}$. The permutations of $\{1, 2, \ldots, n\}^{\times(m_1,m_2,\ldots,m_n)}$, written as a p-tuple with $p = m_1+m_2+\cdots+m_n$, are certain multipermutations of $\{1, 2, \ldots, n\}$. The set of multipermutations $\sigma_n^{\times(m_1,m_2,\ldots,m_n)}$ of the multiset $\{m_1 \cdot 1, m_2 \cdot 2, \ldots, m_n \cdot n\}$ is denoted by $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$. In the special case mentioned above this is abbreviated to $\sigma_n^{\times k}$ and $\mathcal{S}_n^{\times k}$, respectively, and we call the resulting multipermutations *k-permutations* of $\{1, 2, \ldots, n\}$. If $k = 1$, we write simply $\mathcal{S}_n$, the set of permutations of $\{1, 2, \ldots, n\}$.

For a permutation $\sigma_n^{\times(m_1,m_2,\ldots,m_n)}$, let (i, t) denote the tth occurrence (position) of the integer i in $\sigma_n^{\times(m_1,m_2,\ldots,m_m)}$ $(1 \le i \le n,\ 1 \le t \le m_i)$. An $(ordinary)inversion$ of

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$\sigma_n^{\times(m_1,m_2,\ldots,m_m)}$ is defined to be any occurrence (j, i) of a larger integer j preceding a smaller integer i in $\sigma_n^{\times(m_1,m_2,\ldots,m_m)}$. In contrast, a *strong inversion* is an ordered pair: $((j, t), (i, s))$ where the tth occurrence of j precedes the sth occurrence of i, and $j > i$. If $k = 1$, that is, we have $\mathcal{S}_n$, this is equivalent to the usual inversion $j > i$ where j precedes i in the permutation. In general, numerical information on the order of occurrences of the integers involved is taken into account in a strong inversion. The *multiset of ordinary inversions* of $\sigma_n^{\times(m_1,m_2,\ldots,m_n)}$ is denoted by $\mathcal{I}(\sigma_n^{\times(m_1,m_2,\ldots,m_n)})$. The *set of strong inversions* of $\sigma_n^{\times(m_1,m_2,\ldots,m_m)}$ is denoted by $\mathcal{I}^*(\sigma_n^{\times(m_1,m_2,\ldots,m_n)})$. For a k-permutation $\sigma_n^{\times k}$ of $\{1, 2, \ldots, n\}$ these are denoted, respectively, by $\mathcal{I}(\sigma_n^{\times k})$ and $\mathcal{I}^*(\sigma_n^{\times k})$.

It follows that every strong inversion gives an ordinary inversion (use the projection of the first coordinates of the strong inversion pair). For instance, in (2, 1, 2, 1), there are three strong inversions, namely, ((2, 1), (1, 1)), ((2, 1), (1, 2)), and ((2, 2), (1, 2)) and the multiplicity of the weak inversion (2, 1) is three. So $\mathcal{I}^*(\sigma) \subseteq \mathcal{I}^*(\tau)$ implies that $\mathcal{I}(\sigma) \subseteq \mathcal{I}(\tau)$ (as a multiset inclusion). But the converse does not hold. For example, consider $n = 2$ and the multipermutations (a) (2, 1, 1, 2) and (b) (1, 2, 2, 1). The multiset of ordinary inversions in both cases is {(2, 1), (2, 1)}. The sets of strong inversions are (a) {((2, 1), (1, 1)), ((2, 1), (1, 2))} and (b) {((2, 1), (1, 2)), ((2, 2), (1, 2))}.

The *identity multipermutation* in $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ is the multipermutation $\iota_n^{(m_1,m_2,\ldots,m_n)}$ with no decrease; in case that $m_1 = m_2 = \cdots = m_n = k$, this is abbreviated to $\iota_n^{\times k}$. The *anti-identity multipermutation* is the reverse of $\iota_n^{(m_1,m_2,\ldots,m_n)}$ and so has no increase.

Example 1 Let $n = k = 3$ and consider the multipermutation

$$\sigma_3^{\times 3} = (1, 1, 2, 3, 2, 2, 1, 3, 3).$$

Then $\mathcal{I}(\sigma_3^{\times 3})$ equals the multiset

$$\{(2, 1), (2, 1), (2, 1), (3, 2), (3, 2), (3, 1)\},$$

and $\mathcal{I}^*(\sigma_3^{\times 3})$ equals the set

$$\{((3, 1), (2, 2)), ((3, 1), (2, 3)), ((3, 1), (1, 3)), ((2, 1), (1, 3)), ((2, 2), (1, 3)), ((2, 3), (1, 3))\}.$$

For the $identity 3 - permutation$ $\iota_3^{\times 3} = (1, 1, 1, 2, 2, 2, 3, 3, 3)$ of $\{1, 2, 3\}^{\times 3}$, we have $\mathcal{I}^*(\sigma_3^{\times 3}) = \emptyset$; for the *anti-identity* 3-*permutation* $\overleftarrow{\iota_3^{\times 3}} = (3, 3, 3, 2, 2, 2, 1, 1, 1)$ we have
$\mathcal{I}^*(\overleftarrow{\iota_3^{\times 3}}) = \{(j, t), (i, s)) : 1 \leq i < j \leq 3,\ 1 \leq s,\ t \leq 3\}$.

In [18] the *weak Bruhat order* $\preceq_b$ on $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ is defined as follows :

$$\sigma_n^{\times(m_1,m_2,\ldots,m_n)} \preceq_b \tau_n^{\times(m_1,m_2,\ldots,m_n)}$$

provided that

$$\mathcal{I}^*(\sigma_n^{\times(m_1,m_2,\ldots,m_n)}) \subseteq \mathcal{I}^*(\tau_n^{\times(m_1,m_2,\ldots,m_n)}).$$

For $\mathcal{S}_n^{\times k}$ with $k = 1$, this reduces to the weak Bruhat order on $\mathcal{S}_n$, that is, $\mathcal{I}(\sigma_n) \subseteq \mathcal{I}(\tau_n)$. The weak Bruhat order $\preceq_b$ on $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ is proved to be the reflexive and transitive closure of the relation obtained by using *adjacent transpositions* on $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ as specified by

$$\ldots, j, i, \ldots \to \ldots, i, j, \ldots \text{ where } j > i.$$

Thus the strong inversion $((j,t),(i,s))$ for some t and s (only this strong inversion) is deleted from the set of strong inversions of the multipermutation under such an adjacent transposition. The partially ordered set $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}, \preceq_b)$ is also proved to be a lattice, a generalization of the corresponding fact for the *weak Bruhat order* $(\mathcal{S}_n, \preceq_b)$ on permutations to the set of multipermutations of an arbitrary finite multiset. The lattice $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}, \preceq_b)$ is graded by the number of strong inversions. The minimal element is the identity multipermutation with no strong inversions; the maximal element is the anti-identity multipermutation, and so $\sum_{1\le i<j\le n} m_i m_j$ strong inversions.

We now briefly summarize the remaining contents of this paper. Section 2 treats the weak Bruhat order for multipermutations, and contains characterizations of this partially ordered set. One such characterization is in terms of so-called sum matrices. Next, in Sect. 6, we consider 2-permutations and how they give rise to two permutations, called order projections. Construction of 2-permutations from such order projections is presented. Sections 4 and 5 are devoted to Stirling multipermutations, their inversions and characterizations in terms of other combinatorial objects. In Sect. 6 we make some final comments including a brief discussion of a generalization of Stirling permutations called quasi-Stirling permutations.

Notation: In the display of matrices we sometimes omit zeros leaving their positions empty.

2 Weak Bruhat Order and Multipermutations

We now show that a multipermutation of $\{1, 2, \ldots, n\}$ with its set of strong inversions is equivalent to a permutation with its set of inversions where the partial orders agree; a consequence is that the partially ordered sets of the types $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}, \preceq_b)$ can be regarded as sublattices of the partially ordered sets of the types $(\mathcal{S}_p, \preceq_b)$.

First we give an example of the process which can be generalized.

Example 2 Consider the 2-permutation of $\{1, 2, 3, 4\}$:

$$\sigma = (2, 1, 2, 3, 4, 3, 4, 1).$$

Its strong inversions are:

$$((2,1),(1,1)),((2,1),(1,2)),((2,2),(1,2)),((3,1),(1,2)),$$
$$((3,2),(1,2)),((4,1),(3,2),((4,1),(1,2)),((4,2),(1,2)).$$

Now from the 2-permutation (2, 1, 2, 3, 4, 3, 4, 1) we construct a permutation in $\mathcal{S}_8$ by replacing the 1's by 1 and 2, ordered increasingly, and next replace the two 2's by 3 and 4 in that order, etc. This gives the permutation

$$\hat{\sigma} = (3,1,4,5,7,6,8,2),$$

whose inversions are:

$$(3,1),(3,2),(4,2),(5,2),(6,2),(7,6),(7,2),(8,2).$$

We have an injection between the 2-permutations in $\mathcal{S}_4^{\times 2}$ and the permutations in $\mathcal{S}_8$ which preserves weak Bruhat order, that is, $(\mathcal{S}_4^{\times 2}, \preceq_b)$ is isomorphic to a partially ordered subset of $(\mathcal{S}_8, \preceq_b)$. This works in general giving that $(\mathcal{S}_n^{\times k}, \preceq_b)$ is isomorphic to a partially ordered subset of $(\mathcal{S}_{kn}, \preceq_b)$. This is made more precise in Theorem 1. □

Let $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$ be the subset of $\mathcal{S}_p$, where $p = m_1+m_2+\cdots+m_n$, in which each of the sets of integers

$$\begin{aligned}
Y_1 &= \{1,2,\ldots,m_1\}, Y_2 = \{m_1+1, m_1+2,\ldots,m_1+m_2\},\ldots,\\
Y_n &= \{m_1+m_2+\cdots+m_{n-1}+1, m_1+m_2+\cdots\\
&\quad +m_{n-1}+2,\ldots,m_1+m_2+\cdots+m_n\}
\end{aligned}$$

occur in increasing order. A multipermutation $\sigma_n^{\times(m_1,m_2,\ldots,m_n)} \in \mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ thus corresponds to a permutation in $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$ in the way explained in Example 2, i.e., replacing the m_1 1's by the numbers in Y_1, ordered increasingly, and next replacing the m_2 2's by the numbers in Y_2, ordered increasingly, etc. This gives a bijection between $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ and $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$.

Then $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}, \preceq_b)$ is a partially ordered subset of $(\mathcal{S}_p, \preceq_b)$. Notice that all the inversions $a > b$ in a multipermutation in $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$ have a and b in different Y_i's where the integers in each Y_i's are in increasing order. It follows that successive adjacent transpositions applied to a multipermutation in $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$ give a multipermutation that is also in $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$. Hence the inversions in $\mathcal{I}(\sigma_n^{\times(m_1,m_2,\ldots,m_n)\uparrow})$ are identical with the inversions of $\mathcal{I}(\sigma_n^{\times(m_1,m_2,\ldots,m_n)})$ involving $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$. Thus the partial order of $\mathcal{S}_p$ when restricted to $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}$ gives the partially ordered set $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}, \preceq_b)$. Hence we have the following consequence.

Theorem 1 *$(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}, \preceq_b)$ is isomorphic to $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}, \preceq_b)$, and the partially ordered set $(\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)\uparrow}, \preceq_b)$ is a sublattice of $(\mathcal{S}_p, \preceq_b)$ where $p = m_1+m_2+\cdots+m_n$.*

Consider a multipermutation $\sigma = (a_1, a_2, \ldots, a_N) \in \mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$ where $N = m_1 + m_2 + \cdots + m_n$ and $1 \le a_i \le n$ $(i \le N)$. We denote by $\widehat{\sigma}$ the permutation in $\mathcal{S}_N$ obtained from σ as defined in the discussion preceding Theorem 1; $\widehat{\sigma}$ is called the *associated permutation* of the multipermutation σ. Corresponding to this σ is the $N \times n$ (0, 1)-matrix A_σ whose ith row contains a 1 in column a_i $(i \le N)$ and otherwise contains only zeros. We call A_σ the *incidence matrix* of σ. Note that when σ is a permutation, A_σ is the usual permutation matrix associated with σ.

The *sum matrix* $\Sigma(A) = [s_{ij}]$ of any $m \times n$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix defined by $s_{ij} = \sum_{k \le i,\, l \le j} a_{kl}$ $(i \le m,\ j \le n)$; thus s_{ij} is the sum of the entries in the leading $i \times j$ submatrix of A. A well-known characterization of the weak Bruhat order on $n \times n$ permutation matrices is provided by the sum matrix: $P \le_b Q$ if and only if $\Sigma(P) \ge \Sigma(Q)$ (entrywise). The next theorem includes a characterization of the weak Bruhat order for multipermutations in terms of sum matrices.

Theorem 2 (i) *Let* $\sigma \in \mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$. *Then the* k*th column of* $\Sigma(A_\sigma)$ *equals the* $\widehat{k}$*th column of* $\Sigma(A_{\widehat{\sigma}})$ *where* $\widehat{k} = m_1 + m_2 + \cdots + m_k$ $(k \le n)$.

(ii) *Let* $\sigma, \tau \in \mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$. *Then* $\Sigma(A_\sigma) \ge \Sigma(A_\tau)$ *if and only if* $\Sigma(A_{\widehat{\sigma}}) \ge \Sigma(A_{\widehat{\tau}})$.

(iii) *Let* $\sigma, \tau \in \mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$. *Then* $\sigma \preceq_b \tau$ *if and only if* $\Sigma(A_\sigma) \ge \Sigma(A_\tau)$.

Proof (i) The kth column of $\Sigma(A_\sigma)$ is the row sum vector of the submatrix consisting of the k first columns of A_σ, and this vector coincides with the row sum vector of the submatrix consisting of the $\widehat{k}$ first columns in $A_{\widehat{\sigma}}$.

(ii) If $\Sigma(A_{\widehat{\sigma}}) \ge \Sigma(A_{\widehat{\tau}})$, then, by (i), $\Sigma(A_\sigma) \ge \Sigma(A_\tau)$.

Conversely, assume $\Sigma(A_\sigma) \ge \Sigma(A_\tau)$ holds. Let $k < n$ and $\widehat{k} = m_1 + m_2 + \cdots + m_k$ and define $k' = \widehat{k} + m_{k+1}$. Let x and x' denote columns $\widehat{k}$ and k' of $\Sigma(A_{\widehat{\tau}})$, and let y and y' denote columns $\widehat{k}$ and k' of $\Sigma(A_{\widehat{\sigma}})$. The components of these vectors are denoted x_i, x'_i etc. Then, by assumption,

$$x \le y \text{ and } x' \le y'.$$

Let x^* and y^* denote the column $\widehat{k}+1$, i.e., right after column x and y in, respectively, $\Sigma(A_{\widehat{\tau}})$ and $\Sigma(A_{\widehat{\sigma}})$. Then there exists $p \le n$ and $q \le n$ such that

$$x^* = x + e^{(p)} \text{ and } y^* = y + e^{(q)}$$

where $e^{(p)}$ (resp. $e^{(q)}$) is the (0, 1)-vector with a 1 in each position $j \ge p$ (resp. $j \ge q$) and otherwise contains zeros. We now show that $x^* \le y^*$.

If $p \ge q$, then $e^{(p)} \le e^{(q)}$, so $x^* = x + e^{(p)} \le y + e^{(q)} = y^*$ holds. Next, assume that $p < q$ and let $p \le i < q$. Observe that we cannot have $x_i = y_i$, because that would give $x'_i = x_i + 1 > y_i = y'_i$, contradicting $x' \le y'$. Thus, $x_i < y_i$ holds, and then $x^*_i = x_i + 1 \le y_i = y^*_i$. This implies $x^* \le y^*$, and the Claim holds.

We can now repeat this argument, column by column, and by induction, it follows that every column in $\Sigma(A_{\widehat{\tau}})$ is componentwise $\le$ the corresponding column in $\Sigma(A_{\widehat{\sigma}})$. So, $\Sigma(A_{\widehat{\sigma}}) \ge \Sigma(A_{\widehat{\tau}})$, as desired.

(iii) This follows by combining statement (ii) of this theorem and Theorem 1. □

Example 3 Let $n = 3$ and $(m_1, m_2, m_3) = (3, 4, 2)$, so $N = 9$. Consider the following multipermutation $\sigma = (2, 1, 3, 1, 2, 2, 3, 2, 1)$ and its corresponding permutation $\widehat{\sigma} = (4, 1, 8, 2, 5, 6, 9, 7, 3)$. With σ we associate the incidence matrix A_σ and its sum matrix

$$A_\sigma = \left[\begin{array}{c|c|c} & 1 & \\ \hline 1 & & \\ \hline & & 1 \\ \hline 1 & & \\ \hline & 1 & \\ \hline & 1 & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 1 & & \end{array}\right] \quad \text{where} \quad \Sigma(A_\sigma) = \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ \hline 1 & 2 & 2 \\ \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline 2 & 5 & 6 \\ \hline 2 & 5 & 7 \\ \hline 2 & 6 & 8 \\ \hline 3 & 7 & 9 \end{array}\right].$$

With $\widehat{\sigma}$ we have

$$A_{\hat{\sigma}} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} & & & 1 & & & & & \\ \hline 1 & & & & & & & & \\ \hline & & & & & & & 1 & \\ \hline & 1 & & & & & & & \\ \hline & & & & 1 & & & & \\ \hline & & & & & 1 & & & \\ \hline & & & & & & & & 1 \\ \hline & & & & & & 1 & & \\ \hline & & 1 & & & & & & \end{array}\right] \text{ where } \Sigma(A_{\hat{\sigma}}) = \left[\begin{array}{c|c|c|c|c|c|c|c|c} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ \hline 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\ \hline 1 & 2 & 2 & 3 & 4 & 4 & 4 & 4 & 5 \\ \hline 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 & 6 \\ \hline 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 & 7 \\ \hline 1 & 2 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}\right].$$

Note that columns ending in 3,7,9, respectively, are identical in $\Sigma(A_\sigma)$ and $\Sigma(A_{\hat{\sigma}})$. Next, consider the following multipermutation $\sigma = (2, 2, 3, 1, 3, 2, 1, 2, 1)$. Then

$$A_\sigma = \left[\begin{array}{c|c|c} & 1 & \\ \hline & 1 & \\ \hline & & 1 \\ \hline 1 & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline 1 & & \\ \hline & 1 & \\ \hline 1 & & \end{array}\right] \quad \text{where} \quad \Sigma(A_\sigma) = \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ \hline 0 & 2 & 2 \\ \hline 0 & 2 & 3 \\ \hline 1 & 3 & 4 \\ \hline 1 & 3 & 5 \\ \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 2 & 6 & 8 \\ \hline 3 & 7 & 9 \end{array}\right].$$

Then $\Sigma(A_\sigma) \geq \Sigma(A_\tau)$, so $\sigma \preceq_b \tau$ by Theorem 2. □

3 Order Projections

In this section we consider some questions for 2-permutations and start with some motivation.

Example 4 Consider the permutations $\pi_1 = (1, 2, 5, 3, 4)$ and $\pi_2 = (3, 5, 4, 1, 2)$. Consider all 2-permutations σ_n^2 where π_1 corresponds to the first occurrences and π_2 corresponds to the second occurrences of the integers $1, 2, 3, 4, 5$. We can simply follow π_1 by π_2 to get such a 2-permutation: $(1, 2, 5, 3, 4, 3, 5, 4, 1, 2)$. But there are other possibilities, e.g., $(1, 2, 5, 3, 3, 5, 4, 4, 1, 2)$. *How can such compatible 2-permutations be found and how many are there?* □

A motivation for this notion is from the area of *scheduling*, as described next. Two machines each perform n jobs, in some given order represented by the permutations π_1 and π_2. Assume that for each $i \le n$ job i must be done on machine 1 before job i is done on machine 2. A (π_1, π_2)-compatible 2-permutation then specifies a possible job sequence for the $2n$ jobs that is consistent with these restrictions. In [14] one considers job scheduling on two machines with the constraint mentioned above (each job is first performed on machine 1 and later on machine 2) along with certain other constraints on the order of some strings of jobs. We return to this motivating example below.

Let $\sigma_n^2 \in \mathcal{S}_n^{\times 2}$ be a 2-permutation, and let π_1 (resp. π_2) be the permutation in $\mathcal{S}_n$ obtained from the first (resp. last) occurrence of each integer $i \le n$. We call π_1 and π_2 the *order projections* of σ. For instance, $\sigma = (1, 2, 5, 3, 3, 5, 4, 4, 1, 2)$ has order projections $\pi_1 = (1, 2, 5, 3, 4)$ and $\pi_2 = (3, 5, 4, 1, 2)$. Another 2-permutation with the same order projections is clearly $(1, 2, 5, 3, 4, 3, 5, 4, 1, 2)$. Thus, in general, there are many such (π_1, π_2)-*compatible* 2-permutations associated with a given pair π_1, π_2 of permutations. We now investigate such 2-permutations.

In Sect. 4 we give an interpretation of the order projections in terms of certain walks in a tree.

Let π_1 and π_2 be two permutations in $\mathcal{S}_n$. Define $\mathcal{S}_2(\pi_1, \pi_2)$ as the set of (π_1, π_2)-compatible 2-permutations. This set is always nonempty as it contains the concatenation $\sigma = (\pi_1, \pi_2)$. We define an $n \times n$ (0, 1)-matrix $D = [d_{ij}]$ as follows: the ith row consists of 0's followed by 1's, and the first 1 is in column $\pi_1^{-1}(k)$ where $k = \pi_2(i)$ $(i \le n)$. So, for instance, as in Example 5 below, if $\pi_2(1) = 5$, the first row contains its first 1 in column j where j is the position of 5 in π_1; here $j = 3$. Note that the last column of D only contains 1 s. We call D an *order matrix*, and to indicate the dependence on the permutations we write $D = D(\pi_1, \pi_2)$. Define an *increasing path* in $D(\pi_1, \pi_2)$ as a set of positions (i, j_i) such that $d_{ij_i} = 1$ $(i \le n)$ and

$$j_1 \le j_2 \le \cdots \le j_n.$$

Theorem 3 *Let π_1 and π_2 be two permutations in $\mathcal{S}_n$. There is a bijection between $\mathcal{S}_2(\pi_1, \pi_2)$ and the set of increasing paths in the order matrix $D(\pi_1, \pi_2)$.*

Proof Note that any 2-permutation $\sigma \in \mathcal{S}_2(\pi_1, \pi_2)$ may be constructed by starting with π_1 and then inserting the elements in π_2 in that sequence between some of the positions in π_1. This must be done so that each entry in π_2 occurs *after* the same entry in π_1; this assures the desired order projections. This condition is precisely what the entries in $D = D(\pi_1, \pi_2) = [d_{ij}]$ indicate. In fact, let $i \le n$ and assume $d_{ij} = 1$ with j minimal, and let $k = \pi_1(i)$. This means that the integer k is in position j in π_1 and therefore the integer k from π_2 can be inserted after this position, and not before. As a result, the insertion of the n entries of π_2 may be indicated by selecting an entry which is 1 in each row in D. The additional requirement in this choice is that the column index must be weakly increasing; this is to assure that we do not alter the order of the entries in π_2. This discussion verifies the desired bijection. □

As a referee pointed out, the problem treated here is equivalent to a previously studied problem of determining the linear extensions of a poset derived from the permutations π_1 and π_2 as follows. Its elements are $(1, i)$ (corresponding to π_1) and $(2, i)$ (corresponding to π_2) for $i = 1, 2, \ldots, n$ with two chains as given by the elements in the orders given by π_1 and π_2, and with $(1, i) < (2, i)$ for each i followed by the transitive closure to get a poset. The elements of $\mathcal{S}_2(\pi_1, \pi_2)$ correspond to the linear extensions of this poset. For additional details on this perspective, see the references [6, 20] supplied by a referee.

Example 5 Consider $\pi_1 = (3, 1, 5, 2, 4)$ and $\pi_2 = (5, 1, 3, 2, 4)$. The order matrix $D(\pi_1, \pi_2)$ is then

$$\left[\begin{array}{c|c|c|c|c} & & \mathbf{1} & 1 & 1 \\ \hline & 1 & \mathbf{1} & 1 & 1 \\ \hline 1 & 1 & 1 & \mathbf{1} & 1 \\ \hline & & & 1 & \mathbf{1} \\ \hline & & & & \mathbf{1} \end{array}\right].$$

An increasing path is indicated in boldface and the corresponding 2-permutation is $(3, 1, 5, \mathbf{5}, \mathbf{1}, 2, \mathbf{3}, 4, \mathbf{2}, \mathbf{4})$ where the inserted elements are in boldface (and they determine π_2). □

Theorem 3 also makes it possible to compute the cardinality of $\mathcal{S}_2(\pi_1, \pi_2)$. Let $D = D(\pi_1, \pi_2) = [d_{ij}]$. Let $\tilde{D} = \tilde{D}(\pi_1, \pi_2)$ be obtained from D by replacing every 1 with a zero above by 0, repeatedly, row by row. In the example above the entries in positions $(2, 2)$, $(3, 1)$ and $(3, 3)$ are replaced by 0 and we obtain

$$\tilde{D} = \left[\begin{array}{c|c|c|c|c} & & \mathbf{1} & 1 & 1 \\ \hline & & \mathbf{1} & 1 & 1 \\ \hline & & 1 & \mathbf{1} & 1 \\ \hline & & & 1 & \mathbf{1} \\ \hline & & & & \mathbf{1} \end{array}\right].$$

Then $\tilde{D}$ has a support in a Ferrers pattern (justified to the right) with monotone decreasing row sums. Introduce an $n \times n$ matrix $V = [v_{ij}]$ where v_{ij} equals the number of increasing paths, as previously used, from row 1 until position (i, j). Then we must have

$$\begin{array}{ll} v_{1j} = d_{1j} & (j \leq n) \\ v_{ij} = \sum_{k \leq j,\, d_{i-1,k}=1} v_{i-1,k} & (2 \leq i \leq n,\ j \leq n,\ d_{ij} = 1). \end{array} \tag{1}$$

where an empty sum is defined to be zero. Then

$$|\mathcal{S}_2(\pi_1, \pi_2)| = \sum_j v_{nj}. \tag{2}$$

For the permutations in Example 5 we compute

$$V = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix}.$$

so $|\mathcal{S}_2(\pi_1, \pi_2)| = 14$.

Finally, we note that $\mathcal{S}_2(\pi_1, \pi_2)$ contains a unique 2-permutation if and only if $\pi_1(n) = \pi_2(1)$ (and then this 2-permutation is (π_1, π_2)). Moreover, $|\mathcal{S}_2(\pi_1, \pi_2)|$ is maximal when $\pi_1 = \pi_2$.

We now return to the scheduling problem we briefly discussed above. Let σ_n be a 2-permutation of $\{1, 2, \ldots, n\}$ with order projections π_1 and π_2. Then σ_n represents a job sequence for performing n jobs subject to the requirements on two machines to do their part. Let us assume for simplicity that each job takes the same time. Then the two machines might be able to work simultaneously. So $(1, 2, 5, 3, 3, 5, 4, 4, 1, 2)$ with $\pi_1 = (1, 2, 5, 3, 4)$ and $\pi_2 = (3, 5, 4, 1, 2)$ could progress as in the following *activity table* (where the top line indicates time)

	1	2	3	4	5	6	7	8	9	10
I	1	2	5	3	4					
II					3	5	4	1	2	

with 9 time units as opposed to

	1	2	3	4	5	6	7	8	9	10
I	1	2	5	3	4					
II						3	5	4	1	2

with 10 time units. For a given π_1 and π_2 how does one determine the minimum number of time units possible? Let $t^*(\pi_1, \pi_2)$ denote the minimum total time for a 2-permutation with order projections π_1, π_2. Here the total time is defined as the position j of the final entry in π_2 in the activity table. Then, in general, the minimal total time satisfies

$$n + 1 \leq t^*(\pi_1, \pi_2) \leq 2n.$$

The lower bound is attained when $\pi_1 = \pi_2$, and the activity plan just shifts π_2 one column to the right compared to π_1. The upper bound is attained when $\pi_1(n) = \pi_2(1)$, where π_2 is put right after π_1.

Consider the following recursive computation of integers $T_1, T_2, \ldots, T_n$ based on π_1 and π_2: Let $T_0 = 0$ and

$$T_j = \max\{T_{j-1}, \pi_1^{-1}(\pi_2(j))\} + 1 \quad (j = 1, 2 \ldots, n). \tag{3}$$

Proposition 1 *Let π_1 and π_2 be two permutations in $\mathcal{S}_n$. Then $t^*(\pi_1, \pi_2) = T_n$.*

Proof In the activity table the first row contains π_1 followed by blanks. Consider the second row. Let $k = \pi_2(1)$ be the first component of π_1. The first possible column j for k is right after the position of k in the first row, so $j = \pi_1^{-1}(k) + 1$. Similarly, for each $j \le n, \pi_1(j)$ must be placed after column $\pi_1^{-1}(\pi_2(j))$ and also after the position of the previous entry in π_2. Then, by induction on j, the first possible column for the jth entry of π_2 is given by the expression in (3), and the result follows. □

Finally we note that for this scheduling problem, there is no loss in generality in assuming $\pi_1 = (1, 2, \ldots, n)$ (unless some property of the permutations is considered) by replacing π_2 with $\pi_1^{-1}\pi_2$.

4 Stirling Multipermutations

Stirling permutations were introduced by Gessel and Stanley [11] and have many interesting properties (see e.g., [1, 5, 8, 13, 17, 21]). A permutation σ_n in $\mathcal{S}_n^{\times 2}$ is a *Stirling permutation* provided

$$\sigma_n = (\ldots, i, \ldots, j \ldots, i, \ldots) \text{ implies that } j > i.$$

Thus Stirling permutations are 2-permutations of $\{1, 2, \ldots, n\}$ that *avoid* the pattern 212; between the two occurrences of an integer there can only be larger integers. The set of Stirling permutations of $\{2\cdot 1, 2\cdot 2, \ldots, 2\cdot n\}$ is denoted by $\widehat{\mathcal{S}}_n^{\times 2}$. The *identity Stirling permutation* in $\widehat{\mathcal{S}}_n^{\times 2}$ is the ordinary identity 2-permutation $(1, 1, 2, 2, \ldots, n, n)$ and its reversal is the *anti-identity Stirling permutation* $(n, n, \ldots, 2, 2, 1, 1)$. As used above, we usually let the subscript on a multipermutation denote the size of its underlying *set*.

Example 6 Consider the Stirling permutation $\sigma_4 = (2, 4, 4, 2, 1, 3, 3, 1) \in \widehat{\mathcal{S}}_4^{\times 2}$. Using our construction from the previous section, we consider the associated permutation $(3, 7, 8, 4, 1, 5, 6, 2) \in \mathcal{S}_8$. The Stirling property in terms of this associated permutation is that between two integers $a, a+1$ (in that order) where a is odd, only larger integers occur. More generally, we have the next theorem. □

Theorem 4 *Let n be a positive integer. There is a bijection between the set of Stirling permutations in $\widehat{\mathcal{S}}_n^{\times 2}$ and the set $\widehat{\mathcal{S}}_{2n}$ of permutations in $\mathcal{S}_{2n}$ with the property that between two integers a and $a+1$ in that order with a odd, only larger integers occur.*

Proof This is an immediate consequence of our correspondence and the definition of a Stirling permutation. □

The defining property for Stirling permutations can be carried over to multipermutations of $\{1, 2, \ldots, n\}$ [7, 15]. Consider a multipermutation σ_n of $\{a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n\}$ where $a_1, a_2, \ldots, a_n$ are positive integers. Then σ_n is called a *Stirling multipermutation* provided that between two equal integers in σ_n only larger integers occur, equivalently, between the first and last instance of each integer k in σ_n only larger integers occur. Let $\widehat{\mathcal{S}}_n(a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n)$ be the set of Stirling multipermutations of $\{a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n\}$. (If $a_n = 1$, then n can be deleted from σ_n leaving a Stirling permutation σ_{n-1} of $\{a_1 \cdot 1, a_2 \cdot 2, \ldots, a_{n-1} \cdot (n-1)\}$; thus one could assume that $a_n \geq 2$.) If $a_1 = a_2 = \cdots = a_n = k$, then we denote the corresponding set of Stirling multipermutations by $\widehat{\mathcal{S}}_n^{\times k}$ and call these *Stirling k-permutations of* $\{1, 2, \ldots, n\}$. Thus if $k = 1$, then $\widehat{\mathcal{S}}_n^{\times 1} = \mathcal{S}_n$. If $k \geq 2$ then, deleting one instance of each integer j with $1 \leq j \leq n$ in a Stirling k-permutation in $\widehat{\mathcal{S}}_n^{\times k}$, results in a Stirling $(k-1)$-permutation in $\widehat{\mathcal{S}}_n^{\times (k-1)}$. Conversely, if $k \geq 2$ and σ_n is a Stirling k-permutation in $\widehat{\mathcal{S}}_n^{\times k}$, then inserting a new copy of each integer j with $1 \leq j \leq n$ between the first and kth instances of j results in a Stirling $(k+1)$-permutation in $\widehat{\mathcal{S}}_n^{\times (k+1)}$. Thus, if $k \geq 3$, every Stirling k-permutation of $\{1, 2, \ldots, n\}$ can be constructed by starting with a Stirling 2-permutation of $\{1, 2, \ldots, n\}$ and, for each j between 1 and n, inserting anywhere between its two j's, $(k-2)$ more j's.

As for permutations, and unlike multipermutations in general, a Stirling multipermutation is determined by its multiset of (ordinary) inversions.

Theorem 5 *The function sending a Stirling multipermutation σ_n of $\{a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n\}$ into its multiset $\mathcal{I}(\sigma_n)$ of ordinary inversions is an injective function on $\widehat{\mathcal{S}}_n(a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n)$.*

Proof Let $m = a_1 + a_2 + \cdots + a_n$. Let $\sigma_n = (i_1, i_2, \ldots, i_m)$ and $\pi_n = (j_1, j_2, \ldots, j_m)$ be two Stirling multipermutations of $\{1, 2, \ldots, n\}$ in $\widehat{\mathcal{S}}_n(a_1 \cdot 1, a_2 \cdot 2, \ldots, a_n \cdot n)$. Since the a_n n's must be adjacent in both multipermutations, there must be an s and t such that the a_n n's in σ_n and those in π_n are given by

$$i_s = \cdots = i_{s+a_n-1} = n \text{ and } j_t = \cdots = j_{t+a_n-1} = n.$$

If $s \neq t$, there must exist some $p < n$ which occurs less often in exactly one of the subsequences

$$i_{s+a_n}, i_{s+a_n+1}, \ldots, i_m \text{ and } j_{t+a_n}, j_{t+a_n+1}, \ldots, j_m,$$

and that implies that an inversion (n, p) occurs a different number of times in the inversion sets $\mathcal{I}(\sigma_n)$ and $\mathcal{I}(\pi_n)$, a contradiction. Thus $s = t$ (and the two subsequences must be the same).

Now, delete all the n's in both σ_n and π_n giving σ_{n-1} and π_{n-1}, respectively. Then σ_{n-1} and π_{n-1} are multipermutations of an identical multiset. Thus, we can repeat the argument for σ_{n-1} and π_{n-1}, and it follows by induction that $\sigma_n = \pi_n$, proving the desired injectivity. □

As in Sect. 3, with a Stirling permutation σ_n there is naturally associated [5] the pair of order projection (π_n^1, π_n^2) of permutations of $\{1, 2, \ldots, n\}$ where π_n^1 is given by the first occurrences of each integer in σ_n, and π_n^2 is given by the second occurrences of each such integer. The pair of permutations (π_n^1, π_n^2) is called a *Stirling permutation pair*. For example,

$$\sigma_4 = (1, 3, 3, 1, 2, 4, 4, 2) \to \pi_4^1 = (1, 3, 2, 4), \pi_4^2 = (3, 1, 4, 2).$$

In terms of the correspondence given in Theorem 4, given an odd integer a in σ_n^1 and the even integer $a+1$ in σ_n^2, then only larger integers occur between this a and $a+1$ in σ_{2n}. Then π_4^1 corresponds to a permutation of $\{1, 3, 5, 7\}$ and π_4^2 corresponds to a permutation of $\{2, 4, 6, 8\}$. In general, we have the following lemma.

Lemma 1 *If σ_{2n} is the permutation in $\widehat{\mathcal{S}}_{2n}$ corresponding to a Stirling permutation σ_n in $\widehat{\mathcal{S}}_n^{\times 2}$, then its σ_n^1 corresponds to the permutation of the odd integers $\{1, 3, \ldots, 2n-1\}$ in σ_{2n} and its σ_n^2 corresponds to the permutation of the even integers $\{2, 4, \ldots, 2n\}$ in σ_{2n}.*

Stirling permutations in $\widehat{\mathcal{S}}_n^{\times 2}$ are characterized [5] in terms of its corresponding Stirling permutation pair of permutations in $\mathcal{S}_n$ as we review below.

First we recall that a permutation of $\{1, 2, \ldots, n\}$ is 312-*avoiding* provided that it does not contain a subsequence of length 3 in the same relative order as 3,1,2. Being a 312-avoiding permutation places restrictions on the inversions of the permutation. In fact, the permutation $\sigma = (j_1, j_2, \ldots, j_n)$ is 312-avoiding is equivalent to the following property (312) of its set of inversions:

(312) If $1 \le k < l < p \le n$ and (j_k, j_l) and (j_k, j_p) are inversions, then (j_l, j_p) is also an inversion, that is, j_k, j_l, j_p is a decreasing subsequence of σ.

Now let $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$ be two permutations of $\{1, 2, \ldots, n\}$. Then π_2 is a 312-*avoiding permutation relative to* π_1 (or, (π_1, π_2) is a 312-*avoiding permutation pair*) provided the following two properties (312-i) and (312-ii) hold:

- (312-i) $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$, and
- (312-ii) If $1 \le k < l < p \le n$ and (j_k, j_l) and (j_k, j_p) are inversions in $\mathcal{I}(\pi_2) \setminus \mathcal{I}(\pi_1)$, then (j_l, j_p) is also an inversion in $\mathcal{I}(\pi_2) \setminus \mathcal{I}(\pi_1)$.

If (312-i) and (312-ii) hold, then the following property (312-iii) also holds:

- (312-iii) j_k, j_l, j_p is a decreasing subsequence of π_2 and j_p, j_l, j_k is an increasing subsequence of π_1.

We have the following theorem [5].

Theorem 6 *Let π_1 and π_2 be two permutations of $\{1, 2, \ldots, n\}$. Then (π_1, π_2) is a Stirling permutation pair if and only if (π_1, π_2) is a 312-avoiding pair of permutations, that is, $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$ and $\pi_1^{-1}\pi_2$ is 312-avoiding.*

We now show that the weak Bruhat order on $\widehat{\mathcal{S}}_n^{\times 2}$ is determined by ordinary inversions.

Theorem 7 *Let σ_n and π_n be Stirling permutations in $\widehat{\mathcal{S}}_n^{\times 2}$. The $\sigma_n \preceq_b \pi_n$ if and only if the multiset $\mathcal{I}(\sigma_n)$ of ordinary inversions of σ_n is contained in the multiset of ordinary inversions of π_n.*

Proof If $\sigma_n \preceq_b \pi_n$, then the set of strong inversions of σ_n is contained in the set of strong inversions of π_n. Hence the multiset of ordinary inversions of σ_n is contained in the multiset of ordinary inversions of π_n.

Now suppose that the multiset of ordinary inversions of σ_n is contained in the multiset of ordinary inversions of π_n. Consider an ordinary inversion $a > b$ in σ_n. Since σ_n is a Stirling permutation, σ_n is of the form

$(i)\ \sigma_n = (\ldots, a, \ldots, a, \ldots, b, \ldots, b, \ldots)$ or $\sigma_n = (\ldots, b, \ldots, a, \ldots, a, \ldots, b, \ldots)$,

so the inversion $a > b$ has multiplicity (i) 4 or (ii) 2. Since π_n is a Stirling permutation, and since the multiset of ordinary inversions of σ_n is contained in the multiset of ordinary inversions of π_n, π_n has the corresponding forms

$$
\begin{aligned}
&(i)\ \pi_n = \sigma = (\ldots, a, \ldots, a, \ldots, b, \ldots, b, \ldots), \text{ or}\\
&(ii_1)\ \pi_n = (\ldots, b, \ldots, a, \ldots, a, \ldots, b, \ldots). \text{ or}\\
&(ii_2)\ \pi = (\ldots, a, \ldots, a, \ldots, b, \ldots, b, \ldots).
\end{aligned}
$$

In case (i), we have that the subset of strong inversions of σ_n involving a and b equals the subset of strong inversions of π_n involving a and b. In case (ii) with case (ii_1) holding, we have a similar equality. In case (ii) with case (ii_2), the subset of strong inversions of σ_n involving a and b is a proper subset of the strong inversions of π_n. The theorem now follows. □

Corollary 1 *The set $\widehat{\mathcal{S}}_n^{\times 2}$ of Stirling permutations partially ordered by its multiset of ordinary inversions (the weak Bruhat order on $\widehat{\mathcal{S}}_n^{\times 2}$) is a lattice, the weak Bruhat order on $\widehat{\mathcal{S}}_n^{\times 2}$.*

A Stirling permutation in $\widehat{\mathcal{S}}_n^{\times 2}$ where the two occurrences of j are adjacent for every $j \leq n$ will be called a *double-permutation*. They are clearly in one-to-one correspondence with the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$.

Let σ_n be a Stirling permutation in $\widehat{\mathcal{S}}_n^{\times 2}$. Define $\delta_2(\sigma_n)$ as the number of j's such that the two occurrences of j are adjacent in σ_n. So, $\delta_2(\sigma_n) = n$ means that σ_n is a double-permutation. Let $j \leq n$ and assume the two occurrences of j in σ_n are not adjacent. Let σ_n' be the 2-permutation obtained from σ_n by moving the right-most j in σ_n to the position after the left-most j; we call this operation a *left-join*. A *right-join* is defined similarly, but then we move the left-most j to the position after the right-most j.

Theorem 8 (i) *Let σ_n be a Stirling permutation in $\widehat{\mathcal{S}}_n^{\times 2}$. Then we can find a sequence $\sigma_n^{(k)} \in \widehat{\mathcal{S}}_n^{\times 2}$ $(0 \leq k \leq N)$ of Stirling permutations such that $\sigma_n^{(0)} = \sigma_n$, $\sigma_n^{(k)}$ is obtained by a left-join of $\sigma_n^{(k-1)}$ $(1 \leq k \leq N)$, and $\sigma_n^{(N)}$ is a double-permutation. In addition, $\sigma_n^{(k)} \preceq_b \sigma_n^{(k-1)}$ for each k. Moreover, $N \leq n - 1$, $\delta_2(\sigma_n^{(k)}) > \delta_2(\sigma_n^{(k-1)})$*

($1 \leq k \leq N$), *and, for the Stirling permutation pair (order projections)* (π_1, π_2) *of* $\sigma_n^{(N)}$, $\pi_1 = \pi_2$.

(ii) $\widehat{\mathcal{S}}_n^{\times 2}$ *is connected using the operations* (a) *left-join or its inverse, and* (b) *permutations in which two consecutive kk are interchanged with two consecutive jj.*

Proof (i) Let i_1 and i_2, where $i_1 < i_2$, be the two positions of j in σ_n. By assumption $i_1 < i_2 - 1$. Let σ_n' be obtained from σ_n by a left-join of j, so the right-most j in σ_n is moved to the position after the left-most j. Then σ_n' is also a Stirling permutation. In fact, in σ_n, if j is between two occurences of some l, then $j > l$, and the two consecutive j's in σ_n' also satisfy the Stirling property. Moreover, the removal of the original j does not violate the Stirling property. Thus, $\sigma_n^{(1)} := \sigma_n'$ is a Stirling permutation, and clearly $\delta_2(\sigma_n^{(1)}) < \delta_2(\sigma_n)$, as the two j's are now adjacent and any other pair of adjacent p's are unchanged. Then $\delta_2(\sigma_n) - \delta_2(\sigma_n^{(1)}) \in \{1, 2\}$. Thus, we repeat the process, and after at most $n-1$ steps we have reached a double-permutation $\sigma_n^{(N)}$, and its Stirling pair must consist of two equal permutations. That $\sigma_n^{(k)} \preceq_b \sigma_n^{(k-1)}$ for each k follows from the fact that we move the integer j to the left and interchange only with larger numbers, so certain inversions are removed.

(ii) This follows from (i) as each of two Stirling permutations may be transformed double-permutations. We can move between these double-permutations as for n-permutations, as described in the statement in (ii). □

5 Stirling Characterization

There is an interesting connection between Stirling permutations and certain walks in plane trees given in [13], as we describe next. Consider a *plane tree* T which is an embedding of a tree in the plane: the root vertex is placed on top, each of its neighbors are put on the level below, with corresponding edges attached. This is repeated so that successive levels correspond to vertices with the same distance from the root. Let n be the number of edges in T, and label the edges according to the order in which they are added in the construction of the tree (so first we add edges adjacent to the root, then the new edges adjacent to vertices of distance 1, etc.). An example with $n = 5$ is shown in Fig. 1.

Consider depth-first-search (DFS) in T, starting from the root. Thus, one moves down in the tree to a pendant vertex, then backtrack to vertex with an untraversed edge e. Then one moves along e and further down to a pendant vertex, etc. Due to the backtracking, this DFS constructs an "Euler 2-walk" in T in which every edge is traversed exactly twice, once in a downward direction and once in an upward direction. It corresponds to an ordinary Euler walk in the graph obtained from T by doubling each edge. In this Euler 2-walk, the sequence of edges, in the order they are traversed, defines a Stirling permutation. This is because each number $j \leq n$ occurs twice, and between the two occurences of edge j we only traverse edges below j, and they have higher numbers. Moreover, any Stirling permutation can be constructed in this way from some plane tree.

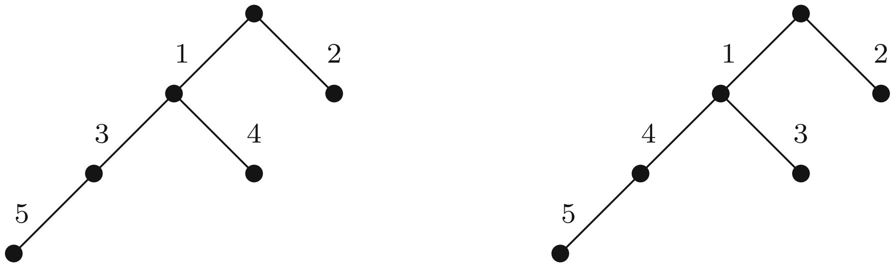


Fig. 1 Stirling permutations and plane trees

As an example consider the Stirling permutation $\sigma_5 = (1, 3, 5, 5, 3, 4, 4, 1, 2, 2)$. The left plane tree T in Fig. 1 gives σ_5 when we use the Euler 2-walk obtained by choosing the left-most alternative in DFS search.

We now observe the following:

- For a given tree T there may be different labelings of the edges, i.e., different sequences of edge additions may result in the same tree T. Therefore different Stirling permutations may correspond to the same tree (but different labelings). See the right plane tree in Fig. 1 which corresponds to the permutation $(1, 4, 5, 5, 4, 3, 1, 2, 2)$.
- Every double-permutation corresponds to a plane tree which is a star (i.e., the root and neighbor vertices). The Stirling permutation $12\cdots nn\cdots 21$ corresponds to a path.
- Let σ_n be a Stirling permutation and let π_1 and π_2 be the corresponding order projections. Let T be a plane tree and W an Euler 2-walk corresponding to σ_n. We then note that π_1 corresponds to the sequence of edges in W that are traversed *downward*, while π_2 corresponds to the sequence of edges in W that are traversed *upward*. In the left example in Fig. 1 we get $\pi_1 = (1, 3, 4, 5, 2)$ (downward) and $\pi_2 = (4, 3, 5, 1, 2)$ (upward).
- The operation used in Theorem 8 to go from $\sigma_n^{(k-1)}$ to $\sigma_n^{(k)}$ where, say, a j is moved to the left, corresponds to a simple modification of the underlying plane tree: shrink the edge uv with label j and replace it by a new pendant edge attached to u, where u is the vertex closer to the root.
- The operation used later in Theorem 9 (denoted s) corresponds to deleting in the plane tree T the edge corresponding to the largest label, and this is a pendant edge.

In order to give a characterization of Stirling permutations we introduce some concepts. We say that a vector $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ is an *AM*-vector (*Adjacent Max*), or simply x is AM, if there is a $k < m$ such that

$$x_k = x_{k+1} > x_j \quad (j \neq k, k+1).$$

Thus a maximum component occurs precisely twice and in adjacent positions. If $x \in \mathbb{R}^m$ is AM, we define a mapping ρ by $\rho(x) \in \mathbb{R}^{m-2}$ is the subvector of x obtained by deleting the two (adjacent) largest components in x. We say that x is *AM-closed* if $x^{(1)} := \rho(x)$ is AM, $x^{(2)} := \rho(x^{(1)})$ is AM etc., i.e., repeated applications of the deleting the largest pair gives only AM vectors until we, finally, have a vector in $\mathbb{R}^2$ with two equal components. Next, let $\sigma_n = (i_1, i_2, \ldots, i_{2n})$ be a 2-permutation of $\{1, 2, \ldots, n\}$. Let $j \le n$ and define the *interval*

$$I_j(\sigma_n) = \{p, p+1, \ldots, q\} \tag{4}$$

where $p < q$ and $i_p = i_q = j$. (Note that p and q are uniquely defined by the 2-permutation and j.) The interval family (4) clearly uniquely determines the 2-permutation. We say that a family $I'_1, I'_2, \ldots, I'_n$ of intervals is *decreasing cross-free* if

$$I'_i \cap I'_j = \emptyset \text{ or } I'_j \subset I'_i \quad (1 \le i < j \le n).$$

Here $\subset$ denotes strict inclusion.

The next theorem characterizes Stirling permutations.

Theorem 9 *Let σ_n be a 2-permutation of $\{1, 2, \ldots, n\}$. Then the following statements are equivalent:*

(i) σ_n is a Stirling permutation.

(ii) σ_n is AM-closed.

(iii) The interval family $I_j(\sigma_n)$ $(j \le n)$ is decreasing cross-free.

Proof (i) $\Leftrightarrow$ (ii): Let $\sigma_n = (i_1, i_2, \ldots, i_{2n})$ be a Stirling permutation. Then σ_n is AM as $\max_k i_k = n$, and n cannot occur in two nonadjacent positions in σ_n. Let $\rho(\sigma_n) = (x_1, x_2, \ldots, x_{2n-2})$. Then $\max_i x_i = n-1$, and $n-1$ cannot occur in two nonadjacent positions in x, because then some smaller number would be between, and this violates that σ_n is a Stirling permutation. By repeating this argument we conclude that σ_n is AM-closed. The converse implication is shown by induction on n. In fact, assume σ_n is AM-closed, and let $x = \rho(\sigma_n)$. Then x is also AM-closed, so, by induction, x is a Stirling permutation in $\widehat{\mathcal{S}}_{n-1}^{\times 2}$. By adding in the two adjacent n's we obtain σ_n which is then a Stirling permutation.

(i) $\Leftrightarrow$ (iii): Let σ_n be a Stirling permutation. Consider its intervals in (4) $I_j = I_j(\sigma_n)$ for $j = 1, 2, \ldots, n$. If (ii) does not hold, then there are two possibilities. Either, for some $i < j$, $I_i \subset I_j$, or, alternatively, I_i and I_j intersect, but neither set is contained in the other. In each of these two cases, σ contains an i between two j's, contradicting the Stirling property. This proves that (i) implies (ii). The converse follows by induction on n by observing that (iii) implies that $I_n = \{k, k+1\}$ for some k. Then we "remove" k and $k+1$, and apply the induction hypothesis. □

Define the iterated mapping $\rho^k(\sigma)$ by applying the mapping ρ k times to a Stirling permutation σ_n $(k = 1, 2, \ldots, n-1)$. Thus, $\rho(\sigma) = \rho^1(\sigma)$ where $\rho^{n-1}(\sigma) = (1, 1)$. We also write $[p, q]$ to indicate the integer interval $\{p, p+1, \ldots, q\}$ where $p < q$.

Example 7 For instance, let $n = 4$ and consider the Stirling permutation $\sigma_4 = 12443321$. Then

$$\rho(\sigma_4) = 123321, \ \rho^2(\sigma_4) = 1221, \ \rho^3(\sigma_4) = 11.$$

Moreover,

$$I_1(\sigma_4) = [1, 8],\ \ I_2(\sigma_4) = [2, 7],\ \ I_3(\sigma_4) = [5, 6],\ \ I_4(\sigma_4) = [3, 4].$$

□

We now consider how Stirling permutations may be constructed, essentially by using the inverse of the operator ρ defined above.

Algorithm 2:

```
Input: natural number n
1.Initialize v: let v = (n, n).
2.for =j = n − 1, n − 2, ..., 1do
    -insert two j's in vsuch that none of these is between any
    two k's (k > j)
Output: vector vof length 2n.
```

Corollary 2 *Algorithm* 2 *produces a Stirling permutation, and any Stirling permutation may be produced in this way.*

Proof The output vector v contains each integer $1 \leq j \leq n$ two times. Step 2 assures that the Stirling property holds in each iteration, and, by induction, the output v is a Stirling permutation.

Next, let σ_n be a Stirling permutation in $\widehat{\mathcal{S}}_n^{\times 2}$. By Theorem 9, σ_n is AM-closed. Thus, in Algorithm 1 we can start by putting the two n's in positions as in σ_n, then delete these and repeat the placement of $n - 1$. The AM-property and induction then assures that the constructed v equals σ. □

For instance, to construct the Stirling permutation $\sigma_4 = (1, 3, 4, 4, 3, 2, 2, 1)$ Algorithm 2 would do the following

$$(i)\ (4, 4),\ (ii)\ (3, 4, 4, 3),\ (iii)\ (3, 4, 4, 3, 2, 2),\ (iv)\ (1, 3, 4, 4, 3, 2, 2, 1) = \sigma.$$

6 Coda

For completeness we briefly discuss a generalization of Stirling permutations.

We call a general multipermutation σ_n of $\{1, 2, \ldots, n\}$ *inversion-even* provided the multiplicities of each of its inversions is even.

In an inversion-even 2-permutation

$$\ldots a \ldots b \ldots a \ldots \text{ with } a > b \text{ implies } \ldots a \ldots b \ldots b \ldots a \ldots .$$

Example 8 The 2-permutations of $\{1, 2\}$ and their number of inversions are given in the table below with identification of those that are Stirling permutations::

2-permutations	number of inversions 21	Stirling permutation
1122	0	Yes
1212	1	No
1221	2	Yes
2211	4	Yes
2121	3	No
2112	2	No

Thus an odd number of inversions in a 2-permutation of $\{1, 2\}$ implies that the two integers 1 and 2 alternate. □

It follows from Example 8 that a 2-permutation σ_n of $\{1, 2, \ldots, n\}$ is inversion-even if and only if it does not contain two integers a and b that alternate in their occurrences, that is, avoid the pattern 1212 and its reverse 2121. Such 2-permutations are called *quasi-Stirling* in [1] and are also considered in [8]. An equivalent definition of a quasi-Stirling permutation is that between any two integers equal to k and for any integer j, either both occurrences of j are between the two k's or neither are. The two 2-permutations 311, 322 and 213, 312 are examples of quasi-Stirling permutations that are not Stirling permutations. The pattern 1212 gives 1 inversion and the pattern 2121 gives 3 inversions. The reverse of a quasi-Stirling permutation is also a quasi-Stirling permutation because 1212 and 2121 are reverses of one another. Thus *quasi-Stirling permutations, as do Stirling permutations, have inversions only of multiplicities* 2*and* 4. The 2-permutations 233112, 322113, 332112, and 321123 have inversions only of multiplicities 2 and 4, and hence they are quasi-Stirling permutations but they are not Stirling permutations, as they contain the pattern 2112. Since a 2-permutation is a Stirling permutation if and only if it avoids the pattern 212, and a 2-permutation is a quasi-Stirling permutation if and only if it avoids the patterns 1212 and 2121, we obtain the following characterization.

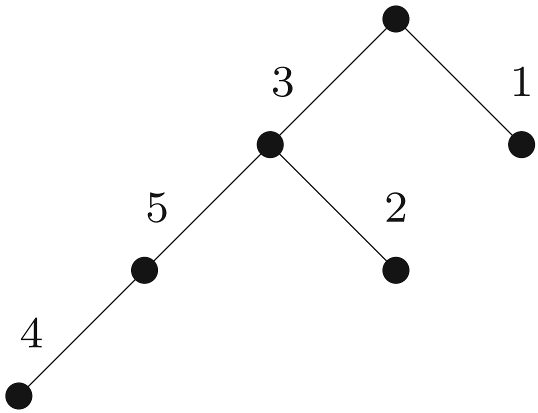
Corollary 3 *A quasi-Stirling permutation is a Stirling permutation if and only if it avoids the pattern* 2112*.*

Recall from Sect. 5 that Stirling permutations may be characterized by Euler 2-walks in labeled trees. Note that the edges of the tree T must be labeled according to the order in which they are added in a construction of the tree. See again the example in Fig. 1. It is natural to ask if also quasi-permutations can be constructed via trees. The following proposition is in [1].

Proposition 2 *A* 2*-permutation is a quasi-Stirling permutation if and only if it corresponds to a closed Euler* 2*-walk in a labeled tree with arbitrary labeling of the edges.*

Proof Let T be a plane tree with an arbitrary edge labeling (i.e., via a bijection from its set of edges E into $\{1, 2, \ldots, n-1\}$). An Euler 2-walk in T gives a 2-permutation

Fig. 2 Quasi-Stirling permutation and plane tree



(as each edge is traversed twice) with the additional property that each edge $e = uv$ is traversed before and after the two times traversal of any edge $e' = pq$ that is *below* e in T. Here "below" means that if e is deleted then e' is disconnected from the root of T. This clearly gives a quasi-Stirling permutation.

Conversely, let σ_n be a quasi-Stirling permutation, and construct a plane tree T as follows, using induction (on n). Let $k = \sigma_n(1)$ and let T consist of a single edge uv where u is the root, and give this edge the label k. Say that the other k is in position $s > 1$, so $\sigma_n(s) = k$. Then, s must be even and in positions $2, 3, \ldots, s-1$ there are $s/2 - 1$ numbers, where each occurs twice, by the quasi-Stirling property. Then, by induction, these numbers can be used as labels on a subtree attached to vertex v. Also, if $s < 2n$, we can extend the tree by another edge attached to the root, with label $\sigma_n(s+1)$ and placed to the right of the edge uv. We continue like this and eventually meet an integer p which is also the last component of σ_n, and then the desired tree T is constructed. □

Example 9 The quasi-Stirling permutation $\sigma_5 = (3, 5, 4, 4, 5, 2, 2, 3, 1, 1)$ corresponds to the Euler 2-walk in the plane tree T in Fig. 2 when choosing the left-most alternative in DFS search. □

Corresponding to a permutation $\sigma_n = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_n$ is a graph called a *permutation graph* $G(\sigma_n)$. The vertices of this graph are $1, 2, \ldots, n$ and there is an edge joining k and l if and only if $i_k > i_l$, $1 \leq k < l \leq n$. Thus the edges correspond to inversions. A characterization of a permutation graph is that both it and its complement with respect to the complete graph K_n are transitively orientable, that is, both are *comparability graphs* (see [9, 10, 12]. We orient the permutation graph $G(\sigma_n)$ by orienting the edge joining k and l by $k \to l$ if $i_k > i_l$.

With a Stirling permutation $\sigma_n \in \widehat{\mathcal{S}}_n$, we can associate a 2-graph[1] as follows. The *inversion* 2-*graph* $G_2(\sigma_n)$ of σ_n is the 2-graph with vertices $1, 2, \ldots, n$ whose edges, as with permutations, correspond to inversions. The multiplicity of an (ordinary) inversion in a Stirling permutation is 2 or 4. We assign the weight 1 to an inversion of multiplicity 2 and weight 2 to an inversion of multiplicity 4; so the edges of $G_2(\sigma_n)$

[1] A 2-graph is a graph whose edges have weight 1 or 2.

have weights 1 or 2. The 2-*complement* of a 2-graph G is taken with respect to the complete 2-graph $2K_n$, obtained by assigning weight 2 to each of the edges of the complete graph K_n, and is denoted by $2K_n \backslash G$. The 2-complement of the 2-graph of a Stirling permutation is also the 2-graph of a Stirling permutation, namely the Stirling permutation obtained by reversing the order of its elements. This leads to the following question.

Question 1 Can Stirling permutations be characterized in terms of their inversion 2-graphs similar to the way that permutations are characterized in term of orientability of their permutation graphs and complements? □

In this regard consider the next example.

Example 10 Consider the $5!! = 5 \cdot 3 \cdot 1 = 15$ Stirling permutations of $\{1, 2, 3\}$ given below:

$$\begin{array}{lll} 112233 & 123321 & 223311 \\ 113322 & 133221 & 233211 \\ 112332 & 133122 & 331122 \\ 122133 & 221133 & 332211 \\ 122331 & 221331 & 331221 \end{array}.$$

The inversion 2-graphs of those Stirling permutations which contain a path of length 2 (otherwise they do not enter into transitivity considerations) are specified in the table below by giving the weights of edges (recall that these weights in the case of Stirling permutations are the number of inversions divided by 2):

$3 \to 2$	$2 \to 1$	$3 \to 1$	Stirling instance
2	2	2	332211
2	1	2	331221
2	1	1	133221
1	2	2	233211
1	1	1	123321
2	2	1	∅
1	2	1	∅
1	1	2	∅

Instances of 2-permutations with all nonzero even weights in the 2-graph that are not included above are (again the number of inversions divided by 2):

$3 \to 2$	$2 \to 1$	$3 \to 1$	non-Stirling instance
2	2	1	∅
1	2	1	322113
1	1	2	233112

Thus we see that 121 and 112 need to be ruled out since they occur for a non-Stirling permutation but not for a Stirling permutation. Each of 212 and 111 occur for both a Stirling permutation and a non-Stirling permutation; thus we need to differentiate Stirling and non-Stirling permutations in these cases among those 2-permutations with all even weights. The pattern 221' is possible for neither Stirling nor non-Stirling permutations so this pattern can't occur. (Note that in the general case of a 2-permutation, we can have odd weights. For example, in 122313, $3 \rightarrow 1$ only occurs once as an inversion.) □

An *orientation* $\vec{G}$ of a 2-graph G is obtained by assigning a direction to each of its edges. Thus each edge of weight 1 or 2 of G becomes a (directed) edge of weight 1 or 2, respectively. (Note well that there are no edges of $\vec{G}$ joining a pair of vertices in opposite directions.) We define a 2-graph G to be *transitively orientable* provided it has an orientation $\vec{G}$ so that the following property holds

$$x \stackrel{a}{\rightarrow} y, y \stackrel{b}{\rightarrow} z \text{ implies } x \stackrel{c \geq \min\{a,b\}}{\longrightarrow} z, \tag{5}$$

where a, b, c denote the weights of 1 or 2 of the corresponding edges of $\vec{G}$. Thus as noted above, the following are possible for a Stirling permutation:

$$\begin{array}{l}
(1)\ x \stackrel{1}{\rightarrow} y, y \stackrel{1}{\rightarrow} z, x \stackrel{1}{\rightarrow} z; \\
(2)\ x \stackrel{1}{\rightarrow} y, y \stackrel{2}{\rightarrow} z, x \stackrel{2}{\rightarrow} z; \\
(3)\ x \stackrel{2}{\rightarrow} y, y \stackrel{1}{\rightarrow} z, x \stackrel{1}{\rightarrow} z; \cdot \\
(4)\ x \stackrel{2}{\rightarrow} y, y \stackrel{1}{\rightarrow} z, x \stackrel{2}{\rightarrow} z; \\
(5)\ x \stackrel{2}{\rightarrow} y, y \stackrel{2}{\rightarrow} z, x \stackrel{2}{\rightarrow} z,
\end{array}$$

Of these, (1), and (4) are possible for a non-Stirling 2-permutation, namely, 321,123 (whose reverse is equal to itself) and 332,112, respectively, while (2), (3), and (5) are not. Thus more then (5) is needed to characterize Stirling permutations. In addition the following are not possible for a Stirling permutation as shown:

$$\begin{array}{ll}
x \stackrel{1}{\rightarrow} y, y \stackrel{1}{\rightarrow} z, x \stackrel{2}{\rightarrow} z; & 233112 \\
x \stackrel{1}{\rightarrow} y, y \stackrel{2}{\rightarrow} z, x \stackrel{1}{\rightarrow} z; & 322113
\end{array}$$

All satisfy the condition (5) for transitive orientability. Note that the triple $x \stackrel{2}{\rightarrow} y, y \stackrel{2}{\rightarrow} z, x \stackrel{1}{\rightarrow} z$ which does not occur for any 2-permutation with all even weights does not satisfy (5).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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