Graphs and Combinatorics (2023) 39:60
https://doi.org/10.1007/s00373-023-02656-6

ORIGINAL PAPER

Minimally *k*-Factor-Critical Graphs for Some Large *k*

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Received: 1 July 2022 / Revised: 11 March 2023 / Accepted: 26 April 2023 /
Published online: 15 May 2023

Abstract
A graph G of order n is said to be k-factor-critical for integers $1 \leq k < n$, if the removal of any k vertices results in a graph with a perfect matching. 1- and 2-factor-critical graphs are the well-known factor-critical and bicritical graphs, respectively. A k-factor-critical graph G is called minimal if for any edge $e \in E(G)$, $G - e$ is not k-factor-critical. In 1998, O. Favaron and M. Shi conjectured that every minimally k-factor-critical graph of order n has the minimum degree $k + 1$ and confirmed it for $k = 1, n - 2, n - 4$ and $n - 6$. In this paper, we use a simple method to reprove the above results. As a main result, the further use of this method enables us to prove the conjecture to be true for $k = n-8$. We also obtain that every minimally $(n-6)$-factor-critical graph of order n has at most $n - \Delta(G)$ vertices with the maximum degree $\Delta(G)$ for $\Delta(G) \geq n - 4$.

Keywords Perfect matching · Minimally k-factor-critical graph · Minimum degree

AMS Subject Classification 05C70 · 05C75

1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the cardinality of $V(G)$. For a vertex x of G, let $d_G(x)$ be the degree of x in G, i.e. the number of edges of G incident with x, and let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of G, respectively.

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A *matching* of G is an edge subset of G in which no two edges have a common end-vertex. A matching M of G is said to be a *perfect matching* or a *1-factor* if it covers all vertices of G. A graph G is called *factor-critical* if the removal of each vertex of G results in a graph with a perfect matching. A graph with at least an edge is called *bicritical* if the removal of each pair of distinct vertices of G results in a graph with a perfect matching. A 3-connected bicritical graph is the so-called *brick*. Factor-critical and bricks were introduced by Gallai [6] and Lovász [7], respectively, which play important roles in Gallai–Edmonds Structure Theorem and in determining the dimensions of perfect matching polytopes and matching lattices; see a detailed monograph due to Lovász and Plummer [11].

Generally, Favaron [3] and Yu [20] independently defined k-factor-critical graphs for any positive integer k. A graph G of order n is said to be *k-factor-critical* for positive integer $k < n$, if the removal of any k vertices of G results in a graph with a perfect matching. They characterized k-factor-critical graphs in Tutte's type and showed that such graphs are $(k+1)$-edge-connected. To date there have been many studies on k-factor-critical graphs; see articles [5, 9, 10, 13, 15, 17, 18, 23] and a monograph [21].

A graph G is called *minimally k-factor-critical* if G is k-factor-critical but $G-e$ is not k-factor-critical for any $e \in E(G)$. O. Favaron and M. Shi [4] studied some properties of minimally k-factor-critical graphs and obtained an upper bound of minimum degree of minimally k-factor-critical graphs as follows.

Theorem 1.1 ([4]) *For a minimally k-factor-critical graph G of order $n \geq k+4$, $\delta(G) \leq \frac{n+k}{2} - 1$. If moreover $n \geq k+6$, then $\delta(G) \leq \frac{n+k}{2} - 2$.*

From Theorem 1.1, the following result is immediate.

Corollary 1.2 ([4]) *Let G be a minimally k-factor-critical graph of order n. If $k = n-2, n-4$ or $n-6$, then $\delta(G) = k+1$.*

Favaron and Shi [4] also pointed out that from the ear decomposition of factor-critical graphs (see [11]), obviously a minimally 1-factor-critical graph has the minimum degree two. Further, since a minimally k-factor-critical graph is $(k+1)$-edge-connected and thus has the minimum degree at least $k+1$, Favaron and Shi asked a problem: does Corollary 1.2 hold for general k?

Similarly, for minimally q-extendable graphs, Lou [8] proved that every minimally q-extendable bipartite graph has minimum degree $q+1$ and Lou et al. [12] conjectured that any minimally q-extendable graph G on n vertices with $n \leq 4q$ has minimum degree $q+1, 2q$ or $2q+1$. Afterward, Zhang et al. [22] showed that a non-bipartite graph of order $n \leq 4q-2$ is q-extendable if and only if it is $2q$-factor-critical, and formally reproposed the following conjecture. Since a q-extendable non-bipartite graph may not be $2q$-factor-critical when $n = 4q$ (see Fig. 1), they pointed out that except the case $n = 4q$, the conjecture of minimum degree of minimally q-extendable graphs in which the value $2q$ can be excluded is actually part of Conjecture 1.3.

Conjecture 1.3 ([4, 22]) *Let G be a minimally k-factor-critical graph of order n with $0 \leq k < n$. Then $\delta(G) = k+1$.*

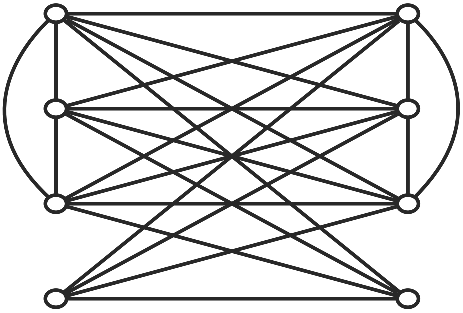

Fig. 1 G is 2-extendable but not 4-factor-critical when $n = 8, q = 2$

From the above discussions, we know that Conjecture 1.3 is true for $k = 1, n-2, n-4, n-6$. To date Conjecture 1.3 remains open for $2 \leq k \leq n-8$ of the same parity as n.

However, recently some great progresses have been made on related bicritical graphs. A brick is *minimal* if the removal of any edge results in a graph that is non-brick. From the construction of a brick, de Carvalho et al. [2] proved that every minimal brick contains a vertex of degree three. Norine and Thomas [16] proved that every minimal brick has at least three vertices of degree three. Later, Lin et al. [14] obtained that every minimal brick has at least four vertices of degree three. At the same time, Bruhn and Stein [1] showed that every minimal brick G has at least $\frac{1}{9}|V(G)|$ vertices of degree at most four.

In this paper, we use a novel and simple method to reprove Corollary 1.2. Continuing this method, we can prove Conjecture 1.3 to be true for $k = n-8$. On the other hand, the only $(n-2)$-factor-critical graph of order n is complete graph K_n. Favaron and Shi [4] characterized minimally $(n-4)$-factor-critical graphs of order n in the degree distribution. Finally, we obtain that every minimally $(n-6)$-factor-critical graph of order n has at most $n - \Delta(G)$ vertices with the maximum degree $\Delta(G)$ for $\Delta(G) \geq n-4$.

2 Some Preliminaries

In this section, we give some graph-theoretical terminologies and notations, and some preliminary results for late use. For a vertex x of a graph G, the *neighborhood* $N(x)$ of x is the set of vertices of G adjacent to x, and the *closed neighborhood* is $N[x] := N(x) \cup \{x\}$. Then $\overline{N[x]} := V(G) \setminus N[x]$ is called the *non-neighborhood* of x in G, which has a critical role in subsequent discussions.

A vertex of a graph G with degree one is called a *pendent vertex*. An *independent set* in a graph is a set of pairwise nonadjacent vertices. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S in G, and $G - S = G[V(G) - S]$. For an edge e of G, $G - e$ stands for the graph with vertex set $V(G)$ and edge set $E(G) - \{e\}$. Similarly, for distinct vertices u and v with $e = uv \notin E(G)$, $G + e$ stands for the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. A *claw* of G is an induced subgraph isomorphic to the star $K_{1,3}$.

A graph G is *trivial* if it has only one vertex. Let $c_o(G)$ be the number of odd components of G. The following is Tutte's 1-factor theorem.

Theorem 2.1 ([19]) *A graph G has a 1-factor if and only if $c_o(G-X) \leq |X|$ for any $X \subseteq V(G)$.*

The following characterization and connectivity of k-factor-critical graphs were obtained by Favaron [3] and Yu [20], independently.

Lemma 2.2 ([3, 20]) *A graph G is k-factor-critical if and only if $C_o(G-B) \leq |B|-k$ for any $B \subseteq V(G)$ with $|B| \geq k$.*

Lemma 2.3 ([3, 20]) *If G is k-factor-critical for some $1 \leq k < n$ with $n+k$ even, then G is k-connected, $(k+1)$-edge-connected and $(k-2)$-factor-critical if $k \geq 2$.*

Favaron and Shi [4] characterized minimally k-factor-critical graphs.

Lemma 2.4 ([4]) *Let G be a k-factor-critical graph. Then G is minimal if and only if for each $e = uv \in E(G)$, there exists $S_e \subseteq V(G) - \{u, v\}$ with $|S_e| = k$ such that every perfect matching of $G - S_e$ contains e.*

Lemma 2.5 ([4]) *Let G be a k-factor-critical graph of order $n > k+2$ and maximum degree $\Delta(G) = n-1$. Then G is minimal if and only if G contains one vertex of degree $n-1$ and $n-1$ vertices of degree $k+1$.*

Plummer and Saito [18] obtained a necessary and sufficient condition of k-factor-critical graphs.

Theorem 2.6 ([18]) *Let G be a graph of order n and let x and y be a pair of nonadjacent vertices of G with $d_G(x)+d_G(y) \geq n+k-1$. Then G is k-factor-critical if and only if $G \cup \{xy\}$ is k-factor-critical.*

Corollary 2.7 *Let G be a minimally k-factor-critical graph of order n, where $n \geq k+5$. If $\Delta(G) = n-2$, then there are at most two vertices with degree $n-2$ and such two vertices are not adjacent.*

Proof If G has three vertices of degree $n-2$, then two of them must be adjacent. So it suffices to show that any two vertices with degree $n-2$ are not adjacent. Suppose to the contrary that there exist two adjacent vertices u, v such that $d_G(u) = d_G(v) = n-2$. Let $G' = G - uv$. Since $n \geq k+5$, we have $d_{G'}(u) + d_{G'}(v) = 2n-6 \geq n+k-1$. By Theorem 2.6, G' is also k-factor-critical, contradicting that G is minimally k-factor-critical graph.

3 A Simple Proof of Corollary 1.2

In this section, we give a different and brief method to reprove Corollary 1.2. We divide our proof into the two cases $k = n-4$ and $n-6$ for $k \geq 2$.

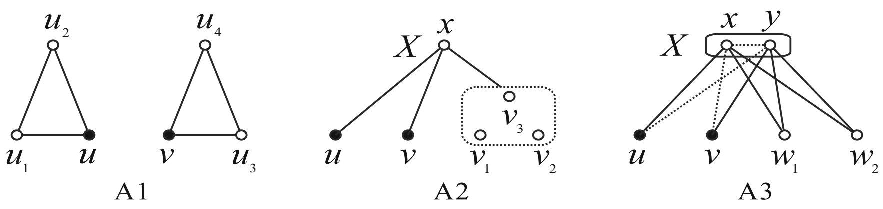

Fig. 2 The three configurations of $G' := G - e - S_e$

Lemma 3.1 ([4]) *A graph G of order $n \geq 6$ is $(n-4)$-factor-critical if and only if it is claw-free and $\delta(G) \geq n-3$.*

Proof of Corollary 1.2 for $k = n - 4$. By Lemma 2.3, $\delta(G) \geq n - 3$. To prove $\delta(G) = n - 3$, suppose to the contrary that $\delta(G) \geq n - 2$. Since G is a minimally $(n-4)$-factor-critical graph, for any $e = uv \in E(G)$, $G - e$ is not $(n-4)$-factor-critical. Since $n = k + 4 \geq 6$ and $\delta(G - e) \geq n - 3$, by Lemma 3.1, $G - e$ must contain a claw. Since G is $(n-4)$-factor-critical, G is claw-free. Hence u and v must be two pendent vertices of the claw. The third pendent vertex of the claw is not adjacent to u and v. So its degree is at most $n - 3$, a contradiction. □

Proof of Corollary 1.2 for $k = n - 6$. Obviously, $\delta(G) \geq n - 5$. Suppose to the contrary that $\delta(G) \geq n - 4$. That is, the non-neighborhood of any vertex in G has at most three vertices. Next we will obtain two claims.

Claim 1: For every $e = uv \in E(G)$, there exists $S_e \subseteq V(G) - \{u, v\}$ with $|S_e| = n - 6$ such that $G - e - S_e$ is one of Configuration $A1$, $A2$ and $A3$ as shown in Fig. 2. (The vertices within a dotted box induce a connected subgraph and the dotted edges indicate optional edges.)

Since G is a minimally $(n-6)$-factor-critical graph, by Lemma 2.4, for any given $e = uv \in E(G)$, there exists $S_e \subseteq V(G) - \{u, v\}$ with $|S_e| = n - 6$ such that every perfect matching of $G - S_e$ contains e. Then $G' := G - e - S_e$ has no perfect matching. By Theorem 2.1, there exists $X \subseteq V(G')$ such that $C_o(G' - X) > |X|$. By parity, $C_o(G' - X) \geq |X| + 2$. So $|X| + 2 \leq C_o(G' - X) \leq |V(G' - X)| = 6 - |X|$. Thus $|X| \leq 2$. Since $G' + e$ has a perfect matching, $C_o(G' - X) = |X| + 2$ and u and v belong separately to distinct odd components of $G' - X$. Since $\delta(G' + e) \geq 2$, $G' + e = G - S_e$ has no pendent vertex. Then G' has no isolated vertex. Now we discuss the following three cases depending on $|X|$.

If $|X| = 0$, then G' must consist of two odd components which each is isomorphic to K_3, and e joins them. So G' is $A1$.

If $|X| = 1$, then $C_o(G' - X) = 3$. If $G' - X$ has three trivial odd components, then $G' + e$ has a pendent vertex, a contradiction. So $G' - X$ has exactly two trivial odd components and one odd component with three vertices. Since $G' + e$ has no pendent vertex, e must join the two trivial odd components. So G' is $A2$.

If $|X| = 2$, then $C_o(G' - X) = 4$. So $G' - X$ consists of exactly four trivial odd components, two of which are joined by e. Let $X = \{x, y\}$. If x or $y \in \overline{N[u]} \cap \overline{N[v]}$, say $y \in \overline{N[u]} \cap \overline{N[v]}$, then $ux, vx \in E(G')$ and $G'[w_1, w_2, y]$ is an odd component of $G' - \{x\}$. Hence it is $A2$. Without loss of generality, assume that $ux, vy \in E(G')$. Then G' is $A3$. So Claim 1 holds.

Next we obtain some properties for the set of vertices of G in both non-neighborhoods of end-vertices of an edge from the three configurations.

Claim 2: (1) If $G-e-S_e$ is $A1$, then $|\overline{N[u]}\cap\overline{N[v]}|\leq 1$, for $u_1,u_2\in N(u)$ but $u_1,u_2\notin N(v)$ and $u_3,u_4\in N(v)$ but $u_3,u_4\notin N(u)$;

(2) If $G-e-S_e$ is $A2$, then $|\overline{N[u]}\cap\overline{N[v]}|=3$ as $\overline{N[u]}\cap\overline{N[v]}=\{v_1,v_2,v_3\}$;

(3) If $G-e-S_e$ is $A3$, then $|\overline{N[u]}\cap\overline{N[v]}|\geq 2$ as $\overline{N[u]}\cap\overline{N[v]}$ contains a pair of non-adjacent vertices w_1 and w_2, which form an independent set of G.

By Claim 1, there are three cases to discuss, where contradictions always happen.

Case 1: $G-e-S_e$ is $A1$.

Consider edge $e'=uu_1$. By Claim 1, there exists $S_{e'}\subseteq V(G)-\{u,u_1\}$ with $|S_{e'}|=n-6$ such that $G-e'-S_{e'}$ is one of Configuration $A1$, $A2$ and $A3$. Since $\overline{N[u_1]}=\{v,u_3,u_4\}$ and $uv\in E(G)$, $\overline{N[u]}\cap\overline{N[u_1]}=\{u_3,u_4\}$. By Claim 2 (1) and (2), $G-e'-S_{e'}$ is neither $A1$ nor $A2$. Since $u_3u_4\in E(G)$, $\{u_3,u_4\}$ is not an independent set of G. So $G-e'-S_{e'}$ is not $A3$. This is a contradiction to Claim 1.

Case 2: $G-e-S_e$ is $A2$.

Since $G-S_e$ has a perfect matching M, without loss of generality, assume that $xv_1,v_2v_3\in M$. Let $e'=ux\in E(G)$. Obviously, $\overline{N[u]}\cap\overline{N[x]}\subseteq\{v_2,v_3\}$. By Claim 2 (2) and Case 1, $G-e'-S_{e'}$ is neither $A1$ nor $A2$ for any $S_{e'}\subseteq V(G)-\{u,x\}$ with $|S_{e'}|=n-6$. Because $v_2v_3\in E(G)$, $\{v_2,v_3\}$ is not an independent set of G. Then $G-e'-S_{e'}$ is also not $A3$. This contradicts Claim 1.

Case 3: $G-e-S_e$ is $A3$.

Without loss of generality, assume that $ux,vy\in E(G)$. Let $e'=ux\in E(G)$. Clearly, $w_1,w_2\in N(x)$ and $w_1,w_2\notin N(u)$. Then $|\overline{N[u]}\cap\overline{N[x]}|\leq 1$. By Claim 2 and Case 1, $G-e'-S_{e'}$ is not $A1$, $A2$ or $A3$ for any $S_{e'}\subseteq V(G)-\{u,x\}$ with $|S_{e'}|=n-6$, which contradicts Claim 1. □

4 The Minimum Degree of Minimally ($n-8$)-Factor-Critical Graphs

Going one step further, we confirm that Conjecture 1.3 is true for $k=n-8$.

Theorem 4.1 *If G is a minimally $(n-8)$-factor-critical graph of order $n\geq 10$, then $\delta(G)=n-7$.*

Proof By Lemma 2.3, $\delta(G)\geq n-7$. Suppose to the contrary that $\delta(G)\geq n-6$.

Claim 1: For every $e=uv\in E(G)$, there exists $S_e\subseteq V(G)-\{u,v\}$ with $|S_e|=n-8$ such that $G-e-S_e$ is one of Configuration $B1$ to $B8$ as shown in Fig. 3. (The vertices within a dotted box induce a connected subgraph.)

Since G is a minimally $(n-8)$-factor-critical graph, by Lemma 2.4, for any $e=uv\in E(G)$, there exists $S_e\subseteq V(G)-\{u,v\}$ with $|S_e|=n-8$ such that every perfect matching of $G-S_e$ contains e. Then $G-e-S_e$ has no perfect matching. Let $G':=G-e-S_e$. By Theorem 2.1, there exists $X\subseteq V(G')$ such that $C_o(G'-X)>|X|$. By parity, $C_o(G'-X)\geq|X|+2$. So $|X|+2\leq C_o(G'-X)\leq|V(G'-X)|=8-|X|$. Thus $|X|\leq 3$. Since $G'+e$ has a 1-factor, $C_o(G'-X)=|X|+2$ and u and v belong

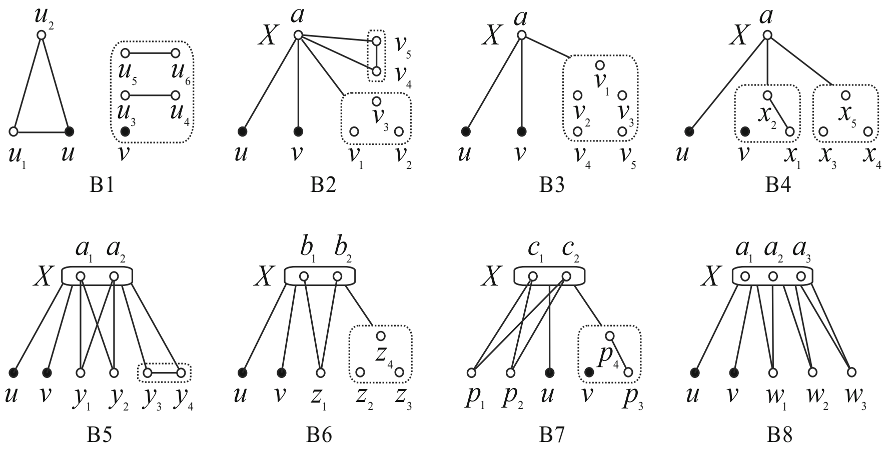

Fig. 3 The eight configurations of $G' := G - e - S_e$

respectively to two distinct odd components of $G' - X$. Moreover, $\delta(G - S_e) \geq 2$. Then $G' + e = G - S_e$ has no pendent vertex and G' has no isolated vertex.

If $|X| = 0$, then G' has exactly two odd components, one of which is K_3 and the other has five vertices. Since $G' + e$ has a 1-factor, e joins the two odd components, and we may assume that u_3u_4, u_5u_6 are two independent edges. So G' is $B1$.

If $|X| = 1$, then $C_o(G' - X) = 3$. Let $X = \{a\}$. $G' - X$ has at most two trivial odd components which are joined by e. Otherwise, $G' + e$ has a pendent vertex, a contradiction. The other odd component has three or five vertices. So G' is $B2$ or $B3$. Specially, if $G' - X$ has exactly one trivial odd component, then the other two nontrivial odd components are both with three vertices. Since $G' + e$ has no pendent vertex, e joins the trivial odd component and a nontrivial odd component with three vertices. Besides, there must exist an edge joining a and the nontrivial odd component, otherwise, it is $B1$. Then G' is $B4$.

If $|X| = 2$, then $C_o(G' - X) = 4$. Hence $G' - X$ has either four trivial odd components or three trivial odd components and one nontrivial odd component with three vertices. Since $G' + e$ has a 1-factor, e joins two of the four odd components of $G' - X$. If $G' - X$ has four trivial odd components, then G' is $B5$. If $G' - X$ has three trivial odd components, then G' is $B6$ or $B7$.

If $|X| = 3$, then $C_o(G' - X) = 5$. Thus $G' - X$ consists of exactly five trivial odd components, two of which are joined by e. So G' is $B8$.

For every $x \in V(G)$, $\overline{N[x]}$ has at most five vertices of G. Then we can obtain the following claim by observing the eight configurations.

Claim 2: (1) If $G - e - S_e$ is $B1$, then $|\overline{N[u]} \cap \overline{N[v]}| \leq 3$. Since $u_1, u_2 \in N(u)$ but $u_1, u_2 \notin N(v)$, $\overline{N[v]}$ has at most three elements in $\overline{N[u]}$;

(2) If $G - e - S_e$ is $B2$ or $B3$, then $|\overline{N[u]} \cap \overline{N[v]}| = 5$;

(3) If $G - e - S_e$ is $B4$, then $3 \leq |\overline{N[u]} \cap \overline{N[v]}| \leq 4$ as $\overline{N[u]} = \{x_1, x_2, x_3, x_4, x_5\}$ and $\{x_3, x_4, x_5\} \subseteq \overline{N[v]}$ but x_1 or $x_2 \in N(v)$;

(4) If $G - e - S_e$ is $B5$ or $B6$, then $|\overline{N[u]} \cap \overline{N[v]}| \geq 4$;

(5) If $G - e - S_e$ is $B7$, then $2 \le |\overline{N[u]} \cap \overline{N[v]}| \le 4$ as $\{p_1, p_2, p_3, p_4\} \subseteq \overline{N[u]}$ and $\{p_1, p_2\} \subseteq \overline{N[v]}$ but p_3 or $p_4 \in N(v)$;

(6) If $G - e - S_e$ is $B8$, then $|\overline{N[u]} \cap \overline{N[v]}| \ge 3$ as $\overline{N[u]} \cap \overline{N[v]}$ contains an independent set $\{w_1, w_2, w_3\}$.

By Claim 1, there are eight cases to distinguish.

Case 1: $G - e - S_e$ is $B1$.

Since $G - S_e$ has a perfect matching M, $uv, u_1u_2 \in M$. So, without loss of generality, we assume that $u_3u_4, u_5u_6 \in M$.

Consider edge $e' = uu_1$. Clearly, $\overline{N[u_1]} = \{v, u_3, u_4, u_5, u_6\}$ and $\overline{N[u]} \cap \overline{N[u_1]} = \{u_3, u_4, u_5, u_6\}$. By Claim 1, there exists $S_{e'} \subseteq V(G) - \{u, u_1\}$ with $|S_{e'}| = n - 8$ such that $G - e' - S_{e'}$ is one of Configuration $B1$ to $B8$. By Claim 2 (1) and (2), $G - e' - S_{e'}$ may not be $B1$, $B2$ or $B3$. Furthermore, since $G[\overline{N[u]} \cap \overline{N[u_1]}]$ contains two independent edges, by Claim 2 (4) and (6), $G - e' - S_{e'}$ can not be $B5$, $B6$ or $B8$. Then $G - e' - S_{e'}$ would be $B4$ or $B7$.

Suppose that $G - e' - S_{e'}$ is $B4$. Since $G[\overline{N[u_1]}]$ is a connected subgraph of G, u (resp. u_1) belongs to the trivial (resp. nontrivial) odd component of $B4 - \{a\}$. The odd component containing u_1 must be K_3. Otherwise, there is a vertex in $\overline{N[u]} \cap \overline{N[u_1]}$ which is not adjacent to the other three vertices, contradicting that u_3u_4, u_5u_6 are two independent edges. But then $|\overline{N[u]} \cap \overline{N[u_1]}| = 3$, a contradiction.

So $G - e' - S_{e'}$ is $B7$. Since $u_3u_4, u_5u_6 \in M$ and $\{p_1, p_2\}$ is an independent set of G, $\{p_1, p_2\} = \{u_3, u_5\}$, $\{p_1, p_2\} = \{u_4, u_6\}$, $\{p_1, p_2\} = \{u_3, u_6\}$ or $\{p_1, p_2\} = \{u_4, u_5\}$. By symmetry, we may assume that $\{p_1, p_2\} = \{u_3, u_5\}$. If u_1 is a trivial odd component of $B7 - \{c_1, c_2\}$, then u_4 or $u_6 \in \{p_3, p_4\}$. But $u_3u_4, u_5u_6 \in E(G)$, a contradiction. Then u (resp. u_1) belongs to the trivial (resp. nontrivial) odd component of $B7 - \{c_1, c_2\}$. The odd component containing u_1 must be K_3. Otherwise, say $u_1p_4 \notin E(G)$, so $p_4 \in \{u_4, u_6\}$, contradicting that $p_1p_4, p_2p_4 \notin E(G)$. But then $|\overline{N[u]} \cap \overline{N[u_1]}| \le 3$, a contradiction.

Case 2: $G - e - S_e$ is $B7$.

We may assume that $uc_1, vp_3 \in E(G)$. Since $G - S$ has a perfect matching, $p_3p_4 \in E(G)$. Let $e' = c_1p_1 \in E(G)$. Obviously, $\overline{N[p_1]} = \{u, v, p_2, p_3, p_4\}$. Then $\overline{N[c_1]} \cap \overline{N[p_1]} \subseteq \{v, p_3, p_4\}$ (see Fig. 4 (1)). By Claim 1, there exists $S_{e'} \subseteq V(G) - \{c_1, p_1\}$ with $|S_{e'}| = n - 8$ such that $G - e' - S_{e'}$ is one of Configuration $B1$ to $B8$. Since $\{v, p_3, p_4\}$ is not an independent set of G, $G - e' - S_{e'}$ is not $B8$. By Claim 2 and Case 1, $G - e' - S_{e'}$ would be $B4$ or $B7$.

If $G - e' - S_{e'}$ is $B4$, then $\overline{N[c_1]} \cap \overline{N[p_1]} = \{v, p_3, p_4\} = \{x_3, x_4, x_5\}$. Hence c_1 (resp. p_1) belongs to the trivial (resp. nontrivial) odd component of $B4 - \{a\}$. Otherwise, $\overline{N[p_1]} = \{x_1, x_2, x_3, x_4, x_5\}$, so $u \in \{x_1, x_2\}$, but $uv \in E(G)$, a contradiction. Moreover, the odd component of $B4 - \{a\}$ containing p_1 must be K_3 as $|\overline{N[c_1]} \cap \overline{N[p_1]}| = 3$. Then $x_1, x_2 \notin \{u, p_2\}$. Since $p_3 \in \{x_3, x_4, x_5\}$, $\{x_1, x_2\} \subseteq \overline{N[p_3]}$. Then $\{c_1, u, p_1, p_2, x_1, x_2\} \subseteq \overline{N[p_3]}$. Hence $d_G(p_3) \le n - 7$, a contradiction (see Fig. 4 (2)).

So $G - e' - S_{e'}$ is $B7$, whose vertices are relabelled by $c_1', c_2', p_1', p_2', p_3', p_4', u', v'$. Then $\{u', v'\} = \{c_1, p_1\}$. Since $\{p_1', p_2'\} \subseteq \overline{N[c_1]} \cap \overline{N[p_1]} \subseteq \{v, p_3, p_4\}$, $vp_4 \notin E(G)$. We may assume $p_1' = v$, $p_2' = p_4$. It is easy to see that $\overline{N[p_4]} = \{u, v, c_1, p_1, p_2\}$. On

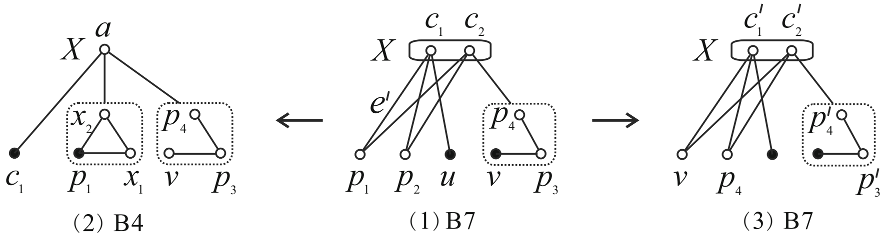

Fig. 4 $G - e' - S_{e'}$ is $B4$ and $G - e' - S_{e'}$ is $B7$

the other hand, $\overline{N[p_4]}=\{v, c_1, p_1, p'_3, p'_4\}$. So $\{u, p_2\} = \{p'_3, p'_4\}$. But $p'_3p'_4 \in E(G)$, $up_2 \notin E(G)$, a contradiction (see Fig. 4 (3)).

Case 3: $G - e - S_e$ is $B5$.

Clearly, $\overline{N[y_1]}=\{u, v, y_2, y_3, y_4\}$. Without loss of generality, assume that $ua_1 \in E(G)$. Let $e' = y_1a_1 \in E(G)$. Then $\overline{N[a_1]} \cap \overline{N[y_1]} \subseteq \{v, y_3, y_4\}$. By Claim 1, there exists $S_{e'} \subseteq V(G) - \{y_1, a_1\}$ with $|S_{e'}| = n - 8$ such that $G - e' - S_{e'}$ is one of Configuration $B1$ to $B8$. Since $y_3y_4 \in E(G)$, $\{v, y_3, y_4\}$ is not an independent set of G. So $G - e' - S_{e'}$ is not $B8$. By Claim 2, Case 1 and Case 2, $G - e' - S_{e'}$ would be $B4$.

If $G - e' - S_{e'}$ is $B4$, then $\overline{N[a_1]} \cap \overline{N[y_1]}=\{v, y_3, y_4\}$, which induces a connected subgraph of G. But $vy_3, vy_4 \notin E(G)$, a contradiction.

Case 4: $G - e - S_e$ is $B4$.

Since $G - S_e$ has a perfect matching M, without loss of generality, assume that $x_1x_2, ax_3, x_4x_5 \in M$. Let $e' = ua \in E(G)$. Obviously, $\overline{N[u]} = \{x_1, x_2, x_3, x_4, x_5\}$. Hence $\overline{N[a]} \cap \overline{N[u]} \subseteq \{x_1, x_2, x_4, x_5\}$. By Claim 1, there exists $S_{e'} \subseteq V(G) - \{u, a\}$ with $|S_{e'}| = n - 8$ such that $G - e' - S_{e'}$ is one of Configuration $B1$ to $B8$. By Claim 2, Cases 1, 2 and 3, $G - e' - S_{e'}$ would be $B4$, $B6$ or $B8$.

Since $x_1x_2, x_4x_5 \in E(G)$, $\{x_1, x_2, x_4, x_5\}$ has no subset, which is an independent set with size three. So $G - e' - S_{e'}$ is not $B8$. Because there are no edge joining $\{x_1, x_2\}$ and $\{x_4, x_5\}$, for any $T \subseteq \{x_1, x_2, x_4, x_5\}$ with $|T| = 3$, $G[T]$ does not induce a connected subgraph of $B4 - \{a\}$ or $B6 - \{b_1, b_2\}$. So $G - e' - S_{e'}$ can not be $B4$ or $B6$. This is a contradiction to Claim 1.

Case 5: $G - e - S_e$ is $B6$.

Without loss of generality, we may assume that $b_1z_2 \in E(G)$. Let $e' = b_1z_1$. Clearly, $\overline{N[z_1]}=\{u, v, z_2, z_3, z_4\}$. By Claim 1, there exists $S_{e'} \subseteq V(G) - \{b_1, z_1\}$ with $|S_{e'}| = n - 8$ such that $G - e' - S_{e'}$ is one of Configuration $B1$ to $B8$. Moreover, $b_1, b_2 \notin \overline{N[u]} \cap \overline{N[v]}$, otherwise, it is $B2$ or $B3$. Assume that $ub_1, vb_2 \in E(G)$. So $\overline{N[b_1]} \cap \overline{N[z_1]} \subseteq \{v, z_3, z_4\}$. By Claim 2, Cases 1, 2 and 4, $G - e' - S_{e'}$ would be $B8$.

If $G - e' - S_{e'}$ is $B8$, then $\overline{N[b_1]} \cap \overline{N[z_1]} = \{v, z_3, z_4\}$ and $\{v, z_3, z_4\}$ is an independent set of G. So $z_3z_4 \notin E(G)$. Thus $z_2z_3, z_2z_4 \in E(G)$. Since $\delta(G - S_e) \geq 2$, $b_2z_3, b_2z_4 \in E(G)$. Now consider edge $e'' = z_2z_3$. We have $\overline{N[z_3]} = \{u, v, b_1, z_1, z_4\}$. Since $b_1z_2, z_2z_4 \in E(G)$, $\overline{N[z_2]} \cap \overline{N[z_3]} = \{u, v, z_1\}$. By Claim 2, Cases 1, 2, 4 and $uv \in E(G)$, $G - e'' - S_{e''}$ is not one of Configuration $B1$ to $B8$ for any $S_{e''} \subseteq V(G) - \{z_2, z_3\}$ with $|S_{e''}| = n - 8$, a contradiction.

Case 6: $G - e - S_e$ is $B2$ or $B3$.

Since $G - S_e$ has a perfect matching M, without loss of generality, assume that $av_1, v_2v_3, v_4v_5 \in M$. Now consider edge $e' = ua$. Then $\overline{N[u]} \cap \overline{N[a]} \subseteq \{v_2, v_3, v_4, v_5\}$. By Claim 2 and Cases 1 to 5, $G - e' - S_{e'}$ would be $B8$ for some $S_{e'} \subseteq V(G) - \{u, a\}$ with $|S_{e'}| = n - 8$ only when $G - e - S_e$ is $B3$. Since $v_2v_3, v_4v_5 \in E(G)$, for any $T \subseteq \{v_2, v_3, v_4, v_5\}$ with $|T| = 3$, T is not an independent set of G. So $G - e' - S_{e'}$ is not $B8$. This is a contradiction to Claim 1.

Case 7: $G - e - S_e$ is $B8$.

Since $\delta(G - S_e) \geq 2$, without loss of generality, assume that $w_1a_1, w_1a_2 \in E(G)$. Let $e' = w_1a_1 \in E(G)$ and $S_{e'} \subseteq V(G) - \{w_1, a_1\}$ with $|S_{e'}| = n - 8$ satisfying Claim 1. Then $\{u, v, w_2, w_3\} \subseteq \overline{N[w_1]}$. By Claim 2 and Cases 1 to 6, $G - e' - S_{e'}$ would be $B8$. Then it suffices to show that there is an independent set S_0' with size three in $G - S_{e'}$ and every vertex in S_0' is not adjacent to w_1 and a_1. Since $S_0' \subseteq \overline{N[w_1]}$, w_2 or $w_3 \in S_0'$, say $w_2 \in S_0'$. Then $w_2a_2, w_2a_3 \in E(G)$ as $d_{G-S_e}(w_2) \geq 2$.

Subcase 7.1: $w_1a_3 \notin E(G)$.

Clearly, $\overline{N[w_1]} = \{u, v, w_2, w_3, a_3\}$. So $a_3 \notin S_0'$ and $w_3 \in S_0'$. Then $w_3a_2, w_3a_3 \in E(G)$. Thus S_0' is either $\{u, w_2, w_3\}$ or $\{v, w_2, w_3\}$, say $\{u, w_2, w_3\}$. Now consider edge $e'' = w_1a_2$. Then $\overline{N[a_2]} \cap \overline{N[w_1]} \subseteq \{u, v, a_3\}$. By Claim 2, Cases 1, 2, 4 and $uv \in E(G)$, $G - e'' - S_{e''}$ is not one of Configuration $B1$ to $B8$ for any $S_{e''} \subseteq V(G) - \{w_1, a_2\}$ with $|S_{e''}| = n - 8$, which contradicts Claim 1.

Subcase 7.2: $w_1a_3 \in E(G)$.

Let $e'' = w_1a_2 \in E(G)$ and $S_{e''} \subseteq V(G) - \{w_1, a_2\}$ with $|S_{e''}| = n - 8$ satisfying Claim 1. We denote the independent set with size three in $G - S_{e''}$ by S_0'' and every vertex in S_0'' is not adjacent to w_1 and a_2. Then $w_3 \in S_0''$. So $w_3a_1, w_3a_3 \in E(G)$. Hence $w_3 \notin S_0'$. Thus u or $v \in S_0'$, say $u \in S_0'$. Then $S_0' = \{u, w_2, w\}$, where $w \in \overline{N[w_1]} \cap \overline{N[a_1]}$. Since $w_2 \notin S_0''$, u or $v \in S_0''$.

If $u \in S_0''$, then $S_0'' = \{u, w_3, w\}$, where $w \in \overline{N[w_1]} \cap \overline{N[a_2]}$. Thus $\{u, w_1, w_2, w_3, a_1, a_2\} \subseteq \overline{N[w]}$. So $d_G(w) \leq n - 7$, a contradiction.

If $v \in S_0''$, then $S_0'' = \{v, w_3, w\}$, where $w \in \overline{N[w_1]} \cap \overline{N[a_2]}$. Thus $\{u, v, w_1, w_2, w_3, a_1, a_2\} \subseteq \overline{N[w]}$. So $d_G(w) \leq n - 8$, a contradiction.

Combining Cases 1 to 7, we complete the proof. □

5 Some Properties of Minimally ($n - 6$)-Factor-Critical Graphs

In this section, we obtain that every minimally $(n-6)$-factor-critical graph G of order n has at most $n - \Delta(G)$ vertices with the maximum degree $\Delta(G)$ for $\Delta(G) \geq n - 4$.

By Lemma 2.5, for any minimally $(n-6)$-factor-critical graph G of order n, G has only one vertex of degree $n-1$ and $n-1$ vertices of degree $n-5$ when $\Delta(G) = n-1$. So we consider the cases of $n - 4 \leq \Delta(G) \leq n - 2$.

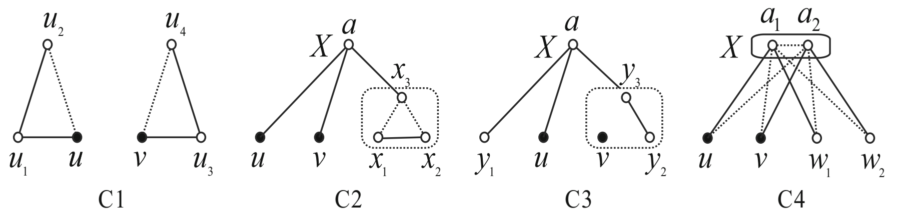

Fig. 5 The four configurations of $G' := G - e - S_e$

Lemma 5.1 *Let G be a minimally $(n-6)$-factor-critical graph of order $n \geq 8$. For every $e = uv \in E(G)$ with $d_G(u) \geq n-4$, $d_G(v) \geq n-4$, there exists $S_e \subseteq V(G) - \{u, v\}$ with $|S_e| = n-6$ such that $G - e - S_e$ is one of Configuration $C1$, $C2$, $C3$ and $C4$ as shown in Fig. 5. (The dotted edge indicates an optional edge.)*

Proof Since G is a minimally $(n-6)$-factor-critical graph, for every $e = uv \in E(G)$ with $d_G(u) \geq n-4$, $d_G(v) \geq n-4$, there exists $S_e \subseteq V(G) - \{u, v\}$ with $|S_e| = n-6$ such that every perfect matching of $G - S_e$ contains e by Lemma 2.4. Then $G - e - S_e$ has no 1-factor. Let $G' := G - e - S_e$. By Theorem 2.1, there exists $X \subseteq V(G')$ such that $C_o(G' - X) > |X|$. By parity, $C_o(G' - X) \geq |X| + 2$. So $|X| + 2 \leq C_o(G' - X) \leq |V(G' - X)| = 6 - |X|$. Then $|X| \leq 2$. Since $G' + e$ has a 1-factor, $C_o(G' - X) = |X| + 2$ and u and v belong respectively to two distinct odd components of $G' - X$. Since $d_G(u) \geq n-4$ and $d_G(v) \geq n-4$, $d_{G-e}(u) \geq n-5$ and $d_{G-e}(v) \geq n-5$. The vertices of $G - e$ distinct from u and v have the same degree as in G. So $\delta(G - e) = \delta(G) = n-5$ by Corollary 1.2. Then $\delta(G - e - S_e) = \delta(G') \geq 1$. Thus G' has no isolated vertex.

If $|X| = 0$, then G' has exactly two odd components, each of which has three vertices. Since $G' + e$ has a 1-factor, e joins the two odd components. So G' is $C1$.

If $|X| = 1$, then $C_o(G' - X) = 3$. Let $X = \{a\}$. $G' - X$ has either two trivial odd components and an odd component with three vertices or three trivial odd components and an even component with two vertices. For the former case, if e joins two trivial odd components, then G' is $C2$. If e joins a trivial odd component and the odd component with three vertices, then G' is $C3$. For the latter case, e joins two of the three trivial odd components, say x_3 belongs to the third trivial odd component and $G[\{x_1, x_2\}]$ is the even component. For concise, we attribute it to $C2$. Then note that the vertices within a dotted box in $C2$ may induce a disconnected subgraph.

If $|X| = 2$, then $C_o(G' - X) = 4$. So $G' - X$ consists of exactly four trivial odd components, two of which are joined by e. Let $X = \{a_1, a_2\}$. Then $a_1, a_2 \notin \overline{N[u]} \cap \overline{N[v]}$, otherwise, it is $C2$ or $C3$. So G' is $C4$. □

Since $d_G(u) \geq n-4$ and $d_G(v) \geq n-4$, there are at most three vertices of G in each of $\overline{N[u]}$ and $\overline{N[v]}$. By a close inspection of the four configurations of Lemma 5.1, we can easily obtain some properties, which play important roles in the proofs of Theorem 5.3, 5.4 and 5.5.

Proposition 5.2 *Let G, e, S_e be as of Lemma 5.1. If $G - e - S_e$ is one of Configuration $C1$, $C2$, $C3$ and $C4$, then the following statements hold:*

(1) If $G - e - S_e$ is $C1$, then $|\overline{N[u]} \cap \overline{N[v]}| \leq 2$, $n - 4 \leq d_G(u), d_G(v) \leq n - 3$ and $|N(u) \cap N(v)| \leq n - 6$.
(2) If $G - e - S_e$ is $C2$, then $|\overline{N[u]} \cap \overline{N[v]}| = 3$ and $d_G(u) = d_G(v) = n - 4$.
(3) If $G - e - S_e$ is $C3$, then $1 \leq |\overline{N[u]} \cap \overline{N[v]}| \leq 2$. Moreover, $d_G(u) = n - 4$, $d_G(v) \geq n - 4$ and $d_G(y_1) = n - 5$.
(4) If $G - e - S_e$ is $C4$, then $2 \leq |\overline{N[u]} \cap \overline{N[v]}| \leq 3$ and $n - 4 \leq d_G(u), d_G(v) \leq n - 3$. Moreover, $\{w_1, w_2\}$ is an independent set of G.

We show only Proposition 5.2 (1). The proof of other cases are similar and thus omitted.

Proof (1) Suppose that $G - e - S_e$ is $C1$. Since $u_1, u_2 \in \overline{N[v]}$ and $u_3, u_4 \in \overline{N[u]}$, we have $d_G(u), d_G(v) \leq n - 3$. Moreover, $u_1 \notin \overline{N[u]}$ and $u_3 \notin \overline{N[v]}$. Then $|\overline{N[u]} \cap \overline{N[v]}| \leq 2$. Since both $\overline{N[u]}$ and $\overline{N[v]}$ have at most one element in S_e, $|N(u) \cap N(v)| \leq n - 6$. □

If n is odd, then both $n - 2$ and $n - 4$ are odd. Thus the total number of vertices of degree $n - 2$ or $n - 4$ is even. If n is even, then both $n - 3$ and $n - 5$ are odd. So the total number of vertices of degree $n - 3$ or $n - 5$ is even. Thus the total number of vertices of degree $n - 2$ or $n - 4$ is also even. Therefore, G contains even total number of vertices of degree $n - 2$ or $n - 4$.

Theorem 5.3 *Let G be a minimally $(n - 6)$-factor-critical graph of order $n \geq 8$. If $\Delta(G) = n - 2$, then G has at most two vertices with degree $n - 2$ and the two vertices are not adjacent. In particular, if G has exactly two vertices with degree $n - 2$, then the other vertices of G have degree $n - 5$. If G has one vertex with degree $n - 2$, then there is only one vertex with degree $n - 4$ and the other vertices of G have degree $n - 5$.*

Proof Firstly, by Corollary 2.7, G has at most two vertices with degree $n - 2$ and the two vertices are not adjacent. Let $d_G(u) = d_G(v) = n - 2$ and $uv \notin E(G)$. If G has a vertex x with degree $n - 3$ or $n - 4$, then $ux, vx \in E(G)$. Consider edge ux. By Lemma 5.1, there exists $S \subseteq V(G) - \{u, x\}$ with $|S| = n - 6$ such that $G - ux - S$ is one of Configuration $C1$, $C2$, $C3$ and $C4$. Then, by Proposition 5.2, $G - ux - S$ would be $C3$ only when $d_G(x) = n - 4$. Hence x (resp. u) belongs to the trivial (resp. nontrivial) odd component of $C3 - \{a\}$. So $y_1 = v$ and $d_G(v) = d_G(y_1) = n - 5$, a contradiction. Thus all vertices of $G - \{u, v\}$ have degree $n - 5$.

If G has only one vertex u with degree $n - 2$, then u is adjacent to every vertex of G except one. Suppose that G has three vertices with degree $n - 4$, say $d_G(v_1) = d_G(v_2) = d_G(v_3) = n - 4$. Then we may assume that $uv_1, uv_2 \in E(G)$. Let $e_1 = uv_1 \in E(G)$ and $S_1 \subseteq V(G) - \{u, v_1\}$ with $|S_1| = n - 6$ satisfying Lemma 5.1. By Proposition 5.2, $G - e_1 - S_1$ is only $C3$. Then v_1 (resp. u) belongs to the trivial (resp. nontrivial) odd component of $C3 - \{a\}$. Assume another trivial odd component of $C3 - \{a\}$ is spanned by $\{w_1\}$. So $uw_1, v_1w_1 \notin E(G)$. Hence $uv_3 \in E(G)$. Let $e_2 = uv_2 \in E(G)$ and $e_3 = uv_3 \in E(G)$. Similar to the discussion above, we have $w_2v_2, w_3v_3 \notin E(G)$, where $\{w_i\}$ spans a trivial odd component of $C3 - \{a\}$ which are corresponding to e_i for $i = 2, 3$. Since $d_G(u) = n - 2$ and $uw_1, uw_2, uw_3 \notin E(G)$, we have $w_1 = w_2 = w_3$. Then $\overline{N[w_1]} = \{u, v_1, v_2, v_3\}$. Therefore, $G[\{u, v_2, v_3\}]$ is the nontrivial odd component of $G - e_1 - S_1$. So $v_1v_2, v_1v_3 \notin E(G)$. Similarly, we have $v_2v_3 \notin E(G)$. But then $G - S_1$ has no 1-factor, a contradiction. Therefore, if $d_G(u) = n - 2$, G has only one vertex with degree $n - 4$.

Suppose that $d_G(u) = n - 2$ and $d_G(v_1) = n - 4$. If G has a vertex y with degree $n - 3$, then $uy \notin E(G)$. Otherwise, by Proposition 5.2, $G - uy - S'$ is not one of Configuration $C1$, $C2$, $C3$ and $C4$ for any $S' \subseteq V(G) - \{u, y\}$ with $|S'| = n - 6$, a contradiction. So $uv_1 \in E(G)$. However, from the above discussions, $G - uv_1 - S_1$ can only be $C3$. Thus y belongs to the trivial odd component of $C3 - \{a\}$. So $d_G(y) = n - 5$, a contradiction. Thus all vertices of $G - \{u, v_1\}$ have degree $n - 5$. □

Theorem 5.4 *Let G be a minimally $(n - 6)$-factor-critical graph of order $n \geq 9$. If $\Delta(G) = n - 3$, then G has at most three vertices with degree $n - 3$. In particular, if $d_G(u) = d_G(v) = d_G(w) = n - 3$, then $uv, uw, vw \notin E(G)$ and the other vertices of G have degree $n - 5$.*

Proof If G has four vertices with degree $n - 3$, then two of them must be adjacent. So it suffices to show that any three vertices with degree $n - 3$ are not adjacent to each other. Assume that $d_G(u) = d_G(v) = d_G(w) = n - 3$. Suppose to the contrary that there is at least one pair of adjacent vertices among $\{u, v, w\}$, say $uv \in E(G)$. Let $e = uv \in E(G)$. By Lemma 5.1, there exists $S \subseteq V(G) - \{u, v\}$ with $|S| = n - 6$ such that $G - e - S$ is one of Configuration $C1$, $C2$, $C3$ and $C4$.

Case 1: $uv \in E(G), uw, vw \notin E(G)$.

Then $|\overline{N[u]} \cap \overline{N[v]}| \leq 2$ and $|N(u) \cap N(v)| \geq n - 5$. By Proposition 5.2, $G - e - S$ would be $C4$.

If $G - e - S$ is $C4$, then $|\overline{N[u]} \cap \overline{N[v]}| = 2$ and $w \in \overline{N[u]} \cap \overline{N[v]}$. Since w is adjacent to every vertex in $V(G) - \{u, v, w\}$, there is not an independent set with size two containing w and a vertex in $V(G) - \{u, v, w\}$. So $G - e - S$ is not $C4$. This contradicts Lemma 5.1.

Case 2: $uv, vw \in E(G), uw \notin E(G)$.

Then $|\overline{N[u]} \cap \overline{N[v]}| \leq 1$ and $|N(u) \cap N(v)| \geq n - 6$. By Proposition 5.2, $G - e - S$ would be $C1$.

If $G - e - S$ is $C1$, then $|N(u) \cap N(v)| = n - 6$ and $|\overline{N[u]} \cap \overline{N[v]}| = 0$. Hence $S = N(u) \cap N(v)$. Moreover, w belongs to the odd component of $G - e - S$ containing v. But w is adjacent to at least one vertex in $N(u) \backslash N[v]$ as $d_G(w) = n - 3$. Thus $G - e - S$ is connected which can not be $C1$. This is a contradiction to Lemma 5.1.

Case 3: $uv, uw, vw \in E(G)$.

Then $|\overline{N[u]} \cap \overline{N[v]}| \leq 2$ and $|N(u) \cap N(v)| \leq n - 4$. We discuss the three subcases.

Subcase 3.1: $|\overline{N[u]} \cap \overline{N[v]}| = 2$.

Let $\overline{N[u]} \cap \overline{N[v]} = \{x, y\}$. Obviously, $|N(u) \cap N(v)| = n - 4$. By Proposition 5.2, $G - e - S$ would be $C4$.

If $G - e - S$ is $C4$, then $\{x, y\}$ is an independent set of $C4$. Thus $xy \notin E(G)$. If $wx \in E(G)$, $wy \notin E(G)$ or $wx \notin E(G)$, $wy \in E(G)$, we consider edge uw or vw the same as Subcase 3.2. If $wx, wy \in E(G)$, we consider edge uw or vw the same as Subcase 3.3. Assume that $wx, wy \notin E(G)$. Since $\delta(G) = n - 5$, x and y are adjacent to every vertex in $V(G)\backslash\{u, v, w, x, y\}$. Let $S' \subseteq V(G)\backslash\{u, v, w, x, y\}$ with $|S'| = n-6$. Then $G-S'$ has no 1-factor, contradicting that G is $(n-6)$-factor-critical.

Subcase 3.2: $|\overline{N[u]} \cap \overline{N[v]}| = 1$.

Let $\overline{N[u]} \cap \overline{N[v]} = \{x\}$. Clearly, $|N(u) \cap N(v)| = n - 5$. By Proposition 5.2, $G - e - S$ can not be Configuration $C1$, $C2$, $C3$ or $C4$, which contradicts Lemma 5.1.

Subcase 3.3: $|\overline{N[u]} \cap \overline{N[v]}| = 0$.

Clearly, $|N(u) \cap N(v)| = n-6$. Next, we consider the cardinalities of $\overline{N[u]} \cap \overline{N[w]}$ and $\overline{N[v]} \cap \overline{N[w]}$.

Subcase 3.3.1: $|\overline{N[u]} \cap \overline{N[w]}| = 1$ or $|\overline{N[v]} \cap \overline{N[w]}| = 1$.

We consider edge uw or vw the same as Subcase 3.2.

Subcase 3.3.2: $|\overline{N[u]} \cap \overline{N[w]}| = 2$ or $|\overline{N[v]} \cap \overline{N[w]}| = 2$.

Without loss of generality, assume that $|\overline{N[u]} \cap \overline{N[w]}| = 2$. Then $|\overline{N[v]} \cap \overline{N[w]}| = 0$ and $|N(u) \cap N(w)| = n-4$. Let $\overline{N[u]} \cap \overline{N[w]} = \{x, y\}$. By Proposition 5.2, $G - e - S$ would be $C1$. Then $S = N(u) \cap N(v)$ and $G[\{v, x, y\}]$ is an odd component of $C1$. Since $G - S$ has a 1-factor, $xy \in E(G)$. However, by Proposition 5.2, $G - uw - S'$ would be $C4$ for some $S' \subseteq V(G) - \{u, w\}$ with $|S'| = n - 6$. Hence $\{x, y\}$ is an independent set of G. So $xy \notin E(G)$, a contradiction.

Subcase 3.3.3: $|\overline{N[u]} \cap \overline{N[w]}| = 0$ and $|\overline{N[v]} \cap \overline{N[w]}| = 0$.

By Proposition 5.2, $G - e - S$ would be $C1$. Hence $S = N(u) \cap N(v)$.

Let $S_1' = N(u)\backslash N[v]$, $S_2' = N(v)\backslash N[u]$, $S_3' = N(u)\backslash N[w]$ and $S_0 = N(u) \cap N(v) \cap N(w)$. Then $|S_1'| = |S_2'| = |S_3'| = 2$ and $|S_0| = n - |S_1'| - |S_2'| - |S_3'| - 3 = n-9$. If $G - e - S$ is $C1$, then $S = N(u) \cap N(v) = S_3' \cup S_0 \cup \{w\}$ and there are no edge joining S_1' and S_2'. Let $e_1 = uw$, $S_1 \subseteq V(G) - \{u, w\}$ with $|S_1| = n-6$ and $e_2 = vw$, $S_2 \subseteq V(G) - \{v, w\}$ with $|S_2| = n - 6$. By Proposition 5.2, both $G - e_1 - S_1$ and $G - e_2 - S_2$ are also $C1$. Then $S_1 = N(u) \cap N(w) = S_1' \cup S_0 \cup \{v\}$ and there are no edge joining S_2' and S_3'. Moreover, $S_2 = N(v) \cap N(w) = S_2' \cup S_0 \cup \{u\}$ and there are no edge joining S_1' and S_3'. Therefore, there are no edge joining S_1', S_2' and S_3' each other. Since $\delta(G) = n - 5$ and $G[S_1' \cup S_2' \cup S_3']$ has a 1-factor, every vertex in S_i' is adjacent to at least $n - 8$ vertices in S_0 for $i = 1, 2, 3$. But $|S_0| = n - 9 < n - 8$, a contradiction.

Therefore, if $d_G(u) = d_G(v) = d_G(w) = n - 3$, then $uv, uw, vw \notin E(G)$.

Now suppose that G has two vertices x, y with degree $n - 4$. Then every vertex in $\{u, v, w\}$ is adjacent to x and y. Let $e'' = ux \in E(G)$ and $S'' \subseteq V(G) - \{u, x\}$ with $|S''| = n - 6$ satisfying Lemma 5.1. Then $|\overline{N[u]} \cap \overline{N[x]}| = 0$ and $|N(u) \cap N(x)| = n - 7$. By Proposition 5.2, $G - e'' - S''$ would be $C1$. Thus $G[\{x, v, w\}]$ must be an odd component of $C1$. Since $vw \notin E(G)$, $G - S''$ has no 1-factor, a contradiction. So $G - e'' - S''$ is not $C1$. This contradicts Lemma 5.1. Therefore all vertices of $G - \{u, v, w\}$ have degree $n - 5$. □

Theorem 5.5 *Let G be a minimally $(n-6)$-factor-critical graph of order $n \geq 11$. If $\Delta(G) = n - 4$, then G has at most four vertices with degree $n - 4$ and the other vertices of G have degree $n - 5$.*

Proof Since G has an even number of vertices with degree $n - 4$, suppose that G has six vertices with degree $n-4$, say $\{u, v, w, x, y, z\}$. Let $S_0 = \{u, v, w, x, y, z\}$. Every vertex in S_0 is adjacent to at most $n-6$ vertices in $V(G) - S_0$. Then $\delta(G[S_0]) \geq 2$. So $G[S_0]$ must contain a cycle. Moreover, $G[S_0]$ has a 1-factor. Thus we can always find

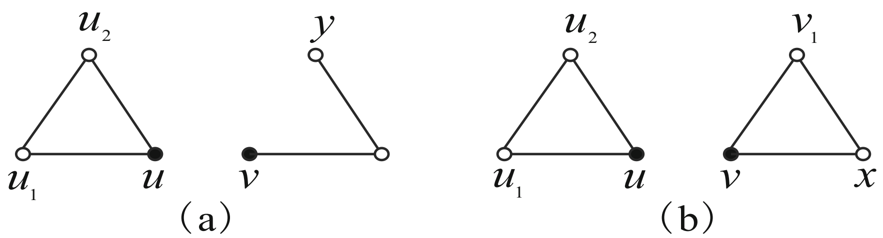

Fig. 6 The two configurations of $C1$

a path P with length three in $G[S_0]$. To prove the theorem, we need only to show that G does not contain such a path P with length three, in which each vertex has degree $n-4$.

Suppose to the contrary that G contains such a path $P = uvxy$. Let $e = uv \in E(G)$. By Lemma 5.1, there exists $S \subseteq V(G) - \{u, v\}$ with $|S| = n-6$ such that $G - e - S$ is one of Configuration $C1$, $C2$, $C3$ and $C4$. We discuss the two cases depending on $|\overline{N[u]} \cap \overline{N[v]}|$.

Case 1: $|\overline{N[u]} \cap \overline{N[v]}| = 1$.

Clearly, $\overline{N[u]} \cap \overline{N[v]} = \{y\}$ and $|N(u) \cap N(v)| = n-7$. Let $N(u) \backslash N[v] = \{u_1, u_2\}$ and $N(v) \backslash (N[u] \cup \{x\}) = \{v_1\}$. By Proposition 5.2, $G - e - S$ would be $C1$ or $C3$.

Subcase 1.1: $G - e - S$ is $C1$.

Then $N(u) \cap N(v) \subseteq S$ and $C1$ must be Configuration (a) or (b) (see Fig. 6).

Suppose $C1$ is (a). Since $d_G(y) = n-4$, yu_1 or $yu_2 \in E(G)$, a contradiction.

Suppose $C1$ is (b). Then $S = N(u) \cap N(v) \cup \{y\}$. Hence x is adjacent to every vertex in $V(G) - \{u, u_1, u_2\}$. Now consider edge $e' = vx$. Clearly, $\overline{N[v]} \cap \overline{N[x]} = \{u_1, u_2\}$ and $|N(v) \cap N(x)| = n-6$. By Proposition 5.2, there exists $S' \subseteq V(G) - \{v, x\}$ with $|S'| = n-6$ such that $G - e' - S'$ would be $C1$ or $C3$.

If $G - e' - S'$ is $C1$, then $S' = N(v) \cap N(x)$. Hence u_1 and u_2 belong respectively to two distinct odd components of $G - e' - S'$. But $u_1u_2 \in E(G)$, a contradiction.

If $G - e' - S'$ is $C3$, then v (resp. x) belongs to the trivial (resp. nontrivial) odd component of $C3 - \{a\}$. Otherwise, $\{y_1, y_2, y_3\} = \{u, u_1, u_2\}$, a contradiction. Since $\overline{N[v]} = \{y, u_1, u_2\}$ and $xy \in E(G)$, $y_1 \in \{u_1, u_2\}$, say $y_1 = u_1$. So $\{y_2, y_3\} = \{u_2, y\}$. Then $u_1u_2 = y_1y_2 \in E(G)$ or $u_1u_2 = y_1y_3 \in E(G)$, a contradiction.

Subcase 1.2: $G - e - S$ is $C3$.

Obviously, $y_1 = y$. If u belongs to the trivial odd component of $C3 - \{a\}$, then $G[\{v, x, v_1\}]$ is the nontrivial odd component of $C3 - \{a\}$. But $xy \in E(G)$, a contradiction. If v belongs to the trivial odd component of $C3 - \{a\}$, then $G[\{u, u_1, u_2\}]$ is the nontrivial odd component of $C3 - \{a\}$. But yu_1 or $yu_2 \in E(G)$ as $d_G(y) = n-4$, a contradiction.

Therefore, $G - e - S$ is not $C1$ or $C3$ when $|\overline{N[u]} \cap \overline{N[v]}| = 1$.

Case 2: $|\overline{N[u]} \cap \overline{N[v]}| = 2$.

Let $\overline{N[u]} \cap \overline{N[v]} = \{y, v_1\}$ and $N(u) \backslash N[v] = \{u_1\}$. Then $\overline{N[u]} = \{x, y, v_1\}$ and $\overline{N[v]} = \{u_1, y, v_1\}$. Obviously, $|N(u) \cap N(v)| = n - 6$. By Proposition 5.2, $G - e - S$ would be $C1$, $C3$ or $C4$.

Subcase 2.1: $G - e - S$ is $C1$.

Then $S = N(u) \cap N(v)$. So $G[\{v, x, y\}]$ and $G[\{u, u_1, v_1\}]$ are two odd components of $C1$. But yu_1 or $yv_1 \in E(G)$ as $d_G(y) = n - 4$, a contradiction.

Subcase 2.2: $G - e - S$ is $C3$.

Let $e' = vx \in E(G)$. By Lemma 5.1, there exists $S' \subseteq V(G) - \{v, x\}$ with $|S'| = n - 6$ such that $G - e' - S'$ is one of Configuration $C1$, $C2$, $C3$ and $C4$.

Subcase 2.2.1: u (resp. v) belongs to the trivial (resp. nontrivial) odd component of $C3 - \{a\}$.

Then $\{y_1, y_2, y_3\} = \{x, y, v_1\}$. Since $xy \in E(G)$, $y_1 = v_1$. Hence xv_1, $yv_1 \notin E(G)$. So $u_1v_1 \in E(G)$ as $d_G(v_1) \geq n - 5$ and $yu_1 \in E(G)$ as $d_G(y) = n - 4$.

(2.2.1.1) $|\overline{N[v]} \cap \overline{N[x]}| = 2$.

Then $\overline{N[v]} \cap \overline{N[x]} = \{u_1, v_1\}$ and $|N(v) \cap N(x)| = n - 6$. By Proposition 5.2, $G - e' - S'$ would be $C1$, $C3$ or $C4$.

If $G - e' - S'$ is $C1$, then $S' = N(v) \cap N(x)$. Hence u_1 and v_1 belong respectively to two distinct odd components of $C1$. But $u_1v_1 \in E(G)$. Then $G - e' - S'$ is connected, which is a contradiction. Moreover, $\overline{N[v]} = \{y, u_1, v_1\}$ and $\overline{N[x]} = \{u, u_1, v_1\}$. But $\{y_1, y_2, y_3\} \neq \{y, u_1, v_1\}$ and $\{y_1, y_2, y_3\} \neq \{u, u_1, v_1\}$. So $G - e' - S'$ is not $C3$. Since $\{u_1, v_1\}$ is not an independent set of G, $G - e' - S'$ is not $C4$.

(2.2.1.2) $|\overline{N[v]} \cap \overline{N[x]}| = 1$.

Then $\overline{N[v]} \cap \overline{N[x]} = \{v_1\}$ and $u_1x \in E(G)$. Let $\{w\} = (N(u) \cap N(v)) \backslash N(x)$. By Proposition 5.2, $G - e' - S'$ would be $C1$ or $C3$.

If $G - e' - S'$ is $C1$, then $S' = N(v) \cap N(x) \cup \{u_1\}$. $G[\{u, v, w\}]$ and $G[\{x, y, v_1\}]$ are two odd components of $C1$. But yw, $v_1w \in E(G)$. So $G - e' - S'$ is connected, a contradiction.

If $G - e' - S'$ is $C3$, then $y_1 = v_1$. Since $\overline{N[v]} = \{y, u_1, v_1\}$ and $u_1v_1 \in E(G)$, v does not belong to the trivial odd component of $C3 - \{a\}$. Moreover, $\overline{N[x]} = \{u, v_1, w\}$ and $v_1w \in E(G)$. Then x does not belong to the trivial odd component of $C3 - \{a\}$. So $G - e' - S'$ is not $C3$.

Subcase 2.2.2: v (resp. u) belongs to the trivial (resp. nontrivial) odd component of $C3 - \{a\}$.

Then $\{y_1, y_2, y_3\} = \{u_1, y, v_1\}$. Because $uu_1 \in E(G)$ and yu_1 or $yv_1 \in E(G)$, $y_1 = v_1$. Then u_1v_1, $yv_1 \notin E(G)$. So $yu_1 \in E(G)$ as $d_G(y) = n - 4$ and $xv_1 \in E(G)$ as $d_G(v_1) \geq n - 5$.

(2.2.2.1) $|\overline{N[v]} \cap \overline{N[x]}| = 0$.

Then $u_1x \in E(G)$. Let $\{w_1, w_2\} = (N(u) \cap N(v)) \backslash N(x)$. By Proposition 5.2, $G - e' - S'$ would be $C1$. However, the odd component of $C1$ containing v must contain w_1 or w_2 and the odd component of $C1$ containing x must contain y or v_1. But yw_1, yw_2, v_1w_1, $v_1w_2 \in E(G)$. In each case, $G - e' - S'$ is connected and can not be $C1$.

(2.2.2.2) $|\overline{N[v]} \cap \overline{N[x]}| = 1$.

Then $u_1x \notin E(G)$ and $\overline{N[v]} \cap \overline{N[x]} = \{u_1\}$. Let $\{w_1\} = (N(u) \cap N(v))\backslash N(x)$. By Proposition 5.2, $G - e' - S'$ would be $C1$ or $C3$.

If $G - e' - S'$ is $C1$, then the odd component of $C1$ containing x must contain y and the odd component of $C1$ containing v must contain u_1 or w_1. But $yu_1, yw_1 \in E(G)$, a contradiction.

If $G - e' - S'$ is $C3$, then $y_1 = u_1$. Since $\overline{N[v]} = \{y, u_1, v_1\}$ and $yu_1 \in E(G)$, v does not belong to the trivial odd component of $C3 - \{a\}$. Because $\overline{N[x]} = \{u, u_1, w_1\}$ and $uu_1 \in E(G)$, x does not belong to the trivial odd component of $C3 - \{a\}$. Then $G - e' - S'$ is not $C3$.

Thus, if $G - e - S$ is $C3$, then there is an edge $e' = vx$ such that $G - e' - S'$ is not one of Configuration $C1$, $C2$, $C3$ and $C4$, which contradicts Lemma 5.1.

Subcase 2.3: $G - e - S$ is $C4$.

Then $\{y, v_1\}$ is an independent set of G. So $yv_1 \notin E(G)$.

If $xv_1 \notin E(G)$, then y and v_1 are adjacent to every vertex in $V(G) - \{u, v, x, y, v_1\}$. We consider edge $e' = vx$ the same as Subcase 2.2.1.

If $xv_1 \in E(G)$, then $|\overline{N[v]} \cap \overline{N[x]}| \leq 1$ and y is adjacent to every vertex in $V(G) - \{u, v, y, v_1\}$. We consider edge $e' = vx$ with similar discussion in Subcase 2.2.2.

Therefore, $G - e - S$ is not $C1$, $C3$ or $C4$ when $|\overline{N[u]} \cap \overline{N[v]}| = 2$.

From the above discussion, for any $e = uv \in E(G)$, there exists no $S \subseteq V(G) - \{u, v\}$ with $|S| = n - 6$ such that $G - e - S$ is one of Configuration $C1$, $C2$, $C3$ and $C4$, which contradicts Lemma 5.1. Then G does not contain a path P with length three, in which each vertex has degree $n - 4$. □

Acknowledgements We would like to thank the referees for their numerous corrections and many useful suggestions. Their comments have helped us to improve the presentation of the paper.

Funding This work is supported by NSFC (Grant No. 12271229).

Data availability All data generated or analysed during this study are included in this article.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

1. Bruhn, H., Stein, M.: Minimal bricks have many vertices of small degree. Eur. J. Comb. **36**, 261–269 (2014)
2. de Carvalho, M.H., Lucchesi, C.L., Murty, U.S.R.: How to build a brick. Discrete Math. **306**, 2386–2410 (2006)
3. Favaron, O.: On k-factor-critical graphs. Discuss. Math. Graph Theory **16**, 41–51 (1996)
4. Favaron, O., Shi, M.: Minimally k-factor-critical graphs. Australas. J. Comb. **17**, 89–97 (1998)
5. Favaron, O., Flandrin, E., Ryjáček, Z.: Factor-criticality and matching extension in DCT-graphs. Discuss. Math. Graph Theory **17**, 271–278 (1997)
6. Gallai, T.: Neuer Beweis eines Tutte'schen Satzes, Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 135–139 (1963)
7. Lovász, L.: On the structure of factorizable graphs. Acta Math. Acad. Sci. Hungar. **23**, 179–195 (1972)
8. Lou, D.: On the structure of minimally n-extendable bipartite graphs. Discrete Math. **202**, 173–181 (1999)

9. Lou, D.: On matchability of graphs. Australas. J. Comb. **21**, 201–210 (2000)
10. Lou, D.: Some conditions for n-extendable graphs. Australas. J. Comb. **9**, 123–136 (1994)
11. Lovász, L., Plummer, M. D.: Matching theory. Ann. Discrete Math. **29** (1986)
12. Lou, D., Yu, Q.: Connectivity of k-extendable graphs with large k. Discrete Appl. Math. **136**, 55–61 (2004)
13. Lou, D., Yu, Q.: Sufficient conditions for n-matchable graphs. Australas. J. Comb. **29**, 127–133 (2004)
14. Lin, F., Zhang, L., Lu, F.: The cubic vertices of minimal bricks. J. Graph Theory **76**, 20–33 (2014)
15. Nishimura, T.: A closure concept in factor-critical graphs. Discrete Math. **259**, 319–324 (2002)
16. Norine, S., Thomas, R.: Minimal bricks. J. Comb. Theory Ser. B **96**, 505–513 (2006)
17. Plummer, M.D.: Degree sums, neighborhood unions and matching extension in graphs. In: Bodendiek, R. (ed.) Contemporary Methods in Graph Theory (B), pp. 489–502. I. Wissenschaftsverlag, Mannheim (1990)
18. Plummer, M.D., Saito, A.: Closure and factor-critical graphs. Discrete Math. **215**, 171–179 (2000)
19. Tutte, W.T.: The factorization of linear graphs. J. Lond. Math. Soc. **22**, 107–111 (1947)
20. Yu, Q.: Characterizations of various matching extensions in graphs. Australas. J. Comb. **7**, 55–64 (1993)
21. Yu, Q., Liu, G.: Graph Factors and Matching Extensions. Higher Education Press, Beijing (2009)
22. Zhang, Z., Wang, T., Lou, D.: Equivalence between extendibility and factor-criticality. Ars Comb. **85**, 279–285 (2007)
23. Zhai, S., Wei, E., Zhang, F.: The characterization of p-factor-critical graphs. Acta Math. Appl. Sin. (English Ser.) **38**, 154–158 (2022)