Graphs and Combinatorics (2023) 39:21
https://doi.org/10.1007/s00373-023-02623-1

ORIGINAL PAPER

Multicolor Ramsey Numbers of Bipartite Graphs and Large Books

Yan Li[1] · Yusheng Li[2] · Ye Wang[3]

Received: 17 September 2022 / Revised: 14 December 2022 / Accepted: 31 January 2023 /
Published online: 11 February 2023

Abstract
For graphs G and H, the Ramsey number $r_{k+1}(G; H)$ is defined as the minimum N such that any edge-coloring of K_N by $k+1$ colors contains either a monochromatic G in the first k colors or a monochromatic H in the last color. A book $B_n^{(m)}$ is a graph that consists of n copies of K_{m+1} sharing a common K_m. We shall give upper bounds for $r_{k+1}(K_{t,s}; B_n^{(m)})$ and $r_{k+1}(C_{2t}; B_n^{(m)})$, some of which are sharp up to the sub-linear term asymptotically.

Keywords Multicolor Ramsey number · Bipartite graph · Book · Even cycle

1 Introduction

For graphs $G_1, \ldots, G_k$, the Ramsey number $r_k(G_1, \ldots, G_k)$ is defined as the minimum N such that if edges of K_N are colored by k colors, then there is a monochromatic G_i in a color i with $1 \le i \le k$. We shall write $r_{k+1}(G, \ldots, G, G_{k+1})$ as $r_{k+1}(G; G_{k+1})$ in short, and write two color Ramsey number $r_2(G_1, G_2)$ as $r(G_1, G_2)$, and $r(G, G)$ as $r(G)$.

Call graph $B_n^{(m)}$ a book that consists of n copies of K_{m+1} that share a common K_m. As usual, we write $B_n^{(2)}$ as B_n. Book graph plays an important role in graph Ramsey theory. It was shown by Rousseau and Sheehan [10] that $r(B_n) = 4n+2$ for infinitely many n, and $r(B_m, K_n)$ was bounded from above by Li and Rousseau [6]. Moreover, Conlon [3] obtained $r(B_n^{(m)}) \sim 2^m n$ as $n \to \infty$.

✉ Ye Wang
ywang@hrbeu.edu.cn

1 College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

2 School of Mathematical Sciences, Tongji University, Shanghai 200092, China

3 College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China

For large n, complete bipartite graph $K_{m,n}$ and book $B_n^{(m)}$ seem to look like each other, and the Ramsey numbers involving them are known to be close in some cases. Chung and Graham [2] established

$$r_k(K_{m,n}) \le (n-1)(k + k^{1/m})^m$$

for $k \ge 2$, $n \ge m \ge 2$, and

$$r_k(K_{2,n}) \le (n-1)k^2 + k + 2.$$

Recently, Wang et al. obtained the following result. For positive functions $f(n)$ and $g(n)$, we write $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ as $n \to \infty$.

Lemma 1 [11] *Let integers $k \ge 1$ and $s \ge t \ge m \ge 1$. Then*

$$r_{k+1}(K_{t,s}; K_{m,n}) \le n + (1+o(1))(s-t+1)^{1/t} kmn^{1-1/t} \tag{1}$$

as $n \to \infty$. There are infinitely many n such that (1) becomes an equality for $r_{k+1}(K_{2,s}; K_{1,n})$, $r_{k+1}(K_{3,3}; K_{1,n})$, $r(K_{2,s}, K_{2,n})$ and $r(K_{3,3}, K_{m,n})$ for $m \le 3$.

Since $r_{k+1}(G; K_{m,n}) \le r_{k+1}(G; B_n^{(m)})$, we shall generalize Lemma 1 by replacing $K_{m,n}$ with $B_n^{(m)}$.

Theorem 1 *Let $s \ge t \ge 2$ and $k, m \ge 1$ be fixed integers. If n is large, then*

$$r_{k+1}(K_{t,s}; B_n^{(m)}) \le n + (1+o(1))(s-t+1)^{1/t} kmn^{1-1/t}. \tag{2}$$

Corollary 1 *Let $s \ge 2$ and $m \ge 1$ be fixed integers. Then, there are infinitely many n such that*

$$r(K_{2,s}, B_n^{(m)}) = n + (1+o(1))m\sqrt{(s-1)n},$$

and $r(K_{2,s}, K_{m,n}) = n + (1+o(1))m\sqrt{(s-1)n}$ as such $n \to \infty$.

Corollary 2 *Let $m \ge 1$ be an integer. Then, there are infinitely many n such that*

$$r(K_{3,3}, B_n^{(m)}) = n + (1+o(1))mn^{2/3},$$

and $r(K_{3,3}, K_{m,n}) = n + (1+o(1))mn^{2/3}$ as such $n \to \infty$.

Ramsey numbers involving cycles and large stars have attracted much attention. Parsons [9] obtained

$$r(C_4, K_{1,n}) \le n + \lceil \sqrt{n} \rceil + 1$$

for any $n \ge 2$, and the equality holds for infinitely many n, and if q is a prime power, then $r(C_4, K_{1,q^2}) = q^2 + q + 1$ and $r(C_4, K_{1,q^2+1}) = q^2 + q + 2$.

Zhang, Chen and Cheng [13] showed that

$$r_{k+1}(C_4; K_{1,n}) \le n + \lceil k\sqrt{n + (k^2+2k-3)/4}\rceil + \frac{k(k+1)}{2},$$

and $r_3(C_4; K_{1,n}) = n + \sqrt{4n+1} + 3$ for infinitely many n.

Liu and Li [8] determined $r(C_{2t+1}, B_n^{(m)}) = 2(m+n-1)+1$ for $t, m \ge 1$, and Lin and Peng [7] obtained $r(C_n, B_n^{(m)}) = (m+o(1))n$ as $m \ge 3$ and $n \to \infty$.

We shall investigate the behavior of $r_{k+1}(C_{2t}; B_n^{(m)})$ for large n as follows.

Theorem 2 *Let k, t and m be positive integers. If n is large, then*

$$r_{k+1}(C_{2t}; B_n^{(m)}) \le n + (1+o(1))c_t kmn^{1/t},$$

where $c_t > 0$ is a constant depends on t only. Furthermore, for each $t \in \{2, 3, 5\}$, there are infinitely many n such that

$$r_{k+1}(C_{2t}; B_n^{(m)})) \ge n + (1-o(1))ckn^{1/t}$$

for such n if n is large, where $c = c(t) > 0$ is a constant.

2 Proofs of Main Results

For a graph G, denote by $v(G)$ and $e(G)$ the numbers of vertices and edges of G, respectively. A graph G is said to be H-free if G contains no H as a subgraph. The Turán number $ex(n, H)$ of H is defined as the maximum $e(G)$ of an H-free graph G of order n.

A well known result of Kövari, Sós and Turán [5] tells us

$$ex(N, K_{t,s}) \le \frac{1}{2}\big[(s-1)^{1/t}N^{2-1/t} + (t-1)N\big] \ \ (s \ge t \ge 1).$$

Füredi [4] showed

$$ex(N, K_{t,s}) \le \frac{1}{2}\big[(s-t+1)^{1/t}N^{2-1/t} + tN + tN^{2-2/t}\big]$$

for $s \ge t \ge 1$. Thus we know that if $s \ge t \ge 2$, then

$$ex(N, K_{t,s}) \le (1+o(1))\frac{1}{2}(s-t+1)^{1/t}N^{2-1/t} \tag{3}$$

as $N \to \infty$. For even cycles C_{2t}, Bondy and Simonovits [1] proved for any $t \ge 2$,

$$ex(N, C_{2t}) \le c_t N^{1+1/t} \tag{4}$$

for large N, where $c_t > 0$.

We also need the following result from [11], for which we can replace the condition $ex(N, H) \sim cN^{2-\eta}$ with $ex(N, H) \geq cN^{2-\eta}$ from the proof as we need a lower bound for $r_{k+1}(H; K_{1,n})$ only. Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of graph G, respectively.

Lemma 2 [11] *Let H be a bipartite graph with $ex(N, H) \geq cN^{2-\eta}$ as $N \to \infty$, where c and η are positive constants. If there are extremal graphs G_N of order N for $ex(N, H)$ such that $\delta(G_N) \sim \Delta(G_N)$ as $N \to \infty$, then*

$$r_{k+1}(H; K_{1,n}) \geq n + (1-\epsilon)2kcn^{1-\eta}$$

for large n, where $\epsilon > 0$.

In following proofs, when we color the edges of K_N by $k+1$ colors, we shall write the monochromatic graph induced by edges in color i as G_i for $1 \leq i \leq k+1$. For a vertex v, we denote by $d_i(v)$ as the degree of v in the graph G_i, and thus $\sum_{i=1}^{k+1} d_i(v) = N - 1$ for any v.

We shall not distinguish $\lceil x \rceil$ and $\lfloor x \rfloor$ from x for large x as the differences are negligible for asymptotic computation.

Proof of Theorem 1 For any $\epsilon > 0$, let

$$\ell_m = (1+\epsilon)km(s-t+1)^{1/t} n^{1-1/t}$$

and $N_m = n + \ell_m$. We shall show

$$r_{k+1}(K_{t,s}; B_n^{(m)}) \leq N_m = n + \ell_m \tag{5}$$

for all large n by induction on m.

To simplify the proof, we shall start at $m = 0$ instead of $m = 1$, in which $K_{0,n}$ is admitted as $\overline{K}_n$ that consists of n vertices without any edge.

For the case $m = 0$, the claimed upper bound follows as any graph of order $N_0 = n$ contains $\overline{K}_n$ as a subgraph.

Next, we assume that (5) holds for any $m \geq 0$, and we shall show that it holds for $m + 1$.

Consider an edge coloring of $K_{N_{m+1}}$ by $k+1$ colors. We shall show that there is a $B_n^{(m+1)}$ in G_{k+1} or a $K_{t,s}$ in some G_i for $1 \leq i \leq k$.

If there is a vertex v such that $d_{k+1}(v) \geq r_{k+1}(K_{t,s}; B_n^{(m)})$, then the neighborhood of v in G_{k+1}, whose edges are colored by $k+1$ colors, contains a subgraph $B_n^{(m)}$ in color $k+1$, hence that and v form a subgraph $B_n^{(m+1)}$ of G_{k+1}, and thus we are done. So we may assume that for each vertex v,

$$d_{k+1}(v) \leq r_{k+1}(K_{t,s}; B_n^{(m)}) - 1 \leq n + \ell_m - 1,$$

in which the second inequality comes from induction hypothesis. So we have

$$e(G_{k+1}) = \frac{1}{2}\sum_v d_{k+1}(v) \leq \frac{1}{2}N_{m+1}(n + \ell_m - 1).$$

Hence

$$\sum_{i=1}^{k} e(G_i) = \binom{N_{m+1}}{2} - e(G_{k+1}) \geq \frac{N_{m+1}}{2}(N_{m+1} - n - \ell_m)$$
$$= \frac{N_{m+1}}{2}(\ell_{m+1} - \ell_m),$$

where

$$\ell_{m+1} - \ell_m = \Big[(1+\epsilon)(m+1) - (1+\epsilon)m\Big] k(s-t+1)^{1/t} n^{1-1/t}$$
$$= (1+\epsilon)k(s-t+1)^{1/t} n^{1-1/t}$$
$$\geq \big(1 + \frac{\epsilon}{2}\big)k(s-t+1)^{1/t} N_{m+1}^{1-1/t}$$

as $n \sim N_{m+1}$ for large n. Therefore, there is a G_j with $1 \leq j \leq k$ such that

$$e(G_j) \geq \frac{1}{k}\sum_{i=1}^{k} e(G_i) \geq \frac{1}{2}\big(1 + \frac{\epsilon}{2}\big)(s-t+1)^{1/t} N_{m+1}^{2-1/t} > ex(N_{m+1}, K_{t,s}),$$

where the last inequality comes from (3). Thus G_j contains a $K_{t,s}$, and then we get the desired upper bound. □

Proof of Corollary 1 Lemma 7 in [11] says that

$$r(K_{2,s}, K_{m,n}) \geq n + (1+o(1))m\sqrt{(s-1)n}$$

for any $s \geq 2$, $m \geq 1$, and infinitely many n, which and Theorem 1 for $k = 1$ and $t = 2$ imply the desired statements since $r(K_{2,s}; B_n^{(m)}) \geq r(K_{2,s}; K_{m,n})$. □

Proof of Corollary 2 Lemma 9 in [11] says that

$$r(K_{3,3}, K_{m,n}) \geq n + (1+o(1))mn^{2/3}$$

for any $m \geq 1$ and infinitely many n, which and Theorem 1 for $k = 1$ imply the desired statements since $r(K_{3,3}; B_n^{(m)}) \geq r(K_{3,3}; K_{m,n})$. □

Proof of Theorem 2 For any $\epsilon > 0$, let

$$\ell_m = (1+\epsilon)c_t kmn^{1/t}$$

and $N_m = n + \ell_m$, where $c_t > 0$ is the constant in (4). We shall show

$$r_{k+1}(C_{2t}; B_n^{(m)}) \leq N_m = n + \ell_m \tag{6}$$

for all large n. The proof is similar to that for Theorem 1 by induction on $m \geq 0$, and we go to the inductive step directly to show (6) for case $m+1$ by considering an edge coloring of $K_{N_{m+1}}$ with $k+1$ colors, in which G_{k+1} contains no $B_n^{(m)}$.

The similar analysis and induction hypothesis imply

$$\sum_{i=1}^{k} e(G_i) \geq \binom{N_{m+1}}{2} - e(G_{k+1}) \geq \frac{N_{m+1}}{2}(\ell_{m+1} - \ell_m),$$

where

$$\ell_{m+1} - \ell_m \geq \left(1 + \frac{\epsilon}{2}\right) c_t k N_{m+1}^{1/t}.$$

So some G_j with $1 \leq j \leq k$ has $e(G_j) > ex(N_{m+1}, C_{2t})$ and G_j contains C_{2t}.

Next, for $t \in \{2, 3, 5\}$, we shall show that there are infinitely many n such that

$$r_{k+1}(C_{2t}; B_n^{(m)}) \geq n + (1-\epsilon)ckn^{1/t}$$

for these n. To this end, the starting point is the result of Wenger [12] as

$$ex(n, C_{2t}) \geq cn^{1+1/t} = cn^{2-(t-1)/t},$$

where $c = c(t) > 0$ is a constant. Thus Lemma 2 implies

$$r_{k+1}(C_{2t}; K_{1,n}) \geq n + (1 - o(1))2ckn^{1-(t-1)/t} = n + (1 - o(1))2ckn^{1/t},$$

and it follows by $r_{k+1}(C_{2t}; B_n^{(m)}) \geq r_{k+1}(C_{2t}; K_{1,n})$ as required. □

Acknowledgements We are grateful to the editors and referees for invaluable suggestions and comments that improve the presentation of the paper greatly.

Funding This work is supported in part by NSFC (11871377,11931002,12101156)

Declarations

Conflict of interest The authors have not disclosed any competing interests.

References

1. Bondy, J., Simonovits, M.: Cycles of even length in graphs. J. Combin. Theory Ser. B **16**, 97–105 (1974)
2. Chung, F., Graham, R.L.: On multicolor Ramsey numbers for complete bipartite graphs. J. Combin. Theory Ser. B **18**, 164–169 (1975)
3. Conlon, D.: The Ramsey number of books. Adv. Combin. **3**, 12pp (2019)
4. Füredi, Z.: An upper bound on Zarankiewicz' problem. Combin. Probab. Comput. **5**, 29–33 (1996)
5. Kövari, T., Sós, V., Turán, P.: On a problem of K. Zarankiewicz. Colloq. Math. **3**, 50–57 (1954)
6. Li, Y., Rousseau, C.C.: On book-complete graph Ramsey numbers. J. Combin. Theory Ser. B **68**, 36–44 (1996)

7. Lin, Q., Peng, X.: Large book-cycle Ramsey numbers. SIAM J. Discrete Math. **35**, 532–545 (2021)
8. Liu, M., Li, Y.: Ramsey numbers of a fixed odd-cycle and generalized books and fans. Discrete Math. **339**, 2481–2489 (2016)
9. Parsons, T.: Ramsey graphs and block designs. I. Trans. Amer. Math. Soc. **209**, 33–44 (1975)
10. Rousseau, C.C., Sheehan, J.: On Ramsey numbers for books. J. Graph Theory **2**, 77–87 (1978)
11. Wang, Y., Li, Y., Li, Y.: Ramsey numbers of several $K_{t,s}$ and a large $K_{m,n}$. Discrete Math. **345**, 112987 (2022)
12. Wenger, R.: Extremal graphs with no C_4's, C_6's, or C_{10}'s. J. Combin. Theory Ser. B **52**, 113–116 (1991)
13. Zhang, X., Chen, Y., Cheng, T.: On three color Ramsey numbers $R(C_4, C_4, K_{1,n})$. Discrete Math. **342**, 285–291 (2019)