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ORIGINAL PAPER

Upper Total Domination in Claw-Free Cubic Graphs

Ammar Babikir[1] · Michael A. Henning[1]

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Abstract
A set S of vertices in a graph G is a total dominating set if every vertex of G is adjacent to some other vertex in S. A total dominating set S is minimal if no proper subset of S is a total dominating set of G. The upper total domination number, $\Gamma_t(G)$, of G is the maximum cardinality of a minimal total dominating set of G. A claw-free graph is a graph that does not contain a claw $K_{1,3}$ as an induced subgraph. It is known, or can be readily deduced, that if $G \neq K_4$ is a connected claw-free cubic graph of order n, then $\frac{1}{3}n \le \alpha(G) \le \frac{2}{5}n$, and $\frac{1}{3}n \le \Gamma(G) \le \frac{1}{2}n$, and these bounds are tight, where $\alpha(G)$ and $\Gamma(G)$ denote the independence number and upper domination number, respectively, of G. In this paper, we prove that if G is a connected claw-free cubic graph of order n, then $\frac{4}{9}n \le \Gamma_t(G) \le \frac{3}{5}n$.

Keywords Total domination · Upper total domination · Claw-free cubic graph

Mathematics Subject Classification 05C69

1 Introduction

An *isolate-free* graph is a graph that contains no isolated vertex, that is, every vertex has degree at least 1 in the graph. Let G be an isolate-free graph. A set S of vertices in G is a *total dominating set*, abbreviated TD-set, of G if every vertex in G is adjacent to some other vertex in S. A *minimal TD-set* in G is a TD-set that contains no TD-set of G as a proper subset. The *total domination number*, $\gamma_t(G)$, of G is the

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✉ Michael A. Henning
mahenning@uj.ac.za

Ammar Babikir
babikirammar@gmail.com

[1] Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa

minimum cardinality of a TD-set of G, while the *upper total domination number*, $\Gamma_t(G)$, of G is the maximum cardinality of a minimal TD-set in G. By definition, we have $\gamma_t(G) \le \Gamma_t(G)$. A TD-set of cardinality $\gamma_t(G)$ is called a *γ_t-set of G*, while a minimal TD-set of cardinality $\Gamma_t(G)$ is called a *Γ_t-set of G*. For recent books on domination and total domination in graphs, we refer the reader to [10–12, 16].

A graph is *F-free* if it does not contain F as an induced subgraph. In particular, if $F = K_{1,3}$, then the graph is *claw-free*, while if $F = K_4 - e$, then the graph is *diamond-free*. An excellent survey of claw-free graphs has been written by Flandrin et al. [7]. Chudnovsky and Seymour recently attracted considerable interest in claw-free graphs due to their excellent series of papers in *Journal of Combinatorial Theory* on this topic (see, for example, their paper [3]). A *cubic graph* (also called a 3-*regular graph*) is a graph in which every vertex has degree 3. Domination and total domination in claw-free cubic graphs has been extensively studied in the literature (see, for example, [5, 6, 8, 9, 13–15, 18, 19, 21, 22] and elsewhere). In this paper, we continue the study of total domination in claw-free cubic graphs. Let G be a connected, claw-free, cubic graph of order n. Since $\gamma_t(G) \ge n/\Delta(G)$ for every isolate-free graph, we observe that $\gamma_t(G) \ge \frac{1}{3}n$, and this bound is sharp. Recently, the authors [1, 2] proved that if we exclude four graphs, then $\gamma_t(G) \le \frac{3}{7}n$. In this paper we prove that $\frac{4}{9}n \le \Gamma_t(G) \le \frac{3}{5}n$.

1.1 Graph Theory Notation and Terminology

For notation and graph theory terminology, we in general follow [16]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. A *neighbor* of a vertex v in G is a vertex u that is adjacent to v, that is, $uv \in E(G)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. We denote the *degree* of v in G by $d_G(v) = |N_G(v)|$. As mentioned earlier, a graph is *isolate-free* if it does not contain an isolated vertex; that is, a vertex of degree 0. For disjoint subsets X and Y of vertices in G, we denote the set of edges between X and Y by $[X, Y]$.

For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. Thus, a set $S \subseteq V(G)$ is a TD-set of G if $N_G(S) = V(G)$. The *open S-private neighborhood* of v is defined by $\mathrm{pn}_G(v, S) = \{w \in V(G) : N_G(w) \cap S = \{v\}\}$. The *open S-external private neighborhood* of v is the set $\mathrm{epn}_G(v, S) = \mathrm{pn}_G(v, S) \setminus S$, while the *open S-internal private neighborhood* of v is defined by $\mathrm{ipn}_G(v, S) = \mathrm{pn}_G(v, S) \cap S$. We note that $\mathrm{pn}_G(v, S) = \mathrm{ipn}_G(v, S) \cup \mathrm{epn}_G(v, S)$. If the graph G is clear from the context, we omit writing it in the above expressions. For example, we simply write $\mathrm{epn}(v, S)$ and $\mathrm{ipn}(v, S)$ rather than $\mathrm{epn}_G(v, S)$ and $\mathrm{ipn}_G(v, S)$, respectively.

A fundamental property of minimal TD-sets was established by Cockayneet al. [4].

Lemma 1 ([4]) *A TD-set S in a graph G is a minimal TD-set in G if and only if for every vertex $v \in S$, $|\mathrm{epn}(v, S)| \ge 1$ or $|\mathrm{ipn}(v, S)| \ge 1$.*

For a set of vertices $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. We denote the subgraph obtained from G be deleting a set S of vertices and all edges incident with vertices in S by $G - S$. In particular, if $S = \{v\}$, then we simply denote $G - S$ by $G - v$ (rather than $G - \{v\}$). We denote the *path*, *cycle*, and *complete graph* on n vertices by P_n, C_n, and K_n, respectively, and we denote the *complete bipartite graph* with partite sets of cardinality n and m by $K_{n,m}$. A *triangle* in G is a subgraph isomorphic to K_3, whereas a *diamond* in G is an induced subgraph of G isomorphic to K_4 with one edge missing, denoted by $K_4 - e$. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \dots, k\}$.

1.2 A Triangle-Diamond Necklace

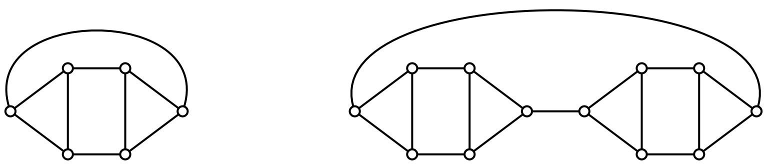
We define in this section what we have coined a *triangle-diamond-necklace*. A *triangle-necklace* was defined by the authors in [2] as follows.

Definition 1 ([2]) For $k \geq 1$ an integer, let F_{2k} be the connected cubic graph constructed as follows. Take $2k$ disjoint copies $T_1, T_2, \dots, T_{2k}$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2k]$. Let

$$\begin{aligned} E_a &= \{x_{2i-1}x_{2i} \,:\, i \in [k]\} \\ E_b &= \{y_{2i-1}y_{2i} \,:\, i \in [k]\} \\ E_c &= \{z_{2i}z_{2i+1} \,:\, i \in [k]\}, \end{aligned}$$

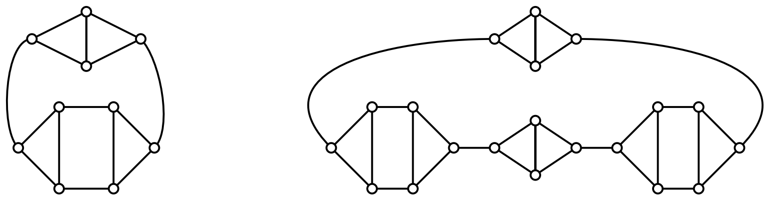
where addition is taken modulo $2k$ (and so, $z_1 = z_{2k+1}$). Let F_{2k} be obtained from the disjoint union of these $2k$ triangle by adding the edges $E_a \cup E_b \cup E_c$. We call the resulting graph F_{2k} a **triangle-necklace with** $2k$ **triangles**. Let $\mathcal{T}_{\text{cubic}} = \{F_{2k} \,:\, k \geq 1\}$. The triangle-necklaces F_2 and F_4 are shown in Fig. 1a, b, respectively.

Definition 2 For $k \geq 1$ an integer, let $D_1, \dots, D_k$ be k disjoint copies of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ where a_ib_i is the missing edge in D_i for $i \in [k]$. Adopting the notation in Definition 1, let G_{2k} be obtained from a triangle-necklace F_{2k} with $2k$ triangles by deleting the k edges $z_{2i}z_{2i+1}$ from F_{2k} for all $i \in [k]$, adding the k diamonds $D_1, \dots, D_k$, and adding the edges a_iz_{2i} and b_iz_{2i+1} for all $i \in [k]$ (where addition is taken modulo $2k$). We call the resulting graph G_{2k} a



(a) F_2 (b) F_4

Fig. 1 The triangle-necklaces F_2 and F_4



(a) G_2 (b) G_4

Fig. 2 The triangle-diamond-necklaces G_2 and G_4

triangle-diamond-necklace with k diamonds. Let $\mathcal{G}_{\text{cubic}} = \{G_{2k} : k \geq 1\}$. The triangle-diamond-necklace G_2 and G_4 are shown in Fig. 2a, b, respectively.

2 Known Results

Li and Virlouvet [17] established the following upper bound on the independence number of a claw-free graph with minimum degree at least δ.

Theorem 1 ([17]) *If G is a claw-free graph of order n, then $\alpha(G) \leq \frac{2n}{\delta(G)+2}$.*

Southey and Henning [20] established upper bounds on the upper domination number $\Gamma(G)$ and the upper total domination number $\Gamma_t(G)$ of a r-regular graph for all $r \geq 1$.

Theorem 2 ([20]) *For $r \geq 1$ if G is an r-regular graph of order n, then $\Gamma(G) \leq \frac{1}{2}n$ and $\Gamma_t(G) \leq \frac{n}{2-\frac{1}{r}}$.*

Both bounds in Theorem 2 are sharp, and the infinite families of graphs that achieve equality in these bounds are characterized in [20].

2.1 The Independence Number and Upper Domination Number

A lower bound on the independence number $\alpha(G)$ of a connected claw-free cubic graph follows from a more general result. For $k \geq 3$, let $G \neq K_{k+1}$ be a connected k-regular graph of order n. By Brooks Coloring Theorem, the chromatic number $\chi(G)$ of G is at most the maximum degree, namely k. Since the independence number $\alpha(G) \geq n/\chi(G)$ for all graphs G, this yields $\alpha(G) \geq n/k$. In the special case when $k = 3$, we have that if $G \neq K_4$ is a connected cubic graph of order n, then $\alpha(G) \geq n/3$. In particular, this yields the following trivial lower bound on the independence number of a claw-free cubic graph, which is certainly known, but we were unable to find a reference.

Observation 1 If $G \neq K_4$ is a connected claw-free graph of order n, then $\alpha(G) \geq \frac{1}{3}n$.

The bound of Observation 1 is tight, as may be seen, for example, by taking a connected claw-free graph that is diamond-free.

As a special case of Theorem 1, we have the following upper bound on the independence number of a connected claw-free cubic graph.

Theorem 3 ([17]) *If G is a connected claw-free cubic graph of order n, then $\alpha(G) \le \frac{2}{5}n$.*

We remark that the bound of Theorem 3 is tight. For example, suppose that $G \in \mathcal{G}_{\text{cubic}}$ is a triangle-diamond-necklace of order n. Thus, $G = G_{2k}$ for some $k \ge 1$, and so G has order $n = 10k$ and $\alpha(G) = 4k = \frac{2}{5}n$. The shaded vertices in Fig. 3a, b are examples of an α-set in the graphs G_2 and G_4, respectively.

Since $\Gamma(G) \ge \alpha(G)$ for all graphs G, the following lower bound on the upper domination number of a claw-free cubic graph follows from Observation 1.

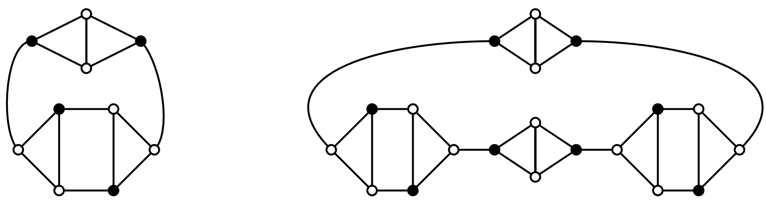
Observation 2 If $G \ne K_4$ is a connected claw-free graph of order n, then $\Gamma(G) \ge \frac{1}{3}n$.

As a consequence of the characterizations given in [20], we can readily deduce the extremal family of connected claw-free graphs with largest possible upper domination number. For $k \ge 1$, let $H_1 = kK_3$ and $H_2 = kK_3$ consist of k disjoint copies of K_3. Let H be the graph obtained from the disjoint union $H_1 \cup H_2$ by adding a perfect matching between $V(H_1)$ and $V(H_2)$ in such a way that the resulting graph H is connected. We note that H is a claw-free cubic graph of order $n = 6k$. Moreover, the set $V(H_1)$ is a minimal dominating set of H, implying that $\Gamma(H) \ge |V(H_1)| = \frac{1}{2}n$. By Theorem 2, $\Gamma(H) \le \frac{1}{2}n$. Consequently, $\Gamma(H) = \frac{1}{2}n$. Let $\mathcal{H}_{\text{cubic}}$ be the family of all such graphs H so constructed.

Theorem 4 ([20]) *If G is a connected claw-free cubic graph of order n, then $\Gamma(G) \le \frac{1}{2}n$, with equality if and only if $G \in \mathcal{H}_{\text{cubic}}$.*

Combining the above lower and upper bounds on the independence number and upper total domination number yields the following known result.

Theorem 5 *If $G \ne K_4$ is a connected claw-free cubic graph of order n, then the following properties hold.*



(a) G_2 (b) G_4

Fig. 3 Claw-free cubic graphs G of order n with $\alpha(G) = \frac{2}{5}n$

(a) $\frac{1}{3}n \le \alpha(G) \le \frac{2}{5}n$, *and*
(b) $\frac{1}{3}n \le \Gamma(G) \le \frac{1}{2}n$.

3 Main Result

Our aim in this paper is to provide lower and upper bounds on the upper total domination number of a claw-free connected cubic graph. We shall prove the following theorem.

Theorem 6 *If G is a connected claw-free cubic graph of order n, then $\frac{4}{9}n \le \Gamma_t(G) \le \frac{3}{5}n$.*

The upper bound in Theorem 6 follows from the more general result given in Theorem 2 that if G is a cubic graph of order n, then $\Gamma_t(G) \le \frac{3}{5}n$. We remark that this bound is tight, even for claw-free cubic graphs. For example, consider a triangle-diamond-necklace $G \in \mathcal{G}_{\rm cubic}$ of order n. Thus, $G = G_{2k}$ for some $k \ge 1$, and so $n = 10k$. Adopting the notation in Definition 2, let $C = \{c_1, \dots, c_k\}$, $D = \{d_1, \dots, d_k\}$, $X = \{x_{2i-1} : i \in [k]\}$, $Y = \{y_{2i} : i \in [k]\}$, and $Z = \{z_1, z_2, \dots, z_{2k}\}$. The set $C \cup D \cup X \cup Y \cup Z$ is a minimal TD-set of G_{2k}, and so $\Gamma_t(G_{2k}) \ge 6k = \frac{3}{5}n$. By Theorem 2, $\Gamma_t(G_{2k}) \le \frac{3}{5}n$. Consequently, $\Gamma_t(G_{2k}) = \frac{3}{5}n$. For example, the shaded vertices in Fig. 4a, b form a Γ_t-set of G_2 and G_3, respectively. We state this formally as follows.

Observation 3 If $G \in \mathcal{G}_{\rm cubic}$ has order n, then $\Gamma_t(G) = \frac{3}{5}n$.

4 Proof of Theorem 6

As remarked earlier, the tight upper bound in Theorem 6 follows from Theorem 2. In this section, we establish the lower bound in Theorem 6. In order to prove this

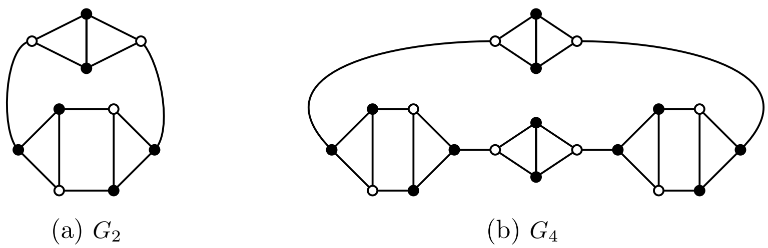


Fig. 4 Claw-free cubic graphs G of order n with $\Gamma_t(G) = \frac{3}{5}n$

lower bound, we need to prove a stronger result. For this purpose, we introduce the concept of a special subcubic graph.

Definition 3 We call a graph G a *special subcubic graph* if the following three properties hold: (i) G is connected, (ii) $\Delta(G) \leq 3$, and (iii) every vertex belongs to a triangle in G.

Every special subcubic graph has minimum degree at least 2, noting that every vertex belongs to a triangle. We note that possibly there are no vertices of degree 2, in which case the special subcubic graph is a claw-free connected cubic graph. Hence, the family of claw-free connected cubic graphs is a subfamily of the family of special subcubic graphs. An identical proof to that presented in [14] yields the following structural property of special subcubic graph.

Lemma 2 *If $G \neq K_4$ is a special subcubic graph, then the vertex set $V(G)$ can be uniquely partitioned into sets each of which induces a triangle or a diamond in G.*

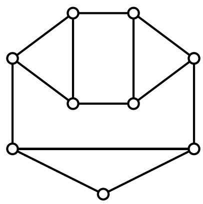
Adopting the notation in [14] used for claw-free cubic graphs, we refer to the unique partition given in Lemma 2 as a *triangle-diamond partition* of G, abbreviated Δ-D-partition. Further we call every triangle and diamond induced by a set in our Δ -D-partition a *unit* of the partition. A unit that is a triangle is called a *triangle-unit* and a unit that is a diamond is called a *diamond-unit*. (We note that a triangle-unit is a triangle that does not belong to a diamond.) We say that two units in the Δ-D-partition are *adjacent* if there is an edge joining a vertex in one unit to a vertex in the other unit. If two triangle-units are joined by two edges, then we call these triangle-units *double-bonded*. The special subcubic graph G_9, for example, shown in Fig. 5 has two-triangle units that are double-bonded.

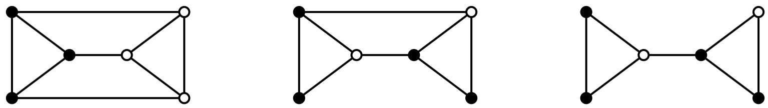
If $G \neq K_4$ is a special subcubic graph of order n with n_t triangle-units and n_d diamond-units, then since triangle-unit contributes 3 to the order and every diamond-unit contributes 4 to the order we note that $n = 3n_t + 4n_d$. We are now in a position to present our key result.

Theorem 7 *If G is a special subcubic graph of order n, then $\Gamma_t(G) \geq \frac{4}{9}n$.*

Proof We proceed by induction on the order $n \geq 3$ of the special subcubic graph. If $n = 3$, then $G = K_3$ and $\Gamma_t(G) = 2 = \frac{2}{3}n > \frac{4}{9}n$. If $n = 4$, then either $G = K_4$ or $G = K_4 - e$. In both cases, $\Gamma_t(G) = 2 = \frac{1}{2}n > \frac{4}{9}n$. Since there is no special subcubic

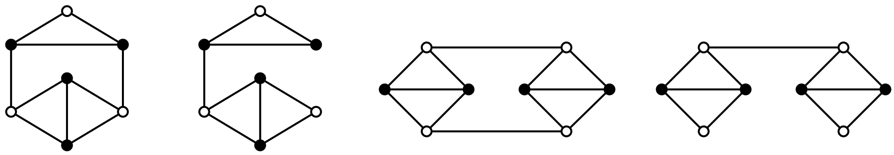
Fig. 5 The graph G_9





(a) $G_{6.1}$ (b) $G_{6.2}$ (c) $G_{6.3}$

Fig. 6 The three special subcubic graphs of order 6



(a) $G_{7.1}$ (b) $G_{7.2}$ (c) $G_{8.1}$ (d) $G_{8.2}$

Fig. 7 The four special subcubic graphs of order 7 and 8

graph on five vertices, $n \neq 5$. Suppose that $n = 6$. In this case, G is one of the three graphs $G_{6.1}$, $G_{6.2}$ and $G_{6.3}$ shown in Fig. 6a–c, respectively, where the shaded vertices are examples of a Γ_t-set in the respective graphs. If $G = G_{6.1}$, then $\Gamma_t(G) = 3$, and if $G = G_{6.2}$ or if $G = G_{6.3}$, then $\Gamma_t(G) = 4$. In all cases, $\Gamma_t(G) \geq 3 = \frac{1}{2}n > \frac{4}{9}n$.

If $n = 7$, then G is one of the two graphs $G_{7.1}$ and $G_{7.2}$ shown in Fig. 7a, b, respectively, where the shaded vertices are examples of a Γ_t-set in the respective graphs. In both cases, $\Gamma_t(G) = 4 = \frac{4}{7}n > \frac{4}{9}n$.

If $n = 8$, then G is one of the two graphs $G_{8.1}$ and $G_{8.2}$ shown in Fig. 7c, d, respectively, where the shaded vertices are examples of a Γ_t-set in the respective graphs. In both cases, $\Gamma_t(G) = 4 = \frac{1}{2}n > \frac{4}{9}n$.

If $n = 9$, then G consists of three triangle-units, with at least two additional edges between the triangle-units. Either $G = G_9$, in which case $\Gamma_t(G) = 4 = \frac{4}{9}n$, or G is obtained from G_9 by removing one or two edges, in which case $\Gamma_t(G) = 6 > \frac{4}{9}n$. This establishes the base cases when $3 \leq n \leq 9$. Let $n \geq 10$ and assume that if G' is a special subcubic graph of order n' where $n' < n$, then $\Gamma_t(G') \geq \frac{4}{9}n'$. We proceed further with a series of claims.

Claim 1 *If G contains a diamond-unit, then* $\Gamma_t(G) > \frac{4}{9}n$.

Proof Suppose that G contains a diamond-unit D. Let $V(D) = \{u_1, u_2, u_3, u_4\}$ where u_1u_2 is the missing edge in D. Since $n \geq 10$, at least one of u_1 and u_2 has degree 3 in G. Let $G' = G - V(D)$, and let G' have order n', and so $n' = n - 4$. We note that either G' is connected, in which case G' is a special subcubic graph, or G' has exactly two components, each of which is a special subcubic graph. Applying the inductive hypothesis to G' if G' is connected, or to the two components of G' if G'

is disconnected, we have by linearity that $\Gamma_t(G') \geq \frac{4}{9}n' = \frac{4}{9}(n-4)$. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices u_3 and u_4, implying that $\Gamma_t(G) \geq \Gamma_t(G') + 2 \geq \frac{4}{9}(n-4) + 2 > \frac{4}{9}n$. □

By Claim 1, we may assume that G contains no diamond-unit, that is, every unit in G is a triangle-unit.

Claim 2 *If G contains double-bonded triangle-units, then $\Gamma_t(G) \geq \frac{4}{9}n$.*

Proof Suppose that G contains two triangle-units T_1 and T_2, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2]$ and where x_1x_2 and y_1y_2 are edges in G. Thus, T_1 and T_2 form double-bonded triangle-units. Since $n \geq 10$, at least one of z_1 and z_2 has degree 3 in G. We may assume that $d_G(z_2) = 3$. Let z_3 be the neighbor of z_2 not in T_2. Let T_3 be the triangle-unit that contains z_3, and let $V(T_3) = \{x_3, y_3, z_3\}$. Let $G' = G - (V(T_1) \cup V(T_2) \cup V(T_3))$, and let G' have order n', and so $n' = n - 9$. We note that G' contains at most three components, and each component of G' is a special subcubic graph. Applying the inductive hypothesis to the components of G', we have by linearity that $\Gamma_t(G') \geq \frac{4}{9}n' = \frac{4}{9}(n-9) = \frac{4}{9}n - 4$. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices x_1, y_1, z_2 and z_3, implying that $\Gamma_t(G) \geq \Gamma_t(G') + 4 \geq \frac{4}{9}n$. □

By Claim 2, we may assume that G contains no double-bonded triangle-units. Thus, by our earlier assumptions, every unit in G is a triangle-unit and every two triangle-units are joined by at most one edge.

Claim 3 *If a triangle-unit of G contains two vertices of degree 2 in G, then $\Gamma_t(G) > \frac{4}{9}n$.*

Proof Suppose that G contains a triangle-unit T that contains two vertices of degree 2 in G. Let $V(T) = \{x, y, z\}$, where x and y have degree 2 in G. Let $G' = G - V(T)$, and let G' have order n', and so $n' = n - 3$. Applying the inductive hypothesis to the special subcubic graph G', we have $\Gamma_t(G') \geq \frac{4}{9}n' = \frac{4}{9}(n-3)$. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices x and y, implying that $\Gamma_t(G) \geq \Gamma_t(G') + 2 \geq \frac{4}{9}(n-3) + 2 > \frac{4}{9}n$. □

By Claim 3, we may assume that every triangle-unit of G contains at most one vertex of degree 2 in G.

Claim 4 *If G contains a vertex of degree 2, then $\Gamma_t(G) \geq \frac{4}{9}n$.*

Proof Suppose that G contains a vertex z_1 of degree 2 in G. Let T_1 be the triangle-unit in G that contains z_1, and let $V(T_1) = \{x_1, y_1, z_1\}$. By assumption, both x_1 and y_1 have degree 3 in G. Let x_2 be a neighbor of x_1 not in T_1. Let T_2 be the triangle-unit in G that contains x_2, and let $V(T_2) = \{x_2, y_2, z_2\}$. By assumption, at least one of y_2 and z_2 have degree 3 in G. Renaming vertices if necessary, we may assume that y_2 has degree 3 in G. Let y_3 be a neighbor of y_2 not in T_2. Let T_3 be the triangle-unit in G that contains y_3, and let $V(T_3) = \{x_3, y_3, z_3\}$. Since G contains no double-bonded triangle-units, the units T_1, T_2 and T_3 are distinct. Let $G' = G - (V(T_1) \cup V(T_2) \cup V(T_3))$, and let G' have order n', and so $n' = n - 9$. We note that G' contains at most four components, and each component of G' is a special subcubic graph. Applying the inductive hypothesis to the components of G', we have by linearity that $\Gamma_t(G') \ge \frac{4}{9}n' = \frac{4}{9}(n-9) = \frac{4}{9}n - 4$. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices x_1, y_2, y_3 and z_1, implying that $\Gamma_t(G) \ge \Gamma_t(G') + 4 \ge \frac{4}{9}n$. □

By Claim 4, we may assume that G is a cubic graph. By our earlier observations, every unit in G is a triangle-unit, and two triangle-units are joined by at most edge. We now construct a graph F from G as follows. For each triangle-unit in G, we associate a vertex of F. If two triangle-units in G are joined by an edge, then add an edge between the corresponding vertices in F. The graph F is called the *contraction graph* of G. We note that F is a cubic graph of order n_t, where recall that n_t denotes the number of triangle-units in G.

Claim 5 *If F is a bipartite graph, then* $\Gamma_t(G) \ge \frac{1}{2}n$.

Proof Suppose that F is a bipartite (cubic) graph. Thus, F has two partite sets X and Y, and these two sets have the same cardinality. Let S be the set of all vertices in G that belong to a triangle-unit associated with the set X, and let $\overline{S} = V(G) \setminus S$. We note that $|S| = |\overline{S}| = \frac{1}{2}n$, and $\overline{S}$ is the set of all vertices in G that belong to a triangle-unit associated with the set Y. Reconstructing the graph G from the contraction graph F, the set S is a dominating set of G. Moreover, every vertex in S is adjacent to a unique vertex in $\overline{S}$, and every vertex in $\overline{S}$ is adjacent to a unique vertex in S; that is, the set of edges $[S, \overline{S}]$ between S and $\overline{S}$ induce a perfect matching in G. In particular, $|\text{epn}(v, S)| = 1$ for every vertex $v \in S$. Since $G[S]$ consists of disjoint copies of K_3, the graph $G[S]$ is a 2-regular graph and is therefore isolate-free. Thus, S is a TD-set of G. As observed earlier, $|\text{epn}(v, S)| = 1$ for every vertex $v \in S$. Therefore by Lemma 1, the set S is a minimal TD-set of G. Hence, $\Gamma_t(G) \ge |S| = \frac{1}{2}n$. □

By Claim 5, we may assume that F is not a bipartite graph, that is, F contains an odd cycle. Let g_{odd} denote the odd girth of F, that is, g_{odd} is the length of a shortest odd cycle in F. We note that g_{odd} is an odd integer at least 3. Let C be a shortest odd cycle in F (of length g_{odd}), and let C be the cycle

$$C : v_1 v_2 \dots v_{g_{\text{odd}}} v_1.$$

By the odd girth condition, the cycle C is an induced cycle in F. Let T_i be the triangle-unit in G corresponding to the vertex v_i in F for $i \in [g_{\text{odd}}]$. Further, let $V(T_i) = \{x_i, y_i, z_i\}$ where $x_i y_{i+1}$ is an edge in G for all $i \in [g_{\text{odd}}]$, where addition is taken modulo g_{odd}, and so $x_{g_{\text{odd}}} y_1$ is an edge in G. Let

$$R = \bigcup_{i=1}^{g_{\rm odd}} V(T_i),$$

and so $|R| = 3g_{\rm odd}$. We consider three cases.

Case 1. $g_{\rm odd} \equiv 3 \,(\mathrm{mod}\, 6)$. In this case, we let $G' = G - R$. Let G' have order n', and so $n' = n - |R| = n - 3g_{\rm odd}$. We note that G' contains at most $g_{\rm odd}$ components, and each component of G' is a special subcubic graph. Applying the inductive hypothesis to the components of G', we have by linearity that $\Gamma_t(G') \ge \frac{4}{9}n' = \frac{4}{9}(n - 3g_{\rm odd}) = \frac{4}{9}n - \frac{4}{3}g_{\rm odd}$. Let

$$S = \bigcup_{i=1}^{\frac{1}{3}g_{\rm odd}} \{x_{3i-2}, x_{3i}, y_{3i-1}, y_{3i}\}.$$

We note that $|S| = \frac{4}{3}g_{\rm odd}$. In the special case when $g_{\rm odd} = 9$, the triangle-units that belong to the set R are illustrated in Fig. 8, and the vertices in the set S are given by the shaded vertices. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices in the set S, implying that

$$\Gamma_t(G) \ge \Gamma_t(G') + |S| \ge \left(\frac{4}{9}n - \frac{4}{3}g_{\rm odd}\right) + \frac{4}{3}g_{\rm odd} = \frac{4}{9}n.$$

Case 2. $g_{\rm odd} \equiv 5 \,(\mathrm{mod}\, 6)$. Let v_1' be the neighbor of v_1 in F that does not belong to the cycle C. Let T_1' be the triangle-unit in G corresponding to the vertex v_1' in F, and let $V(T_1') = \{x_1', y_1', z_1'\}$ where $z_1 z_1'$ is an edge in G. In this case, let $R' = R \cup V(T_1')$ and let $G' = G - R'$. Let G' have order n', and so $n' = n - |R'| = n - 3(g_{\rm odd} + 1)$. We note that G' contains at most $g_{\rm odd} + 1$ components, and each component of G' is a special subcubic graph. Applying the inductive hypothesis to the components of G', we have by linearity that $\Gamma_t(G') \ge \frac{4}{9}n' = \frac{4}{9}n - \frac{4}{3}(g_{\rm odd} + 1)$. Let

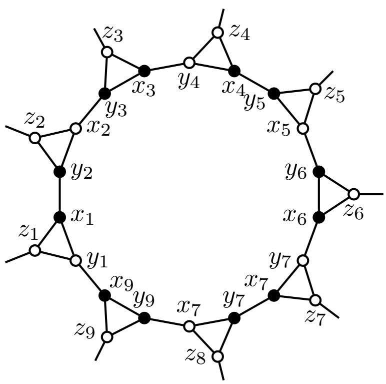


Fig. 8 Case 1 when $g_{\rm odd} = 9$

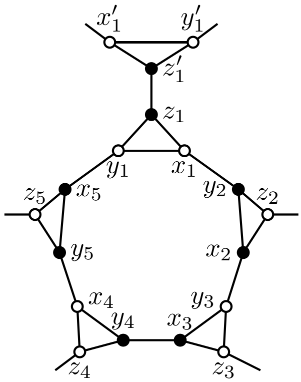


Fig. 9 Case 2 when $g_{\text{odd}} = 5$

$$S = \{z_1, z_1', x_{g_{\text{odd}}}, y_{g_{\text{odd}}}\} \cup \bigcup_{i=1}^{\frac{1}{3}(g_{\text{odd}}-2)} \{x_{3i-1}, x_{3i}, y_{3i-1}, y_{3i+1}\}.$$

We note that $|S| = \frac{4}{3}(g_{\text{odd}} - 2) + 4$. In the special case when $g_{\text{odd}} = 5$, the triangle-units that belong to the set R' are illustrated in Fig. 9, and the vertices in the set S are given by the shaded vertices. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices in the set S, implying that

$$\Gamma_t(G) \geq \Gamma_t(G') + |S| \geq \left(\frac{4}{9}n - \frac{4}{3}(g_{\text{odd}} + 1)\right) + \left(\frac{4}{3}(g_{\text{odd}} - 2) + 4\right) = \frac{4}{9}n.$$

Case 3. $g_{\text{odd}} \equiv 1 \pmod 6$. By the odd girth condition, there must exists two vertices at distance 2 apart on the cycle C that have no common neighbor in $V(F) \setminus V(C)$. Renaming the vertices of C, if necessary, we may assume that v_2 and $v_{g_{\text{odd}}}$ are two such vertices on the cycle. Let v_2' and $v_{g_{\text{odd}}}'$ be the neighbors of v_2 and $v_{g_{\text{odd}}}$, respectively, in F that do not belong to the cycle C. By assumption, $v_2' \neq v_{g_{\text{odd}}}'$. Let T_2' and $T_{g_{\text{odd}}}'$ be the triangle-units in G corresponding to the vertices v_2' and $v_{g_{\text{odd}}}'$ in F, and let $V(T_i') = \{x_i', y_i', z_i'\}$ where $z_i z_i'$ is an edge in G for $i \in \{2, g_{\text{odd}}\}$. In this case, let

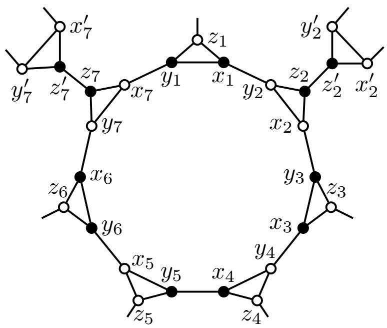


Fig. 10 Case 3 when $g_{\text{odd}} = 7$

$R' = R \cup V(T'_2) \cup V(T'_{g_{\rm odd}})$ and let $G' = G - R'$. Let G' have order n', and so $n' = n - |R'| = n - 3(g_{\rm odd} + 2)$. We note that G' contains at most $g_{\rm odd} + 2$ components, and each component of G' is a special subcubic graph. Applying the inductive hypothesis to the components of G', we have by linearity that $\Gamma_t(G') \ge \frac{4}{9}n' = \frac{4}{9}n - \frac{4}{3}(g_{\rm odd} + 2)$. Let

$$S = \{x_1, x_3, y_1, y_3, z_2, z'_2, z_{g_{\rm odd}}, z'_{g_{\rm odd}}\} \cup \bigcup_{i=1}^{\frac{1}{3}(g_{\rm odd}-4)} \{x_{3i+1}, x_{3i+3}, y_{3i+2}, y_{3i+3}\}.$$

We note that $|S| = \frac{4}{3}(g_{\rm odd} - 4) + 8$. In the special case when $g_{\rm odd} = 7$, the triangle-units that belong to the set R' are illustrated in Fig. 10, and the vertices in the set S are given by the shaded vertices. Every Γ_t-set of G' can be extended to a minimal TD-set of G by adding to it the vertices in the set S, implying that

$$\Gamma_t(G) \ge \Gamma_t(G') + |S| \ge \left(\frac{4}{9}n - \frac{4}{3}(g_{\rm odd} + 2)\right) + \left(\frac{4}{3}(g_{\rm odd} - 4) + 8\right) = \frac{4}{9}n.$$

In all three cases, we have $\Gamma_t(G) \ge \frac{4}{9}n$, which proves the desired lower bound. □

We remark that the lower bound in Theorem 7 is achieved, for example, by the special subcubic graph $G = G_9$ shown in Fig. 5b, noting that in this case $\Gamma_t(G) = 4 = \frac{4}{9}n$.

Recall the statement of the lower bound in Theorem 6: If G is a claw-free connected cubic graph of order n, then $\Gamma_t(G) \ge \frac{4}{9}n$. As observed earlier, every claw-free connected cubic graph is a special subcubic graph. Hence, the lower bound in Theorem 6 is an immediate consequence of Theorem 7.

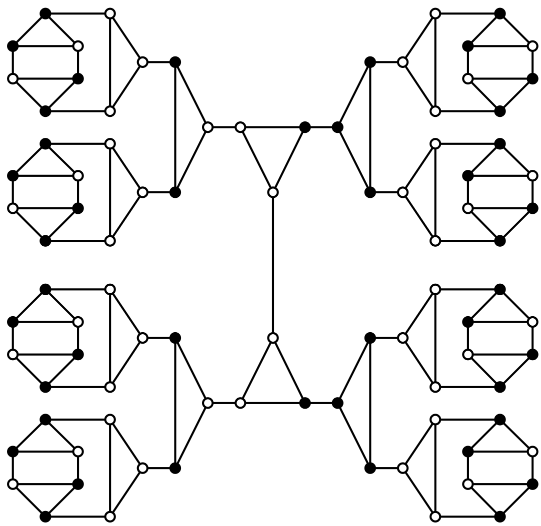


Fig. 11 A claw-free cubic graph G_{90} of order $n = 90$ with $\Gamma_t(G) = 44 = \frac{22}{45}n$

5 Concluding Remarks

We close with the following problem that we have yet to settle. Let $\mathcal{F}_{\text{cubic}}$ denote the family of all connected claw-free cubic graphs.

Problem 1 Determine or estimate the best possible constant c_{tdom} such that $\Gamma_t(G) \ge c_{\text{tdom}} \cdot n(G)$ for all $G \in \mathcal{F}_{\text{cubic}}$.

By Theorem 7, $c_{\text{tdom}} \ge \frac{4}{9}$. One can prove (or use a computer) that the claw-free cubic graph $G = G_{90}$ of order $n = 90$ shown in Fig. 11 satisfies $\Gamma_t(G) = 44 = \frac{22}{45}n$, where the shaded vertices are an example of a Γ_t-set of G. This yields the following lower and upper bounds on the constant c_{tdom}. It would be interesting to determine the exact value of c_{tdom}.

Theorem 8 $\frac{4}{9} \le c_{\text{tdom}} \le \frac{22}{45}$.

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Declarations

Conflict of interest We declare that there is no conflict of interests with our submission.

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