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ORIGINAL PAPER

Block-Transitive 3-Designs with Block Size At Most 6

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Abstract

In this paper we consider block-transitive automorphism groups of a 3-design with small block size. Let G be a block-transitive automorphism group of a nontrivial 3-(v, k, λ) design $\mathcal{D}$ with $k \leq 6$. Then one of the following occurs:

(i) if G is point-primitive then G is of affine or almost simple type;
(ii) if G is point-imprimitive then G has rank 3 or 4, and $\mathcal{D}$ is a 3-$(16, 6, \lambda)$ design with

$$\lambda \in \{4, 8, 12, 16, 24, 28, 32, 48, 56, 64, 80, 84, 96, 112, 128, 140, 160\}.$$

Keywords Block-transitive · Automorphism group · 3-design · Point-primitive · Point-imprimitive

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1 Introduction

Definition 1 For positive integers $t < k < v - 1$ and λ, a nontrivial t-(v, k, λ) design $\mathcal{D}$ is an incidence structure $(\mathcal{P}, \mathcal{B})$ satisfying the following properties:

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(i) $\mathcal{P}$ is a set of v elements, called *points*,
(ii) $\mathcal{B}$ is a set of b k-subsets of $\mathcal{P}$, called *blocks*,
(iii) Every t-subset of $\mathcal{P}$ is contained in exactly λ blocks.

Since all the blocks have the same size k, it follows that each point belongs to the same number of blocks and we denote that number by r. If no two blocks are identical, then we speak of a *simple* t-design. All of the t-designs in this paper will be simple and nontrivial. A *flag* of $\mathcal{D}$ is a point-block pair (α, B), that is, $\alpha \in \mathcal{P}$ and $B \in \mathcal{B}$ such that $\alpha \in B$.

An *automorphism* of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The *full automorphism group* of $\mathcal{D}$ consists of all automorphisms of $\mathcal{D}$ and is denoted by $\mathrm{Aut}(\mathcal{D})$. In the following, we will call a group $G \leq \mathrm{Aut}(\mathcal{D})$ *block-transitive* (respectively *flag-transitive*, *point-primitive*, *point-imprimitive*) if G acts transitively on the blocks (respectively transitively on the flags, primitively on the points, imprimitively on the points) of $\mathcal{D}$. For short, $\mathcal{D}$ is said to be, e.g., block-transitive if $\mathcal{D}$ admits a block-transitive group of automorphisms. A set of blocks of $\mathcal{D}$ is called a set of *base blocks* with respect to an automorphism group G of $\mathcal{D}$ if it contains exactly one block from each G-orbit on the block set. In particular, if G is a block-transitive automorphism group of $\mathcal{D}$, then any block B is a base block of $\mathcal{D}$.

Block-transitivity is just one of many conditions that can be imposed on the automorphism group G of a t-design $\mathcal{D}$. It is well known that if G is a block-transitive automorphism group of a t-design $\mathcal{D}$ with $t \geq 2$, then G is also transitive on points by Block's Lemma [3]. For a 2-(v, k, 1) design $\mathcal{D}$, by a result of Camina and Gagen [8], if G acts transitively on the blocks of $\mathcal{D}$ and if $k \mid v$, then G is point-primitive. For 2-(v, k, λ) designs, it is elementary that if $\mathcal{D}$ satisfies $v > (\frac{1}{2}k(k-1)-1)^2$ then the block-transitivity of G implies its point-primitivity [11]. Note also that, this implication remains true if $\mathcal{D}$ is flag-transitive and $\lambda > (r, \lambda)[(r, \lambda) - 1]$ (see [12, (2.3.7)]). For t-(v, k, λ) designs with $t \geq 4$, the block-transitivity of $G \leq \mathrm{Aut}(\mathcal{D})$ has an even stronger implication due to the following assertion by Cameron and Praeger [7, Theorem 2.1]:

Proposition 1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design with $t \geq 2$. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then G also acts $\lfloor t/2 \rfloor$-homogeneously on $\mathcal{P}$.*

The first step to classify the point-primitive designs is to give a reduction of primitive automorphism groups. The O'Nan-Scott Theorem partitions the finite primitive permutation groups into a number of types, and here we use the 5-type subvision introduced in [16], that is

(i) Affine type.
(ii) Almost simple type.
(iii) Product type.
(iv) Simple diagonal type.
(v) Twisted wreath product type.

In a beautiful classical work, Camina and Gagen [8] showed that if G acts as a block-transitive group of automorphisms of a 2-(v, k, 1) design and k divides v, then $G/S(G)$ (where $S(G)$ denotes the maximal soluble normal subgroup of a group G) has a simple socle. Inspired by the proof of [8, Theorem 2], several others [5, 10, 19] generalised the result in [8] to prove that groups acting flag-transitively on 2-(v, k, 1) designs are affine or almost simple. It is worth nothing that both Davies [10] and Zieschang [19] generalised the result to the situation of 2-designs with $(r, \lambda) = 1$. According to Proposition 1, if G acts block-transitively on a t-(v, k, λ) design $\mathcal{D}$ with $t \geq 4$ then G is either point-primitive of affine or almost simple type as G is 2-homogeneous on the points of $\mathcal{D}$. Therefore, it is necessary to study the block-transitive t-(v, k, λ) designs with $t \leq 3$.

The main aim of this paper is to study 3-(v, k, λ) designs admitting a block-transitive automorphism group G. Firstly, we analyse the case in which the automorphism group G is point-primitive, and we prove a reduction theorem for small values of k.

Theorem 1 *Let G be a block-transitive automorphism group of a nontrivial* 3-(v, k, λ) *design with $k \leq 6$. If G is point-primitive, then G is of affine type, or almost simple type.*

In fact, there exist many 3-designs admitting a block-transitive, point-primitive automorphism group of affine or almost simple type. Here are some examples (see [14]):

Example 1

(i) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ and $\mathcal{B}$ are the points and planes of the affine space $AG(d, 2)$ with $d \geq 3$. Then $\mathcal{D}$ is isomorphic to the 3-$(2^d, 4, 1)$ design admitting $G = AGL(d, 2)$ as a flag-transitive (block-transitive), point-primitive automorphism group of affine type.

(ii) Let $\mathcal{D}$ be the Mathieu-Witt 3-(22, 6, 1) design, and $M_{22} \trianglelefteq G \leq M_{22} : 2$. Then G is a flag-transitive (block-transitive), point-primitive automorphism group of $\mathcal{D}$ with almost simple action.

From Proposition 1, we know that the block-transitivity does not necessarily imply point-primitivity when $t = 2, 3$. For example, let $\mathcal{D}$ be a 2-design consisting of the points and hyperplanes of any Desarguesian projective space $PG(n, q)$ where $n \geq 2$ and $(q^{n+1} - 1)/(q - 1)$ is not a prime, and take G as the group generated by a Singer cycle (see [11]).

For the point-imprimitive case, Delandtsheer and Doyen have shown in [11] that if $\mathcal{D}$ is a t-(v, k, λ) design admitting a block-transitive, point-imprimitive automorphism group G then $v \leq (\binom{k}{2} - 1)^2$. Assume that G has a system of l classes of imprimitivity each of size c. In [6, Corollaries 3.2 and 3.4], it was shown that for a block-transitive, point-imprimitive 3-(v, k, λ) design with $l = 2$ or $c = 2$ then $v \leq \binom{k}{2} + 1$. Moreover, Mann and Tuan [17] gave a stronger result that any block-transitive, point-imprimitive 3-design satisfies $v \leq \binom{k}{2} + 1$. Thus for a fixed block size k, there are only finitely many t-(v, k, λ) designs with a block-transitive automorphism group which is point-imprimitive.

Secondly, the other purpose of this paper is to study 3-(v, k, λ) designs admitting a block-transitive, point-imprimitive automorphism group and prove the following theorem:

Theorem 2 *Let G be a block-transitive automorphism group of a nontrivial 3-(v, k, λ) design $\mathcal{D}$ with $k \leq 6$. If G is point-imprimitive then G has rank 3 or 4, and $\mathcal{D}$ is a 3-$(16, 6, \lambda)$ design, where λ is one of the following*:

$$\{4_6, 8_2, 12_{17}, 16_8, 24_5, 28_1, 32_4, 48_7, 56_1, 64_2, 80_1, 84_1, 96_4, 112_1, 128_1, 140_1, 160_1\}.$$

Remark 1 The notation $\lambda = a_n$ means that $\lambda = a$ and there are exactly n pairwise non-isomorphic such 3-$(16, 6, \lambda)$ designs. The base block and a corresponding automorphism group of each design are given in Table 5.

The paper is organized as follows. In Sect. 2, we introduce some preliminary results that are important for the remainder of the paper. In Sects. 3 and 4, we shall give the proofs of the Theorems 1 and 2 respectively.

2 Preliminaries

The notation and terminology used is standard and can be found in [9, 12] for design theory and in [13, 15, 18] for group theory. In particular, if G is a permutation group on point set $\mathcal{P}$, and $\alpha \in B \subseteq \mathcal{P}$, then G_α denotes the stabilizer of a point α in G, and G_B denotes the setwise stabilizer of B in G, and $G_{\alpha B}$ denotes the stabilizer of a flag (α, B) in G.

Lemma 1 [9, 1.2, 1.9] *The parameters v, b, r, k, λ of a 3-design satisfy the following conditions*:

(i) $vr = bk$.
(ii) $\lambda v(v-1)(v-2) = bk(k-1)(k-2)$.

From above elementary counting arguments, we derive furthermore arithmetic condition:

$$r = \frac{\lambda(v-1)(v-2)}{(k-1)(k-2)}. \tag{1}$$

The following useful lemma will be used throughout this paper.

Lemma 2 *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a nontrivial 3-(v,k,λ) design. Let $G\leq \mathrm{Aut}(\mathcal{D})$ be a block-transitive group, and let d be a nontrivial subdegree of G. Then the following statements hold*:

(i) r divides $k|G_\alpha|$;

(ii) r divides $k\lambda\binom{d}{2}$;

(iii) $(v-1)(v-2)$ divides $k(k-1)(k-2)\binom{d}{2}$.

Proof Let B be a block of $\mathcal{D}$ containing the point α. The point-transitivity and block-transitivity of G imply

$$|G:G_{\alpha B}|=|G:G_\alpha||G_\alpha:G_{\alpha B}|=v|G_\alpha:G_{\alpha B}|,$$

and

$$|G:G_{\alpha B}|=|G:G_B||G_B:G_{\alpha B}|=b|G_B:G_{\alpha B}|.$$

By Lemma 1(i) we have

$$|G_\alpha:G_{\alpha B}|=\frac{r|G_B:G_{\alpha B}|}{k} \tag{2}$$

and so part (i) holds.

Suppose that G_α has q orbits $\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_q$ on pencil $P(\alpha)$ (i.e. blocks containing a given point α). From Eq. (2) we have $r\mid k|\mathcal{O}_i|$ where $i\in\{1,2,\ldots,q\}$. Let $\Gamma\neq\{\alpha\}$ be a nontrivial G_α-orbit with $|\Gamma|=d$. Since G_α is transitive on $\mathcal{O}_i$, each block of $\mathcal{O}_i$ and Γ intersect in a constant number of points, say μ_i. That is, $\mu_i=|\Gamma\cap B_i|$ where $B_i\in\mathcal{O}_i$. Counting the size of set $\{(\{\beta,\gamma\},B)\mid\{\beta,\gamma\}\in B\cap\Gamma$ and $B\in P(\alpha)\}$ in two ways, we get

$$\sum_{i=1}^{q}|\mathcal{O}_i|\binom{\mu_i}{2}=\lambda\binom{d}{2}.$$

Combining this with $r\mid k|\mathcal{O}_i|$, we obtain that $r\mid k\sum_{i=1}^{q}|\mathcal{O}_i|\binom{\mu_i}{2}$. Thus r divides $k\lambda\binom{d}{2}$.

Part (iii) follows from part (ii) and Eq. (1). □

In the study of point-imprimitive case, the basis of our method is the following elementary result.

Lemma 3 [6, *Proposition* 1.1] *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a t-(v,k,λ) design, admitting a block-transitive automorphism group G. Let H be a permutation group with $G\leq H\leq S_v$, and $\mathcal{B}^*=\mathcal{B}^H$ the set of images of blocks in $\mathcal{B}$ under H. Then $(\mathcal{P},\mathcal{B}^*)$ is a t-(v,k,λ^*) design, for some λ^*, admitting the block-transitive automorphism group H.*

Lemma 4 [15, *Theorem* 1.6.2] *Let G be a transitive permutation group on $\mathcal{P}$. Let Σ be a nontrivial G-invariant partition of $\mathcal{P}$. Then $G \leq \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(\Sigma)$, where $\Delta \in \Sigma$, and the normal subgroup $G \cap \mathrm{Sym}(\Delta)^{|\Sigma|}$ of G fixes every block in Σ.*

3 Primitivity

The principal tool used in the proof is the O'Nan-Scott Theorem for finite primitive groups proved by Liebeck, Praeger and Saxl in [16]. We will prove Theorem 1 by dealing with the cases of product action, simple diagonal action and twisted wreath product action separately. In this section, suppose that we have a pair $(\mathcal{D}, G)$ satisfying the following hypothesis:

Hypothesis: Let $\mathcal{D}$ be a nontrivial 3-(v, k, λ) design with $k \leq 6$. Assume that G is a block-transitive, point-primitive automorphism group of $\mathcal{D}$.

3.1 Product Action

Here, we suppose that G has a product action on $\mathcal{P}$. Then $G \leq K^m \rtimes S_m = K \wr S_m$ with $m \geq 2$, where K is a primitive group (of almost simple or diagonal type) on Ω of size $v_0 \geq 5$, and $\mathcal{P} = \Omega^m$.

Proposition 2 *Let $(\mathcal{D}, G)$ satisfy Hypothesis. Then G is not of product action type.*

Proof Assume the contrary, suppose that $H \cong K \wr S_m$ with S_m acting on the set $M = \{1, 2, \ldots, m\}$. Let $\alpha = (\gamma, \gamma, \ldots, \gamma)$ and $\beta = (\delta, \gamma, \ldots, \gamma)$ (where δ is in the shortest orbit of K_γ) be two distinct points of $\mathcal{P}$, then $d = |\beta^{G_\alpha}|$ is a subdegree of G. Since G is a subgroup of H, it follows that

$$d = |G_\alpha : G_{\alpha\beta}| \leq |H_\alpha : H_{\alpha\beta}| < m\frac{v_0 - 1}{s - 1}.$$

where s is the rank of K on Ω, and hence $d \leq m\frac{v_0-1}{s-1}$. From Lemma 2(iii) we have

$$2(v-1)(v-2) \leq k(k-1)(k-2) \cdot m\frac{v_0 - 1}{s - 1} \cdot (m\frac{v_0 - 1}{s - 1} - 1).$$

Note that $v = v_0^m$ and $k \leq 6$, and so

$$(v_0^m - 1)(v_0^m - 2) \leq 60m^2v_0^2.$$

Thus $m = 2$ and $5 \leq v_0 \leq 15$. A simple calculation, we get all possible (v_0, m, s) as in Table 1.

First, assume that $(m, s) = (2, 2)$. Here $H = K \wr S_2$ and K acts 2-transitively on Ω.

Table 1 All possible values of v_0, m, s with $k \leq 6$

	$k = 4$	$k = 5$	$k = 6$
$(m, s) = (2, 2)$	$v_0 \in \{5\}$	$v_0 \in \{5, 6, \ldots, 9\}$	$v_0 \in \{5, 6, \ldots, 14\}$
$(m, s) = (2, 3)$	$\emptyset$	$\emptyset$	$v_0 \in \{5, 6\}$

Let $\alpha = (\gamma, \gamma)$ be a point of $\mathcal{P} = \Omega \times \Omega$. Then $H_\alpha = K_\gamma \wr S_2$ has two nontrivial suborbits on the point set $\mathcal{P}$ as follows:

(1) $\Gamma_1 = \{(\delta_1, \gamma) \mid \delta_1 \in \Omega \setminus \{\gamma\}\} \cup \{(\gamma, \delta_2) \mid \delta_2 \in \Omega \setminus \{\gamma\}\}$;
(2) $\Gamma_2 = \{(\delta_1, \delta_2) \mid \delta_1, \delta_2 \in \Omega \setminus \{\gamma\}\}$.

Thus, H_α has rank 3 with subdegrees $1, 2(v_0 - 1), (v_0 - 1)^2$ on $\mathcal{P}$. Note that $G \le H$, so each subdegree of H is the sum of some subdegrees of G. From Lemma 2(iii), we conclude that $(v_0^2 - 1)(v_0^2 - 2)$ divides $k(k-1)(k-2)(v_0 - 1)(2v_0 - 3)$, it is impossible for all pairs (v_0, k) listed in Table 1.

For the case $(m, s) = (2, 3)$, K is a primitive group with rank 3 on Ω. From [9, 9.62 Table], there is no such group K with a primitive action (of almost simple or diagonal type) and rank 3 on a set Ω of size $v_0 \in \{5, 6\}$. □

3.2 Simple Diagonal Action

Suppose that G is a primitive group of simple diagonal type. Then $M = \mathrm{Soc}(G) = T_1 \times \cdots \times T_m \cong T^m$ and $M_\alpha \cong T$ is a diagonal subgroup of M, where $T_i \cong T$ is a non-abelian finite simple group, for $i = 1, \ldots, m$ and $m \ge 2$. Here G_α is isomorphic to a subgroup of $\mathrm{Aut}(T) \times S_m$ and has an orbit Γ in $\mathcal{P} - \{\alpha\}$ with $|\Gamma| \le m|T|$ (see [16]). We state the following lemma that we will use in the proof of our Proposition 3.

Lemma 5 *There does not exist a non-abelian finite simple group T satisfying*

$$(|T| - 1)(|T| - 2) \le 480|\mathrm{Out}(T)|. \tag{3}$$

Proof If $T = A_5$, then $|T| = 60$ and $|\mathrm{Out}(T)| = 2$, it does not satisfy (3). Similarly, the group $T = A_6$ does not satisfy (3) as $|T| = 360$ and $|\mathrm{Out}(T)| = 4$. If there is a non-abelian finite simple group satisfying $(|T| - 1)(|T| - 2) \le 480|\mathrm{Out}(T)|$, then $|T| > 60$ (in fact $|T| \ge 168$) and $|\mathrm{Out}(T)| \ge 1$ and so

$$60|T| < (|T| - 1)(|T| - 2) \le 480|\mathrm{Out}(T)|,$$

which implies that $|T| < 8|\mathrm{Out}(T)| \le 8|\mathrm{Out}(T)|^3$. This is contrary to [1, Lemma 2.3]. □

Proposition 3 *Let $(\mathcal{D}, G)$ satisfy Hypothesis. Then G is not of simple diagonal type.*

Proof If G is of simple diagonal type, then $|\mathcal{P}| = |T|^{m-1}$ and G has a subdegree d less than $m|T|$. From Lemma 2(iii), we have

$$(|T|^{m-1} - 1)(|T|^{m-1} - 2) \le k(k-1)(k-2)\binom{m|T|}{2}.$$

It is easy to get $m = 2$ as $k \le 6$ and $|T| \ge 60$.

Also by Lemma 2(i), we have that r divides $k|G_\alpha|$, and so $k \cdot |\mathrm{Aut}(T)| \cdot |S_2|$ is divisible by r. Since $\mathrm{Out}(T) \cong \mathrm{Aut}(T)/\mathrm{Inn}(T)$ and $\mathrm{Inn}(T) \cong T/Z(T)$, it yields that r divides $2k \cdot |T| \cdot |\mathrm{Out}(T)|$ as T is a non-abelian simple group. Combining this with Lemma 1(ii), we get that $(|T|-1)(|T|-2)$ divides $2k(k-1)(k-2) \cdot |T| \cdot |\mathrm{Out}(T)|$. Then $(|T|, |T|-1) = 1$ and $(|T|, |T|-2) = 2$ imply

$$(|T|-1)(|T|-2) \leq 4k(k-1)(k-2) \cdot |\mathrm{Out}(T)| \leq 480|\mathrm{Out}(T)|.$$

This violates Lemma 5. □

3.3 Twisted Wreath Product Action

Next, we suppose that G is a primitive group of twisted wreath product type on $\mathcal{P}$. Let $\alpha \in \mathcal{P}$. Then $G \cong_Q B \rtimes P$, where $P = G_\alpha$ is a transitive permutation group on $\{1, \ldots, m\}$ with $m \geq 6$ (see [13, 4.8(ii)]), and ${}_QB = \mathrm{Soc}(G) = T_1 \times \cdots \times T_m \cong T^m$ is regular for some nonabelian simple group T. Thus, $v = |\mathcal{P}| = |T|^m$. Moreover, G_α has an orbit Γ with $|\Gamma| \leq m|T|$ (cf. [16]).

Proposition 4 *Let $(\mathcal{D}, G)$ satisfy* Hypothesis. *Then G is not of twisted wreath product type.*

Proof If G is of twisted wreath product type, then the argument here is similar to the proof of Proposition 3. By Lemma 2, we easily observe that

$$(|T|^m - 1)(|T|^m - 2) \leq k(k-1)(k-2)\binom{m|T|}{2}.$$

Then the inequalities $k \leq 6$ and $|T| \geq 60$ imply $m = 1$, this contradicts the fact that $m \geq 6$. □

Proof of Theorem 1 It follows from Propositions 2–4.

4 Imprimitivity

Suppose that G is an imprimitive group on the point set $\mathcal{P}$. Then $\mathcal{P}$ can be partitioned into l nontrivial classes of imprimitivity $\Delta_j, j = 1, \ldots, l$, each of size c, so $G \leq S_c \wr S_l$ by Lemma 4 and $v = |\mathcal{P}| = cl$ with $c > 1, l > 1$. Let B be a k-set of $\mathcal{P}$, and let $\mathcal{B}^* = B^{G^*}$ where $G^* = S_c \wr S_l$. Then the sizes of the intersections of each element of $\mathcal{B}^*$ with the imprimitivity classes determine a partition of k, say $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ with $x_1 \geq x_2 \geq \cdots \geq x_l$ and $\sum_{i=1}^{l} x_i = k$. Set $b_t = \sum_{i=1}^{l} x_i(x_i - 1) \cdots (x_i - t + 1)$. Note that $b_1 = k$. By [6, Proposition 2.2], the following lemma holds.

Lemma 6 *Let $\mathcal{D}^* = (\mathcal{P}, \mathcal{B}^*)$. Then*

(i) $\mathcal{D}^*$ is a 2-design if and only if

$$b_2 = \sum_{i=1}^{l} x_i(x_i - 1) = \frac{k(k-1)(c-1)}{(v-1)}.$$

(ii) $\mathcal{D}^*$ is a 3-design if and only if it is a 2-design and

$$b_3 = \sum_{i=1}^{l} x_i(x_i - 1)(x_i - 2) = \frac{k(k-1)(k-2)(c-1)(c-2)}{(v-1)(v-2)}.$$

We note here that if $\mathcal{D}^* = (\mathcal{P}, \mathcal{B}^*)$ is a 2-design (or a 3-design) then $\mathcal{D}^*$ is called a Cameron–Praeger design $\mathcal{D}(c, l; \mathbf{x})$.

Proposition 5 *Let $\mathcal{D}^* = (\mathcal{P}, \mathcal{B}^*)$ be a nontrivial 3-(v, k, λ^*) design with $k \leq 6$ admitting a block-transitive, point-imprimitive automorphism group G^*. Then $\mathcal{D}^*$ is one of the following:*

(i) a Cameron–Praeger design $\mathcal{D}(8, 2; (4, 2))$;
(ii) a Cameron–Praeger design $\mathcal{D}(2, 8; (2, 1, 1, 1, 1, 0, 0, 0))$.

Proof If $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ is a partition of k, then $x_1 \geq 2$ and $x_2 \geq 1$ as $\mathcal{D}^*$ is a 2-design and $c \geq 2, l \geq 2$.

Obvioulsy, $x_1 \leq c$. Moreover, if $c \geq 3$, every triple inside a class is contained in a block, thus every block contains such a triple and $x_1 \geq 3$. Thus the contraposite: if $x_1 = 2$ then $c = 2$. A similar argument using triples of points in 3 different classes leads to: if $x_3 = 0$ then $l = 2$. By using the equation of Lemma 6(i) and a simple calculation, we obtain all possible $\mathbf{x}$ and corresponding pairs (c, l) which are listed in Table 2.

All cases in Table 2 can be ruled out by the equation of Lemma 6(ii) and the nontriviality of $\mathcal{D}^*$ except the following two cases:

(1) $\mathbf{x} = (4, 2)$ and $(c, l) = (8, 2)$;
(2) $\mathbf{x} = (2, 1, 1, 1, 1, 0, 0, 0)$ and $(c, l) = (2, 8)$.

Thus, we have $v = cl = 16$ and $k = 6$.

If $\mathbf{x} = (4, 2)$ and $(c, l) = (8, 2)$ then we have that $\mathcal{D}^*$ is a Cameron–Praeger design $\mathcal{D}(8, 2; (4, 2))$ by Lemma 6(ii). Let Δ_1, Δ_2 be two classes of imprimitivity. Since every block of $\mathcal{D}^*$ is a 6-set of $\mathcal{P}$ with 4 points in one class of imprimitivity

Table 2 All possible $\mathbf{x}$ and corresponding pairs (c, l)

$\mathbf{x}$	(c, l)	$\mathbf{x}$	(c, l)
(3, 1, 1)	(7, 3)	(3, 3)	(3, 2)
(3, 2)	(3, 2)	(3, 1, 1, 1)	(4, 4)
(2, 2, 1)	(2, 3)	(2, 2, 2)	(2, 3)
(4, 2)	(8, 2)	(2, 1, 1, 1, 1, 0, 0, 0)	(2, 8)

and 2 points in the other class, also $G^* = S_8 \wr S_2$ acts transitively on $\{\Delta_1, \Delta_2\}$ and 8-transitively on Δ_i $(i = 1, 2)$. Let

$$\begin{aligned}\mathcal{B}_1 =& \{B \subset \mathcal{P} \mid |B \cap \Delta_1| = 4 \text{ and } |B \cap \Delta_2| = 2\};\\ \mathcal{B}_2 =& \{B \subset \mathcal{P} \mid |B \cap \Delta_2| = 4 \text{ and } |B \cap \Delta_1| = 2\}.\end{aligned}$$

Then $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2$ and so

$$|\mathcal{B}^*| = \binom{2}{1}\binom{8}{4}\binom{8}{2} = 3920.$$

We further obtain $\lambda^* = 140$.

If $\mathbf{x} = (2, 1, 1, 1, 1, 0, 0, 0)$ and $(c, l) = (2, 8)$ then $\mathcal{D}^*$ is a Cameron–Praeger design $\mathcal{D}(2, 8; (2, 1, 1, 1, 1, 0, 0, 0))$ by Lemma 6(ii). Let $\Delta_1, \Delta_2, \ldots, \Delta_8$ be eight classes of imprimitivity with size 2. Since $G^* = S_2 \wr S_8$ acts transitively on $\{\Delta_1, \Delta_2, \ldots, \Delta_8\}$ and also acts transitively on Δ_i $(i = 1, 2, \ldots, 8)$. Then

$$\begin{aligned}\mathcal{B}^* =& \{B \subset \mathcal{P} \mid |B \cap \Delta_{i_1}| = 2, |B \cap \Delta_{i_2}| = 1, |B \cap \Delta_{i_3}| = 1, |B \cap \Delta_{i_4}| = 1,\\ & |B \cap \Delta_{i_5}| = 1\},\end{aligned}$$

where i_1, i_2, i_3, i_4, i_5 are distinct numbers of $\{1, 2, \ldots, 8\}$. So

$$|\mathcal{B}^*| = \binom{8}{5}\binom{5}{1}\binom{2}{1}^4 = 4480,$$

and so $\lambda^* = 160$ by Lemma 1(ii). □

As a consequence, we have the following results:

Corollary 1 *Let $\mathcal{D}$ be a block-transitive, point-imprimitive nontrivial 3-(v, k, λ) design. Then $v = 16$ and $k = 6$.*

Proof It can be immediately obtained by Lemma 3.

In the following study, we make use of the software package Magma [4]. By using command `N:=TransitiveGroups(16)` we know that there are 1954 transitive groups on 16 points, exactly 22 of which are primitive. Here we only consider the remaining 1932 imprimitive groups. The command $G := N[i]$ returns the i-th transitive group in the list of the Magma-library of transitive groups with degree 16. By this command, we get the transitive permutation representations of G acting on the set $\mathcal{P} = \{1, 2, 3, \ldots, 16\}$.

Corollary 2 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 3-$(16, 6, \lambda)$ design admitting a block-transitive, point-imprimitive automorphism group G. Then one of the following holds:*

(i) $\operatorname{rank}(G) = 3$ with subdegrees $\{1, 7, 8\}$, and

$$\lambda \in \{4, 12, 16, 24, 28, 48, 56, 64, 84, 96, 112, 140\}.$$

(ii) $\operatorname{rank}(G) = 3$ with subdegrees $\{1, 1, 14\}$, and

$$\lambda \in \{4, 8, 12, 16, 24, 32, 64, 96, 128, 160\}.$$

(iii) $\mathrm{rank}(G) = 4$ with subdegrees $\{1, 1, 7, 7\}$, and $\lambda \in \{4, 8, 12, 16, 24, 48, 80\}$.

Proof Let Δ be a class of imprimitivity of G, and let $\alpha \in \Delta$. Clearly, $\Delta^{G_\alpha} = \Delta$, so $|\Delta|$ is sum of some subdegrees d of G. If d is a nontrivial subdegree of G, it follows from Lemma 2(ii) that 7 divides $\binom{d}{2}$ and then we easily observe that $d = 7, 8$ or 14. Thus the subdegrees of G is one of $\{1, 7, 8\}$, $\{1, 1, 7, 7\}$, $\{1, 1, 1, 1, 1, 1, 1, 1, 8\}$ and $\{1, 1, 14\}$. By using Magma we obtain all imprimitive subgroups with these subdegrees listed in Table 3.

Obviously, $\Delta_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\Delta_2 = \{9, 10, 11, 12, 13, 14, 15, 16\}$ are imprimitive parts of $S_8 \wr S_2 = N[1952]$, and $\Delta_1 = \{1, 2\}$, $\Delta_2 = \{3, 4\}$, $\Delta_3 = \{5, 6\}$, $\Delta_4 = \{7, 8\}$, $\Delta_5 = \{9, 10\}$, $\Delta_6 = \{11, 12\}$, $\Delta_7 = \{13, 14\}$, $\Delta_8 = \{15, 16\}$ are imprimitive parts of $S_2 \wr S_8 = N[1948]$. From Proposition 5 we can easily get that $B_1 = \{1, 2, 3, 4, 9, 10\}$ is a base block of the 3-(16, 6, 140) design, and $B_2 = \{1, 2, 3, 5, 7, 9\}$ is a base block of the 3-(16, 6, 160) design.

Let B be a base block of $\mathcal{D}$ then $B \in B_1^{S_8 \wr S_2} \cup B_2^{S_2 \wr S_8}$, also G is a group listed in Table 3. A simple calculation by using Magma-command `Design<3,16|B^G>`shows that the result holds. Moreover, for a given 3-$(16, 6, \lambda)$ design $\mathcal{D}$, we list all block-transitive point-imprimitive automorphism groups $G = N[i]$ in Table 4. □

Proof of Theorem 2 It follows from Proposition 5 and Corollaries 1 and 2.

By using command `IsIsomorphic(D1,D2)`, we give the following result:

Corollary 3 *Up to isomorphism, there are* 63 *different* 3-$(16, 6, \lambda)$ *designs* $\mathcal{D}$ *admitting a block-transitive automorphism group which is not primitive on points. The base block of* $\mathcal{D}$ *and a corresponding automorphism group* $G = N[i]$ *are listed in Table*5.

Remark 2 There are some descriptions of 3-$(16, 6, \lambda)$ designs listed in Table 5.

Table 3 All subgroups $N[i]$ with different subdegrees

Degrees	i
$\{1, 7, 8\}$	1075, 1078, 1501, 1502, 1505, 1693, 1798, 1799, 1801, 1802, 1860, 1861, 1882, 1883, 1903, 1940, 1949, 1950, 1951, 1952
$\{1, 1, 14\}$	1036, 1076, 1077, 1503, 1506, 1507, 1694, 1768, 1800, 1803, 1804, 1805, 1841, 1842, 1843, 1844, 1878, 1902, 1916, 1938, 1944, 1945, 1946, 1948
$\{1, 1, 7, 7\}$	196, 712, 713, 714, 715, 1035, 1504, 1838, 1839, 1873
$\{1, 1, 1, 1, 1, 1, 1, 1, 8\}$	211, 289, 325, 370, 388

Table 4 All automorphism groups $G = N[i]$ of a given 3-$(16, 6, \lambda)$ design

λ	i
4	196, 712, 713, 714, 715, 1075, 1078, 1501, 1502, 1505, 1507, 1802
8	1035, 1036
12	712, 713, 714, 715, 1036, 1501, 1502, 1505, 1507, 1801
16	1075, 1076, 1077, 1078, 1501, 1502, 1503, 1504, 1505, 1506, 1507, 1801, 1804
24	1035, 1036, 1802
28	1693, 1798, 1799, 1860, 1861, 1940
32	1076, 1077, 1503, 1506, 1507, 1694, 1768, 1800, 1803, 1805, 1841, 1844, 1902, 1916
48	1501, 1502, 1504, 1505, 1802
56	1882, 1883, 1903
64	1802, 1842, 1843, 1878
80	1838, 1839, 1873
84	1861, 1882, 1883, 1903
96	1503, 1506, 1507, 1800, 1801, 1803, 1841, 1842, 1843, 1844, 1878
112	1693, 1798, 1799, 1860, 1940
128	1804, 1805, 1902, 1916
140	1949, 1950, 1951, 1952
160	1938, 1944, 1945, 1946, 1948

(i) The number n listed in column Nr means that there are n pairwise non-isomorphic 3-$(16, 6, \lambda)$ designs admitting a block-transitive point-imprimitive automorphism group $G = N[i]$.

(ii) The 3-(16, 6, 4) design with base block $\{2, 6, 7, 8, 9, 10\}$ and automorphism group $G = N[1505]$ has the full group $2^4.A_7$ (which is 3-transitive). Its blocks are the minimal weight words from the Nordstrom-Robinson code (see [2, Proposition 3]). Another orbit of the same group on 6-sets gives one of designs with $\lambda = 12$ (this design also has the full group $2^4.A_7$). Now if we take the union of these two orbits we get one of our designs with $\lambda = 4 + 12 = 16$. Although $2^4.A_7$ is not block-transitive for this design, the full automorphism group $2^4.PSL(4, 2)$ is.

(iii) Note the other two orbits of $2^4.A_7$ on 6-sets yield a block-transitive 3-(16, 6, 30) design and a block-transitive 3-(16, 6, 240) design respectively, which are examples for the primitive affine case (but these do not admit a block-transitive imprimitive subgroup).

Table 5 Base block B and automorphism group G of 3-$(16, 6, \lambda)$ designs

λ	Nr	Base block B	G	λ	Nr	Base block B	G
4	6	$\{2, 10, 11, 14, 15, 16\}$	$N[714]$	24	5	$\{2, 3, 6, 7, 11, 15\}$	$N[1802]$
		$\{1, 6, 9, 11, 12, 14\}$	$N[1075]$			$\{4, 5, 7, 11, 12, 16\}$	$N[1035]$
		$\{3, 8, 11, 12, 15, 16\}$	$N[1075]$			$\{2, 4, 5, 6, 8, 10\}$	$N[1035]$
		$\{1, 4, 5, 8, 12, 15\}$	$N[1075]$			$\{1, 4, 8, 9, 13, 14\}$	$N[1035]$
		$\{2, 4, 9, 11, 12, 14\}$	$N[1078]$			$\{3, 7, 8, 9, 11, 15\}$	$N[1036]$
		$\{2, 6, 7, 8, 9, 10\}$	$N[1505]$	28	1	$\{2, 4, 9, 11, 12, 14\}$	$N[1693]$
8	2	$\{1, 7, 9, 10, 11, 16\}$	$N[1035]$	32	4	$\{1, 3, 6, 9, 13, 14\}$	$N[1076]$
		$\{1, 4, 5, 6, 11, 14\}$	$N[1036]$			$\{1, 2, 4, 6, 11, 15\}$	$N[1076]$
12	17	$\{2, 3, 6, 7, 11, 15\}$	$N[1501]$			$\{2, 6, 7, 8, 9, 11\}$	$N[1506]$
		$\{1, 2, 4, 7, 11, 15\}$	$N[1501]$			$\{4, 7, 8, 11, 13, 16\}$	$N[1507]$
		$\{2, 3, 6, 7, 11, 15\}$	$N[1502]$	48	7	$\{3, 6, 7, 8, 10, 12\}$	$N[1501]$
		$\{3, 8, 11, 12, 15, 16\}$	$N[1502]$			$\{1, 8, 10, 13, 14, 15\}$	$N[1501]$
		$\{1, 5, 6, 8, 9, 14\}$	$N[1505]$			$\{3, 6, 7, 8, 10, 12\}$	$N[1502]$
		$\{3, 8, 10, 11, 13, 14\}$	$N[712]$			$\{3, 8, 10, 11, 13, 14\}$	$N[1502]$
		$\{2, 6, 7, 8, 9, 11\}$	$N[712]$			$\{1, 2, 4, 6, 11, 15\}$	$N[1504]$
		$\{2, 8, 9, 10, 13, 15\}$	$N[712]$			$\{3, 8, 10, 11, 13, 14\}$	$N[1505]$
		$\{2, 3, 4, 8, 14, 16\}$	$N[712]$			$\{3, 6, 7, 8, 10, 12\}$	$N[1505]$
		$\{2, 6, 8, 11, 12, 14\}$	$N[713]$	56	1	$\{2, 4, 5, 7, 10, 11\}$	$N[1882]$
		$\{5, 7, 10, 12, 15, 16\}$	$N[713]$	64	2	$\{2, 4, 5, 7, 10, 11\}$	$N[1802]$
		$\{1, 5, 6, 8, 14, 15\}$	$N[713]$			$\{4, 5, 7, 11, 12, 16\}$	$N[1842]$
		$\{2, 6, 8, 11, 12, 14\}$	$N[714]$	80	1	$\{4, 5, 7, 11, 12, 16\}$	$N[1838]$
		$\{5, 7, 10, 12, 15, 16\}$	$N[714]$	84	1	$\{3, 6, 7, 8, 10, 12\}$	$N[1861]$
		$\{2, 8, 9, 10, 13, 15\}$	$N[714]$	96	4	$\{3, 6, 7, 8, 10, 12\}$	$N[1801]$
		$\{1, 5, 6, 8, 14, 15\}$	$N[715]$			$\{1, 3, 6, 9, 13, 14\}$	$N[1503]$
		$\{8, 9, 11, 13, 14, 16\}$	$N[715]$			$\{1, 3, 6, 9, 13, 14\}$	$N[1506]$
16	8	$\{3, 6, 7, 8, 10, 12\}$	$N[1075]$			$\{3, 6, 7, 8, 10, 12\}$	$N[1507]$
		$\{4, 5, 10, 11, 12, 14\}$	$N[1075]$	112	1	$\{3, 6, 7, 8, 10, 12\}$	$N[1693]$
		$\{5, 6, 9, 10, 14, 15\}$	$N[1075]$	128	1	$\{1, 3, 6, 9, 13, 14\}$	$N[1804]$
		$\{3, 6, 7, 8, 10, 12\}$	$N[1078]$	140	1	$\{1, 2, 3, 4, 13, 14\}$	$N[1952]$
		$\{1, 2, 5, 6, 11, 15\}$	$N[1505]$	160	1	$\{1, 2, 3, 5, 7, 9\}$	$N[1948]$
		$\{2, 7, 9, 10, 11, 14\}$	$N[1076]$				
		$\{4, 7, 9, 10, 11, 16\}$	$N[1076]$				
		$\{4, 7, 9, 10, 11, 16\}$	$N[1077]$				

(iv) Let $\mathcal{P}$ be two disjoint copies of the affine space $AG(3, 2)$ and $\mathcal{B}_1$ be all the 6-sets containing a pair of points from one copy and 4 points on a plane from the other copy. Then $(\mathcal{P}, \mathcal{B}_1)$ is isomorphic to the unique 3-(16, 6, 28) design. Let $\mathcal{B}_2$ be all the 6-sets containing a pair of points from one copy and 4 points not on a plane from the other copy. Then $(\mathcal{P}, \mathcal{B}_2)$ is isomorphic to the unique 3-(16, 6, 112) design. Their full group is $AGL(3, 2) \wr S_2$.

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Declarations

Conflict of interest There is no conflict of interest.

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