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A proof of a conjecture on the paired-domination subdivision number

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Abstract

A paired-dominating set of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number is the minimum cardinality of a paired-dominating set of G. The paired-domination subdivision number is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the paired-domination number. It was conjectured that the paired-domination subdivision number is at most $n-1$ for every connected graph G of order $n \geq 3$ which does not contain isolated vertices. In this paper, we settle the conjecture in the affirmative.

Keywords Paired-domination number · Paired-domination subdivision number

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1 Introduction

Throughout the paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). Let $|V(G)| = n$ denote the *order* of G. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. When no confusion arises, we will delete the subscript G in N_G and $\deg_G$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A support vertex is said to be *strong* if it is adjacent to at least two leaves. We call the *core* of a graph G, denoted by $C(G)$, the set of vertices of G which are neither leaves nor support vertices. For any two vertices u and v in G, the *distance* $d(u, v)$ equals the length (number of edges) of a shortest path between u and v in G. For any integer $k \geq 0$, let $N_k(v)$ denote the set of vertices of G that are at distance k from v. Clearly, $N_0(v) = \{v\}$ and $N_1(v) = N(v)$. The *eccentricity* $ecc(v)$ of a vertex v in a connected graph G is the distance between v and a vertex furthest from v. A *matching* in a graph G is a set of pairwise non-intersecting edges, while a *perfect matching* in G is a matching that covers each vertex.

A *dominating set* of G is a subset S of V such that every vertex in $V - S$ has at least one neighbor in S. A subset S of V is a *paired-dominating set* of G, abbreviated PDS, if S is a dominating set and the subgraph induced by the vertices of S contains a perfect matching. The *paired-domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a PDS of G. If S is a paired-dominating set with a perfect matching M, then two vertices u and v are said to be *partners (or paired)* in S if the edge $uv \in M$. We call a PDS of minimum cardinality a *$\gamma_{pr}(G)$-set*. Note that every graph G without isolated vertices has a PDS since the endvertices of any maximal matching in G form such a set. Paired-domination was introduced by Haynes and Slater [9] and is studied, for example, in [2, 3, 5, 8, 10–12].

The main focus of our paper is the study of the *paired-domination subdivision number* introduced by Favaron et al. in [6] and defined as follows. The paired-domination subdivision number $\mathrm{sd}_{\gamma_{pr}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired-domination number of G. Observe that since the paired-domination subdivision number of the graph K_2 remains unchanged when its only edge is subdivided, we will assume in our study that the graph G has order at least 3. The paired-domination subdivision number has been studied by several authors [1, 4, 7, 13, 14].

Favaron et al. [6] posed the following conjecture.

Conjecture 1 ([6]) *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq n - 1$.*

In connection with Conjecture 1, Egawa et al. [4] proved that for every connected graph G of order $n \geq 4$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2n - 5$. Moreover, if further G has an edge uv such that u and v are not partners in any $\gamma_{pr}(G)$-set, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq n - 1$.

In this paper, we settle Conjecture 1 in the affirmative. The proof will be given in Section 3. But first it is useful to recall some results and give other lemmas and notations that are necessary for our investigation.

Proposition 1 *[6] Let G be a connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdividing the edge e, then $\gamma_{pr}(G') \geq \gamma_{pr}(G)$.*

Proposition 2 *[6] For every graph G of order $n \geq 3$, if $\gamma_{pr}(G) = 2$, then* $1 \leq \mathrm{sd}_{\gamma_{pr}}(G) \leq 3$.

Proposition 3 *[6] If G contains either a strong support vertex or adjacent support vertices, then* $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2$.

2 Preliminaries

We begin by giving some definitions. Let $k \geq 1$ be an integer, G a connected graph and v a vertex of G with $ecc(v) \geq k$. For each $0 \leq j \leq k$, a jth *weak-clique set* of v is a subset $S \subseteq N_j(v)$ such that (i) for each $x \in S$, $N(x) \cap N_{j+1}(v) \neq \emptyset$, and (ii) for each pair of distinct vertices $x, y \in S$, $N(x) \cap N(y) \cap N_{j+1}(v) = \emptyset$. The jth *weak-clique number* of v, $WC_j(v)$, is the maximum cardinality of a jth weak-clique set of v. We note that $WC_0(v) = 1$ while if $WC_1(v) = 0$, then $V(G) = N[v]$. Moreover, if v is a leaf of a graph of order at least three, then $WC_1(v) = 1$.

For each $0 \leq j \leq k$, let $S^j = \{x_1^j, \ldots, x_{\ell_j}^j\}$ be a jth weak-clique set of a vertex v and let $N_j(v) - S^j = \{x_{\ell_j+1}^j, \ldots, x_{s_j}^j\}$ if $N_j(v) - S^j \neq \emptyset$. A jth *weak-separator* of v is the set

$$\{x_i^j u \mid 1 \leq i \leq \ell_j \text{ and } u \in N_{j+1}(v)\}$$

if $N_j(v) = S^j$ and the set

$$\{x_i^j u \mid 1 \leq i \leq \ell_j \text{ and } u \in N_{j+1}(v)\} \cup \left(\bigcup_{r=\ell_j+1}^{s_j} \{x_r^j u \mid u \in (N(x_r^j) \cap N_{j+1}(v)) \setminus (\cup_{i=1}^{r-1} N(x_i^j))\} \right),$$

otherwise. If $WS_j(v)$ is a jth weak-separator of v for each j, then clearly

$$n(G) \geq 1 + \sum_{j=0}^{k} |WS_j(v)|.$$

Now, we can state some lemmas.

Lemma 1 *Let G be a connected graph, and let $xy \in E(G)$. Let G' be a graph obtained from G by subdividing some edges. If there exists a minimum paired-dominating set D of G' with $x, y \in D$ and the partners of x and y are subdivision vertices, then $\gamma_{pr}(G) < \gamma_{pr}(G')$.*

Proof Let F be the set of edges subdivided to obtain G' from G, and let F_1 be the set of all edges in F whose subdivision vertices in G' are not in D. Let $x'$and y' be the subdivision vertices that are partners of x and y in D, respectively. In addition, suppose that x' and y' are resulting from the subdivision of the edges e_1, e_2, respectively. If G_1 denotes the graph obtained from G by subdividing all edges in $F - (F_1 \cup \{e_1, e_2\})$, then the set $D \setminus \{x', y'\}$ in which x and y as partners is a PDS of G_1, and therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G_1) < \gamma_{pr}(G')$. □

Lemma 2 *Let G be a connected graph, and let xyz be a triangle. Let G' be a graph obtained from G by subdividing some edges, and assume that all edges incident with x and all edges in $\{yw \mid w \in N_G(y) \cap N_2(x)\}$ are subdivided. If there exists a minimum paired-dominating set D of G' with $x, y, z \in D$ and the partner of y is z, then $\gamma_{pr}(G) < \gamma_{pr}(G')$.*

Proof Let $N_G(x) = \{y = x_1, z = x_2, \ldots, x_k\}$, $N_G(y) \cap N_2(x) = \{y_1, \ldots, y_r\}$ and let F be the set of all subdivided edges of G. In particular, let $x'_1, \ldots, x'_k$ be the subdivision vertices of the edges $xx_1, \ldots, xx_k$, respectively, and assume, without loss of generality, that the edges $x_1y_1, \ldots, x_1y_r$ are subdivided with new vertices $y'_1, \ldots, y'_r$, respectively. Let D be a minimum paired-dominating set of G' containing x, y, z in which y and z are partners. Let x'_i be the partner of x and let $F' \subseteq F$ be the set of edges whose subdivision vertices do not belong to D. Now, if G_1 denotes the graph obtained from G by subdividing all edges in $F - (F' \cup \{xx_i\})$, then certainly the set $D - \{x_1, x'_i\}$ in which x and x_2 as partners, is a PDS of G_1 smaller than D and therefore yielding the desired result. □

Lemma 3 *Let G be a connected graph with $\gamma_{pr}(G) = 4$ and let $v \in V(G)$ be a vertex of degree three with $N_G(v) = \{x_1, x_2, x_3\}$ such that $x_2x_3 \in E(G)$, $N_G(x_1) \nsubseteq N_G[v]$ and $x_1x_2, x_1x_3 \notin E(G)$. Then*

$$\mathrm{sd}_{\gamma_{pr}}(G) \leq 3 + \deg(x_1).$$

Proof Since $N_G(x_1) \nsubseteq N_G[v]$, assume that $N_G(x_1) - N[v] = \{y_1, \ldots, y_k\}$. Let G' be the graph obtained from G by subdividing the edges $vx_1, vx_2, vx_3, x_2x_3, x_1y_1, \ldots, x_1y_k$ with new vertices $x'_1, x'_2, x'_3, z, y'_1, \ldots, y'_k$, respectively. Let D be a $\gamma_{pr}(G')$-set. To dominate z, we may assume, without loss of generality, that $x_2 \in D$. Suppose α is the partner of x_2.

Firstly, assume that $v \in D$ and let β be the partner of v in D. If $\beta \neq x'_1$, then to dominate x_1 we must have $D \cap \{x_1, x'_1, y'_1, \ldots, y'_k\} \neq \emptyset$ and thus $\gamma_{pr}(G') \geq 5 > \gamma_{pr}(G)$. Hence we assume that $\beta = x'_1$. If $x_1 \in D$, then clearly $\gamma_{pr}(G') \geq 5 > \gamma_{pr}(G)$. Thus, let $x_1 \notin D$. Then to dominate $y'_1, \ldots, y'_k$ we must have $\{y_1, \ldots, y_k\} \subseteq D$. Now, if $k \geq 2$, then clearly $\gamma_{pr}(G') \geq 5 > \gamma_{pr}(G)$. Hence assume that $k = 1$, and thus $y_1 \in D$. Now, if $y_1 \neq \alpha$, then obviously $\gamma_{pr}(G') \geq 5 > \gamma_{pr}(G)$. Thus let $y_1 = \alpha$. But then $D - \{v, x'_1\}$ is a PDS of G, and so $\gamma_{pr}(G') > \gamma_{pr}(G)$.

Secondly, assume that $v \notin D$. Then to dominate x'_1, x'_2, x'_3, we must have

$x_1, x_2, x_3 \in D$. Since x_2 and x_3 have different partners in D, it follows that $\gamma_{pr}(G') = |D| \geq 5 > \gamma_{pr}(G)$. In either case, $\mathrm{sd}_{\gamma_{pr}}(G) \leq 3 + \deg(x_1)$. □

Lemma 4 *Let G be a connected graph of order at least three and let $v \in V(G)$ be a vertex of degree at least two. If v has a neighbor x_1 such that*:

(i) x_1 is an isolated vertex in $G[N(v)]$,
(ii) each vertex in $N(x_1)$ has at least one neighbor in $N(v)$ different from x_1,
(iii) for each vertex $z \in N(v) - \{x_1\}$, $|N(x_1) \cap N(z) \cap N_2(v)| \geq 1$,Then for any vertex $z \in N(v) - \{x_1\}$,

$$\mathrm{sd}_{\gamma_{pr}}(G) \leq |N(v)| + |N(x_1) \cap N_2(v)| + |N(z) \cap N_2(v)|.$$

Proof Let $N(v) = \{x_1, x_2, \ldots, x_k\}$, and assume that $N(x_1) - N[v] = \{y_1, \ldots, y_r\}$ and $N(x_2) - N[v] = \{y_1 = z_1, \ldots, y_\ell = z_\ell, z_{\ell+1}, \ldots, z_s\}$, where $\ell = |N(x_1) \cap N(x_2) \cap N_2(v)|$. Suppose G' is the graph obtained from G by subdividing the edges $vx_1, \ldots, vx_k$ with new vertices $x'_1, \ldots, x'_k$, respectively, the edges $x_1y_1, \ldots, x_1y_r$ with new vertices $y'_1, \ldots, y'_r$, respectively, and the edges $x_2z_1, \ldots, x_2z_s$ with new vertices $z'_1, \ldots, z'_s$, respectively. Let F be the set of all subdivided edges of G. Clearly $|F| = |N(v)| + |N(x_1) \cap N_2(v)| + |N(x_2) \cap N_2(v)|$.

Next, we shall show that $\gamma_{pr}(G') > \gamma_{pr}(G)$. Let D be a $\gamma_{pr}(G')$-set and let F_1 be the set of all edges in F whose subdivision vertices in G' are not in D. We want to construct a PDS smaller than D of a graph G_i obtained from G by subdividing the edges of a subset of F. Clearly, by Proposition 1 we will have $\gamma_{pr}(G') > \gamma_{pr}(G_i) \geq \gamma_{pr}(G)$, which leads to the result. So let us consider the following two cases.

Case 1. $v \in D$.

Let x'_i be the partner of v in D. We first consider the case $x'_j \in D$ with $j \neq i$. Then x_j will be the partner of x'_j. Let G_1 be the graph obtained from G by subdividing all edges in $F - \{vx_i, vx_j\}$. By Lemma 1 we have $\gamma_{pr}(G') > \gamma_{pr}(G_1)$ as desired. Hence we assume that $D \cap \{x'_1, \ldots, x'_k\} = \{x'_i\}$. If $x_1 \in D$ and y'_j is the partner of x_1 in D, then let G_2 be the graph obtained from G by subdividing all edges in $F - \{x_1v, vx_i, x_1y_j\}$. Lemma 1 implies that $\gamma_{pr}(G') > \gamma_{pr}(G_2)$ as desired. Hence we assume that $x_1 \notin D$. Then to dominate vertices $y'_1, \ldots, y'_r$ we must have $y_1, \ldots, y_r \in D$.

First let $x'_i \neq x'_1$. Since $N(x_1) \cap N(v) = \emptyset$, to dominate x_1 we may assume that $y'_j \in D$ for some j. Suppose G_3 is the graph obtained from G by subdividing all edges in $F - (F_1 \cup \{vx_i, x_1y_j\})$. Since each vertex in $N(v)$ has a neighbor in $\{y_1, \ldots, y_r\}$, we deduce that $(D - \{v, x'_i, y'_j\}) \cup \{x_1\}$, with x_1 and y_j as partners, is a PDS of G_3 smaller than D as desired.

Now let $x'_i = x'_1$. If $y'_j \in D$ for some j, then let G_4 be the graph obtained from G by subdividing all edges in $F - \{vx_1, \ldots, vx_k, x_1y_j\}$ and clearly $(D - \{v, x'_1, y'_j\}) \cup \{x_1\}$ with x_1 and y_j as partners, is a PDS of G_4 smaller than D. Hence we assume that $D \cap \{y'_1, \ldots, y'_r\} = \emptyset$. If $x_j \in D$ for some $j \in \{2, \ldots, k\}$, then let G_5 be the obtained

from G by subdividing all edges in $F - \{vx_1, \ldots, vx_k, x_1y_1, \ldots, x_1y_r\}$ and clearly $D - \{v, x'_1\}$ is a PDS of G_5 smaller than D. Thus, suppose that $x_2, \ldots, x_k \notin D$. To dominate x_2, we must have $z'_j \in D$ for some j. In this case, let G_6 be the obtained from G by subdividing all edges in $F - \{vx_1, \ldots, vx_k, x_1y_1, \ldots, x_1y_r, x_2z_j\}$. Clearly $(D - \{v, x'_1, z'_j\}) \cup \{x_2\}$, with x_2 and z_j as partners, is a PDS of G_6 smaller than D.
Case 2. $v \notin D$.
Let F_1 be the set of all edges in F whose subdivision vertices in G' are not in D. To dominate the vertices $x'_1, \ldots, x'_k$, we must have $N_G(v) \subseteq D$, in addition to $x'_i \in D$ for some i to dominate v. If $x'_i \neq x'_1$ and y'_j is the partner of x_1, then let G_7 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{x_1y_j, vx_1, vx_i\})$. It is easy to see that $(D - \{y'_j, x_1, x'_i\}) \cup \{v\}$ with v and x_i as partners, is a PDS of G_7 smaller than D (note that by item (ii), each vertex in $N_G(x_1) - \{v\}$ has a neighbor in $N_G(v) - \{x_1\}$). Assume now that $x'_i = x'_1$. Thus x'_1 and x_1 are partners in D. If $x'_j \in D$ for some $j \neq 1$, then x_j, x'_j are partner in D and $(D - \{x'_j, x_1, x'_1\}) \cup \{v\}$ with v and x_j as partners, is a PDS of G_8 smaller than D where G_8 is the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_1, vx_j\})$. First let z'_j be a partner of x_2, and let G_9 be the obtained from G by subdividing the edges of $F - (F_1 \cup \{vx_1, x_2z_j\})$. Clearly $(D - \{x_1, x'_1, z'_j\}) \cup \{v\}$ with v and x_2 as partners is a PDS of G_9 smaller than D. Finally, assume that x_j is the partner of x_2 in D, for some $j \in \{3, \ldots, k\}$. Let G_{10} be obtained from G by subdividing the edges of $F - (F_1 \cup \{vx_1\})$. Clearly $(D - \{x_1, x'_1, x_2\}) \cup \{v\}$ with v and x_j as partners is a PDS of G_{10} smaller than D, and this completes the proof. □

Lemma 5 *Let G be a connected graph of order at least 4 and let $v \in V(G)$ be a vertex of degree at least two.*

(i) If $WC_1(v) = 0$, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq 3$.
(ii) If $WC_1(v) \geq 2$ and the induced subgraph $G[N(v)]$ is isolated-free, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq \deg(v) + |N_2(v)| + |N_3(v)| \leq n - 1$.
(iii) If the induced subgraph $G[N(v)]$ has an isolated-vertex z and there is a weak-clique set of v, S^1, such that $z \in S^1$ and $|S^1| \geq 2$, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq \deg(v) + |N_2(v)| + |N_3(v)| \leq n - 1$.

Proof **(1)-** If $WC_1(v) = 0$, then $V(G) = N[v]$ and so $\gamma_{pr}(G) = 2$ and we deduce from Proposition 2 that $\mathrm{sd}_{\gamma_{pr}}(G) \leq 3$.

To prove the remaining two items, let $N(v) = \{x_1, x_2, \ldots, x_k\}$, S^1 be a 1th weak-clique set of v of size $WC_1(v)$ and let $WS_1(v)$ be a 1th weak-separator of v. We may assume, without loss of generality, that $S^1 = \{x_1, x_2, \ldots, x_\ell\}$. Moreover, if $N_3(v) \neq \emptyset$, then let S^2 be a 2th weak-clique set of v of size $WC_2(v)$ and let $WS_2(v)$ be a 2th weak-separator of v.

(2)- Let $WC_1(v) \geq 2$ and let the induced subgraph $G[N(v)]$ be isolated-free. Suppose $N(x_1) - N[v] = \{y_1, \ldots, y_r\}$ and $N(x_2) - N[v] = \{z_1, \ldots, z_s\}$. Since $WC_1(v) \geq 2$, we have $(N(x_1) - N[v]) \cap (N(x_2) - N[v]) = \emptyset$. Let G' be the graph obtained from G by subdividing the edges $vx_1, \ldots, vx_k$ with subdivision vertices

$x_1', \ldots, x_k'$, respectively, the edges $x_1y_1, \ldots, x_1y_r$ with subdivision vertices $y_1', \ldots, y_r'$, respectively, the edges $x_2z_1, \ldots, x_2z_s$ with subdivision vertices $z_1', \ldots, z_s'$, respectively, and all other edges in $WS_1(v) \cup WS_2(v)$. Set $F = \{vx_1, \ldots, vx_k\} \cup WS_1(v) \cup WS_2(v)$. Note that $|F| = \deg(v) + |N_2(v)| + |N_3(v)|$. Let D be a $\gamma_{pr}(G')$-set and let F_1 be the set of all edges of G whose subdivision vertices in G' are not in D. We consider two cases.

Case 1. $v \in D$.

Let x_i' be the partner of v in D. First let $D \cap S^1 \neq \emptyset$. We may assume, without loss of generality, that $x_1 \in D \cap S^1$, and let α be the partner of x_1 in D. If $\alpha \in \{x_1', y_1', \ldots, y_r'\}$, then by Lemma 1 we have $\gamma_{pr}(G') > \gamma_{pr}(G)$, and the desired result follows. Hence we assume that $\alpha \in N_G(v)$. Let G_1 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_i\})$. Clearly $D - \{x_1, x_i'\}$, with v and α as partners, is a PDS of G_1 smaller than D. In the sequel we can assume that $D \cap S^1 = \emptyset$. It follows that $N_2(v) \cap \left(\bigcup_{i=1}^{\ell} N_G(x_i)\right) \subseteq D$.

Now we prove that $N_G(v) \cap D = \emptyset$. If $x_j \in N_G(v) \cap D$ and its partner α is a subdivision vertex, then we deduce from Lemma 1 that $\gamma_{pr}(G') > \gamma_{pr}(G)$. If $x_j \in N_G(v) \cap D$ and its partner α belongs to $N(v)$, then let $\alpha = x_p$ and let G_1'' be the graph obtained from G by subdividing all edges in $F - (F_1 \cup \{vx_i\})$. Assume, without loss of generality, that $p > j$. We claim that the set $D' = D - \{\alpha, x_i'\}$ with v and x_j as partner, is a PDS of G_1''. It is enough to show that any vertex y dominated by α in G, is dominated by D' in G_1''. We note that any vertex y in $N(v) - D$ dominated by α in G', is simply dominated by v in G_1'' (since the edge vy belongs to F_1). On the other hand, any subdivision vertex adjacent to α in G_1'' belongs to D' (by definition of G_1''). Now assume that $y \in N_2(v)$ and is dominated by α in G'. Let m be the smallest integer such that $x_my \in E(G)$. Obviously the edge x_my is subdivided in G' by the definition of 1th weak separator. Let z be the subdivision vertex of the edge x_my. To paired-dominate z, we must have $D \cap \{y, x_m, z\} \neq \emptyset$. Now if $y \in D$ or $z \in D$, then we are done. Otherwise, $x_m \in D$ and $z \notin D$ and thus the edge $yx_m \in E(G_1'')$ implying that y is dominated by $x_m \in D'$ in G_1''. Hence D' is a PDS of G_1''. For the remaining situation, if $x_j \in N_G(v) \cap D$ and its partner α belongs to $N_2(v)$, then let G_2 be the graph obtained from G by subdividing all edges in $F - (F_1 \cup \{vx_i\})$. We also claim that $D_1 = D - \{\alpha, x_i'\}$ with v and x_j as partner, is a PDS of G_2. To see this, it is enough to show that any vertex y dominated by α in G', is dominated by D' in G_2. Clearly, any vertex y in $N(v) - D$ dominated by α in G' , is dominated by v in G_2 (since as above, the edge $vy \in F_1$) . Also, any subdivision vertex adjacent to α in G_2 belongs to D_1 (by the definition of G_2). Now, if $y \in N_2(v)$, then by the definition of 1th weak separator y is adjacent to a subdivision vertex β of an edge yw where $w \in N(v)$. Now to paired-dominate β in G' we must have $|D \cap \{w, y, \beta\}| \geq 1$. Clearly $D \cap \{\beta, w, y\} = D_1 \cap \{\beta, w, y\}$ and so y is dominated by a vertex of D_1 in G_2 as desired. Finally, we assume that $y \in N_3(v)$. By the definition of 2th weak separator, y is adjacent to a subdivision vertex β of an edge yw where $w \in N_2(v)$. To paired-dominate β we must have $|D \cap \{w, y, \beta\}| \geq 1$. If $y \in D$ or $\beta \in D$, then we are done. Otherwise, $w \in D$ and $\beta \notin D$ and thus the edge $wy \in E(G_2)$ implying that y is dominated by $w \in D_1$ in G_2.

Hence assume that $N_G(v) \cap D = \emptyset$. It follows that the partner of v, namely x_i', is the unique vertex of $\{x_1', \ldots, x_k'\}$ belonging to D. Suppose, without loss of generality, that $x_i' \neq x_1'$. To dominate x_1 we must have for some $j \in \{1, \ldots, r\}$, $y_j' \in D$ paired with y_j. Let G_3 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_i, x_1y_j\})$. Note that $vx_1 \in F_1$. We claim that $D' = (D - \{x_i', y_j, y_j'\}) \cup \{x_1\}$ with v and x_1 as partners is a PDS of G_3 smaller than D. It is enough we show that any vertex u dominated by y_j in G, is dominated by D' in G_3. If $u \in N(v)$, then u is dominated by v in G_3. If $u \in N_2(v)$, then by definition of 1th weak separator u is adjacent to a subdivision vertex β of an edge uw where $w \in N(v)$. Now to paired-dominate β we must have $|D \cap \{w, u, \beta\}| \geq 1$. Since $w \notin D$, we have $u \in D$ and thus $u \in D'$ as desired. Hence assume that $u \in N_3(v)$. Clearly $u \neq y_j$. By definition of 2th weak separator u is adjacent to a subdivision vertex β of an edge uw where $w \in N_2(v)$. To paired-dominate β we must have $|D \cap \{w, u, \beta\}| \geq 1$. If $u \in D$ or $\beta \in D$, then we are done. Otherwise, u is dominated by w in D' as desired.

Case 2. $v \notin D$.

To dominate the vertices $x_1', \ldots, x_k'$, we must have $N_G(v) \subseteq D$. Also, D must contain x_i' for some i to dominate v. Since $WC_1(v) \geq 2$, we may assume that $x_i' \neq x_1'$. Let G_4 be the graph obtained from G by subdividing the edges in $F - (F_2 \cup \{vx_i\})$, and consider the set $D_1 = D - \{x_i, x_i'\}$. Note that since $G[N(v)]$ is isolate-free, vertex x_i has a neighbor in $N(v)$ and such a neighbor belongs to D_1. Now let $u \in N_2(v)$ be a neighbor of x_i. If the edge x_iu is subdivided in G' (note that x_iu may belong or not to F_1), then clearly u is a dominated by D_1. Hence we assume that x_iu is not subdivided in G'. We deduce that u is adjacent to some vertex in $\{x_1, \ldots, x_{i-1}\}$. Let j be the smallest integer such that $ux_j \in E(G)$. Then the edge x_ju should be subdivided (with new vertex w) according to the definition of weak-separator. Hence u is dominated by w or x_j in D_1. Therefore, we conclude that D_1 is PDS of G_4 smaller than D.

(3)- We proceed similarly as for item (2), in particular, let G' and D as defined above. Assume, without loss of generality, that $z = x_1$. In the case $v \in D$, the proof is the same as in (2). Hence let us assume that $v \notin D$. To dominate the vertices $x_1', \ldots, x_k'$, we must have $N_G(v) \subseteq D$. Also, D must contain x_i' for some i to dominate v. If x_i is not an isolated vertex in $G[N(v)]$, then the result follows as in Case 2 of item (2). Hence, let x_i be an isolated vertex in $G[N(v)]$. If $x_i \neq x_1$ and z' is the partner of $z = x_1$ resulting from the subdivision of the edge e, then let G_5 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_i, e\})$. Set $D_2 = (D - \{x_i, x_i', z'\}) \cup \{v\}$ with v and x_1 as partners. Also, if x_i has a neighbor u in $N_2(v)$ such that the edge x_iu is not subdivided in G', then by a similar argument to that used in Case 2 of item (2), we can see that there is a vertex $x_j \in S^1$ such that edge x_ju is subdivided with a new vertex w, and thus u is either dominated by w (if $x_ju \notin F_1$) or x_j (if $x_ju \in F_1$). Therefore, D_2 is a PDS of G_5 smaller than D. Hence assume that $x_1 = x_i$. Since $x_2 \in D$, let y be the partner of x_2. If y is resulting from the subdivision of some edge e, then let G_6 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_1, e\})$. Clearly $(D - \{x_1, y, x_1'\}) \cup \{v\}$ with v and x_2 as partners is a PDS of G_6 smaller than D. Finally, assume that $y \in N(x)$. In this case, let G_7 be the graph obtained from G by subdividing the edges in $F - (F_1 \cup \{vx_1\})$. Note that $\{vx_2, vy\} \subset F_1$, and thus $(D - \{x_2, x_1, x_1'\}) \cup \{v\}$ with v and y as partners is a PDS of G_7 smaller than D, which completes the proof. $\square$

3 Proof of Conjecture 1

Now, we are ready to state our main result which settles Conjecture 1.

Theorem 1 *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq n - 1$.*

Proof If $\gamma_{pr}(G) = 2$, then the result is immediate from Proposition 2. Hence we may assume that $\gamma_{pr}(G) \geq 4$. If G contains a strong support vertex or two adjacent support vertices, then the result follows from Proposition 3. Thus we may assume that G has at least one vertex x of degree at least 2 which is not a support vertex. If such a vertex x satisfies one of the conditions of Lemma 5, then we are done. Hence we may assume that for each non-support vertex x of G with $\deg(x) \geq 2$, either $WC_1(x) = 1$ or $WC_1(x) \geq 2$ and x has a neighbor y such that y is an isolated vertex in $G[N(x)]$ and y belongs no 1th weak-clique S with $|S| \geq 2$. Among all non-support vertices of G with degree at least two, let v be one having maximum degree. Moreover, if $|WC_1(v)| = 1$, then among the neighbors of v, let x_1 be one chosen so that:

(C_1) x_1 has exactly one neighbor in $N_2(v)$.
(C_2) If (C_1) does not occur, then x_1 is an isolated vertex in $G[N(v)]$.
(C_3) If (C_1) and (C_2) do not occur, then x_1 is an arbitrary vertex in $N(v)$ so that $|N(x_1) \cap N_2(v)| \geq 1$.

Otherwise, among all isolated vertices in $G[N(v)]$ belonging to no 1th weak-clique set of size at least two, let x_1 be one chosen so that $|N(x_1) \cap N_2(v)|$ is minimized.

Assume that $N(v) = \{x_1, \ldots, x_k\}$ and $N(x_1) \cap N_2(v) = \{y_1, \ldots, y_r\}$. Let $S^0 = \{v\}$, S^j be a jth weak-clique set of v and $WS_j(v)$ be a jth weak-separator of v corresponding to S^j, where $j \in \{1, 2, 3\}$. Consider the following two cases.

Case 1. $WC_1(v) = 1$.

Hence $S^1 = \{x_1\}$. Let G' be the graph obtained from G by subdividing the edges $vx_1, \ldots, vx_k$ with subdivision vertices $x_1', \ldots, x_k'$, respectively, the edges $x_1y_1, \ldots, x_1y_r$ with subdivision vertices $y_1', \ldots, y_r'$, respectively, and all the other edges in $WS_1(v) \cup WS_2(v) \cup WS_3(v)$. We show that$\gamma_{pr}(G') > \gamma_{pr}(G)$. Let D be a $\gamma_{pr}(G')$-set. Also, let F be the set of all subdivided edges of G and $F' \subseteq F$ be the set of edges whose subdivision vertices do not belong to D. We distinguish two situations.

Subcase 1.1. $v \in D$.

Let x_i' be the partner of v. If $x_j' \in D$ for some $j \neq i$, then x_j is its partner in D and by Lemma 1 we have $\gamma_{pr}(G') > \gamma_{pr}(G)$. Thus, we may assume that

$$D \cap \{x'_1, \ldots, x'_k\} = \{x'_i\}. \tag{1}$$

If $x_1 \in D$ and y'_j is the partner of x_1 for some j , then Lemma 1 implies that $\gamma_{pr}(G') > \gamma_{pr}(G)$. If $x_1 \in D$ and x_j is the partner of x_1 for some $j \in \{2, \ldots, k\}$, then we deduce from Lemma 2 that $\gamma_{pr}(G') > \gamma_{pr}(G)$. Hence we may assume that $x_1 \notin D$. It follows that

$$\{y_1, \ldots, y_r\} \subseteq D.$$

To dominate x_1, we must have $D \cap \{x'_1, x_2, \ldots, x_k, y'_1, \ldots, y'_r\} \neq \emptyset$. First, if $y'_j \in D$ for some j, then $(D - \{y_j, y'_j, x'_i\}) \cup \{x_1\}$ with v and x_1 as partners, is a PDS of the graph G_1 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i, x_1y_j\})$ smaller than D (a similar discussion to that of item 2 of Lemma 5 can be applied for this situation). Suppose now that $x_j \in D$ for some $2 \leq j \leq k$ and let β be the partner of x_j. If β is a subdivision vertex of an edge, then by Lemma 1 we have $\gamma_{pr}(G') > \gamma_{pr}(G)$. Assume that $\beta = y_\ell$ for some ℓ. We claim that $D' = D - \{y_\ell, x'_i\}$ with v and x_j as partners, is a PDS of the graph G_2 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i\})$. Note that any vertex y in $N(v) - D$ is dominated by v (since the edge vy belongs to $F_1 \cup \{vx_i\}$). Now assume that $y \in N_2(v)$ and is dominated by y_ℓ in G'. Let m be the smallest integer such that $x_my \in E(G)$. Obviously the edge x_my is subdivided in G' by the definition of 1th weak separator. Let z be the subdivision vertex of the edge x_my. To paired-dominate z, we must have $D \cap \{y, x_m, z\} \neq \emptyset$. Now if $y \in D$ or $z \in D$, then we are done. Otherwise, $x_m \in D$ and $z \notin D$ and thus the edge $yx_m \in E(G_2)$ implying that y is dominated by $x_m \in D'$ in G_2.Finally, we assume that $y \in N_3(v)$. By the definition of 2th weak separator, y is adjacent to a subdivision vertex η of an edge yw where $w \in N_2(v)$. To paired-dominate η we must have $|D \cap \{w, y, \eta\}| \geq 1$. If $y \in D$ or $\eta \in D$, then we are done. Otherwise, $w \in D$ and $\eta \notin D$ and thus the edge $wy \in E(G_2)$ implying that y is dominated by $w \in D'$ in G_2. If $\beta = x_\ell$, then $D - \{x_j, x'_i\}$ with v and x_ℓ as partners, is a PDS of the graph G_3 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i\})$ smaller than D. Assume now that x'_i dominates x_1, that is $x'_1 = x'_i$. Using a similar argument as above, we may assume that

$$D \cap N(v) = \emptyset. \tag{2}$$

If there is an edge $x_jy \in WS_1(v)$ whose subdivision vertex is in D , then as above we can get the result. Hence we assume that no subdivision vertex of any edge of $WS_1(v)$ belongs to D and so we have

$$N_2(v) \subseteq D.$$

Since $i = 1$, we deduce from (1), (2) and above assumption that each x_j $(2 \leq j \leq k)$ is dominated by a vertex in $N_2(v)$ and we conclude from $|WC_1(v)| = 1$ that each vertex in $N(v)$ has a neighbor in $\{y_1, \ldots, y_r\}$. Let β_j be the partner of y_j in D for each

$j \in \{1, \ldots, r\}$. If β_j is a subdivision vertex of an edge e for some j, then the set $(D - \{\beta_j, v, x'_1\}) \cup \{x_1\}$ with x_1 and y_j as partners, is a PDS of the graph G_4 obtained from G by subdividing all edges in $F - (F' \cup \{vx_1, x_1y_j, e\})$ smaller than D. If $\beta_j \in N_2(v) - N_G(x_1)$, then the set $D' = D - \{x'_i, v, \beta_j\} \cup \{x_1\}$ with x_1 and y_j as partners, is a PDS of the graph G_5 obtained from G by subdividing all edges in $F - (F' \cup \{vx_1\})$ smaller than D. We note that in the previous situation, each vertex in $N_1(v)$ has a neighbor in $\{y_1, \ldots, y_r\}$ (since $D \cap N(v) = \emptyset$) and so it is dominated. Moreover, each vertex in $N_2(v) - \{\beta_j\}$ is in D', and if a vertex $y \in N_3(v)$ is adjacent to β_j, then by definition, y is on a subdivided edge e_1 between y and $N_2(v)$ and to dominate the subdivision vertex one of the endvertices of e_1 must belong to D and so y is dominated by D'). If $\beta_j \in N_3(v)$, then the set $D' = (D - \{x'_i, v, \beta_j\}) \cup \{x_1\}$ with x_1 and y_j as partners, is a PDS of the graph G_6 obtained from G by subdividing all edges in $F - (F' \cup \{vx_1\})$, smaller than D (note that $N_2(v) \subseteq D'$, β_j is dominated by y_j, each vertex z in $N_3(v) - \{\beta_j\}$ is either an endvertex of a subdivided edge e_1 between z and $N_2(v)$ and so z is dominated by the subdivision vertex of e_1 or the other end-point of e_1 in G_6, and finally if a vertex $z \in N_4(v)$ was dominated by β_j , then z is adjacent to an endvertex of a subdivided edge e_2 between z and $N_3(v)$ and so z is dominated by either the subdivision vertex of e_2 or the other endvertex of e_2. Finally let $\beta_j \in \{y_1, y_2, \ldots, y_r\}$ for each $1 \le j \le r$. Hence x_1 has at least two neighbors in $N_2(v)$. We shall show that $D' = (D - \{v, x'_i, \beta_1\}) \cup \{x_1\}$ with x_1 and y_1 as partners, is a PDS of the graph G_7 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i, x_1y_1\})$, smaller than D. As above we can see that every vertex in $N_2(v) \cup N_3(v)$ is dominated by D' in G_7. Now let $y \in N_1(v)$ be a vertex which was dominated by β_1 in G'. By the choice of x_1, we have that y has at least two neighbors in $N_2(v)$. If $yz \in E(G)$ where $z \in N_2(v) - \{\beta_1\}$, then y will be dominated by z or the subdivision vertex of the edge yz in D'. Thus D' is a PDS of the graph G_7 smaller than D.

Subcase 1.2. $v \notin D$.

Clearly $N(v) \subseteq D$, and to dominate v we must have for some i, $x'_i \in D$ paired with x_i. If x_i has a neighbor in $N(v)$, then one can easily see that $D - \{x_i, x'_i\}$ is a PDS of the graph G_8 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i\})$, smaller than D. Hence we assume that x_i has no neighbor in $N(v)$. Since v is not a support vertex, we have $N(x_i) \cap N_2(v) \neq \emptyset$.

First let $x_1 \neq x_i$ and let β be the partner of x_1 in D . If β is a subdivision vertex of an edge e, then it is easy to see that $(D - \{x_i, x'_i, \beta\}) \cup \{v\}$ with x_1 and v as partners, is a PDS of the graph G_9 obtained from G by subdividing all edges in $F - (F' \cup \{vx_i, vx_1, e\})$ smaller than D. Hence we assume that $\beta \in N(v)$. Since x_i has no neighbor in $N(v)$, by the choice of x_1, x_1 has exactly one neighbor in $N_2(v)$. If $\beta = x_j$ for some $j \neq 1$ and $N(x_j) \subseteq N(v)$, then we can see that $(D - \{x_i, x'_i, x_j\}) \cup \{v\}$ with x_1 and v as partners, is a PDS of the graph G_{10} obtained from G by subdividing all edges in $F - (F' \cup \{vx_i\})$ smaller than D . If $\beta = x_j$ for some $j \neq 1$ and $N(x_j) \not\subseteq N(v)$, then $x_jy_1 \in E(G)$ and we can see that $(D - \{x_i, x'_i, x_1\}) \cup \{y_1\}$ with x_j and y_1 as partners, is a PDS of the graph G_{10} smaller than D.

Now let $x_1 = x_i$. If $y_j \in D$ for some $j \in \{1, \ldots, r\}$, then let G_{11} be the graph obtained from G by subdividing all edges in $F - (F' \cup \{vx_1\})$. Clearly $D - \{x_1, x'_1\}$ is a PDS of G_{11} smaller than D. Hence we assume that $D \cap \{y_1, \ldots, y_r\} = \emptyset$, and thus $D \cap \{y'_1, \ldots, y'_r\} = \emptyset$. If some y_j has no neighbor in $N(v) - \{x_1\}$, then by choosing x_1 instead of v we will be in a position of Item 3 form Lemma 5 and the result follows. Hence we can assume that each y_j has a neighbor in $N(v) - \{x_1\}$, and thus x_1 is not a support vertex. Now, for each $j \in \{2, \ldots, k\}$, let β_j be the partner of x_j in D. If β_j is a subdivision vertex of an edge $e \in E(G)$, then let G_{12} be the graph obtained from G by subdividing all edges in $F - (F' \cup \{e, vx_j, vx_1\})$. Clearly $(D - \{x_1, x'_1, \beta_j\}) \cup \{v\}$ with v and x_j as partners, is a PDS of G_{12} smaller than D. If $\beta_j \in N_2(v)$, then let G_{13} be obtained from G by subdividing all edges in $F - (F' \cup \{vx_j, vx_1\})$. As in Subcase 1.1, we can see that $(D - \{x_1, x'_1, \beta_j\}) \cup \{v\}$ with v and x_j as partners, is a PDS of G_{13} smaller than D. Thus we assume that $\beta_j \in N_1(v)$ for each $j \in \{2, \ldots, k\}$. If some x_j has no a private neighbor in $N_2(v) \cap N(x_1)$ with respect to $N(v) - \{x_1\}$, then let G_{14} be the graph obtained from G by subdividing all edges in $F - (F' \cup \{vx_j, vx_1\})$. Clearly $(D - \{x_1, x'_1, x_j\}) \cup \{v\}$ with v and β_j as partners, is a PDS of G_{14} smaller than D. Thus we may assume that each x_j has a private neighbor in $N_2(v) \cap N(x_1)$ with respect to $N(v) - \{x_1\}$. Therefore, $\deg(x_1) \geq \deg(v)$. But since x_1 is not a support vertex with $|WC_1(v)| = 1$, we deduce from the choice of v that $\deg(x_1) = \deg(v)$. It follows that for each $j \in \{2, \ldots, k\}$, $|N_2(v) \cap N_G(x_1) \cap N_G(x_j)| = 1$. Now, Lemma 4 implies that

$$\mathrm{sd}_{\gamma_{pr}}(G) \leq |N_G(v)| + |N_2(v) \cap (N_G(x_1) \cup N_G(x_2))| + 1. \tag{3}$$

If $\gamma_{pr}(G) > 4$, then we must $|N_2(v) \setminus (N_G(x_1) \cup N_G(x_2))| \geq 1$ and we deduce from the inequality (3) that $sd_{\gamma_{pr}}(G) \leq n - 1$. Hence we assume that $\gamma_{pr}(G) = 4$. If $\deg(v) \geq 4$, then we conclude from $N(v) \subseteq D$ and the fact that x_1 is isolated in $G[N(v)]$ that $\gamma_{pr}(G') > \gamma_{pr}(G)$. Thus $\deg(v) = 3$, and therefore the result follows from Lemma 3.

Case 2. $WC_1(v) \geq 2$, and v has a neighbor y which is isolated in $G[N(v)]$ and y belongs to no 1th weak-clique S with $|S| \geq 2$.

The proof is similar to Case 1 and so we omit the details. But here are a few hints.

We first consider the graph G' obtained from G by subdividing the edges $vx_1, \ldots, vx_k$, $x_1y_1, \ldots, x_1y_r$ and all the other edges in $WS_1(v) \cup WS_2(v) \cup WS_3(v)$. Sets D, F and F' are defined as in Case 1. Subsequently, we distinguish two subcases according to whether or not v belongs to D. In particular, the subcase $v \in D$, will be divided into two situations: $|N_2(v) \cap N(x_1) \cap N(x_j)| = 1$ for some $j \in \{2, \ldots, k\}$ or $|N_2(v) \cap N(x_1) \cap N(x_j)| \geq 2$ for each $j \in \{2, \ldots, k\}$. The goal is to show in either situation of the proof that $\gamma_{pr}(G') > \gamma_{pr}(G)$. □

We conclude this paper with the following conjecture.

Conjecture 2 *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq \gamma_{pr}(G) + 1$.*

It is well-known that connected graphs of order $n \geq 6$ with minimum degree at least two have paired-domination number bounded above by $2n/3$. Therefore for such class of graphs the bound of Theorem 1 would be improved if Conjecture 2 is true.

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