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Mean Color Numbers of Some Graphs

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Abstract

Let $\mu(G)$ denote the mean color number of a graph G. Dong proposed two mean color conjectures. One is that for any graph G and a vertex w in G with $d(w) \geq 1$, if H is a graph obtained from G by deleting all but one of the edges which are incident to w, then $\mu(G) \geq \mu(H)$. The other is that for any graph G and a vertex w in G, $\mu(G) \geq \mu((G-w) \cup K_1)$. In this paper, we show that the two conjectures hold under the condition that w is a simplicial vertex in G. And when G is a connected (n, m)-graph and w is not a cut vertex in G with $d(w) = n-1$, if $m \leq (\frac{\sqrt{2}}{2} + 2)n - 4.5 - \sqrt{2}$, the second conjecture holds too. The two conjectures also hold for some special cases, such as wheels and chordal graphs (Dong in J Combin Theory Ser B 87: 348–365, 2003).

Keywords Graph · Chromatic polynomial · Mean color number · Simplicial vertex · Wheel

Mathematics Subject Classification 05C15

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1 Introduction

In this paper, all graphs are finite and simple. Throughout this paper, n and m will always denote, respectively, the number of vertices and the number of edges in a graph G. The readers are assumed familiar with graph theory terminology as in Bondy and Murty [2], for example.

For any graph G, let $V(G)$, $E(G)$ and $v(G)$ be the vertex set, edge set and order of G, respectively. For a positive integer λ, a proper λ-coloring, or simply a λ-colorings of G is a map $\phi : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $\phi(u) \neq \phi(v)$ where u and v are adjacent vertices. The chromatic polynomial of G, denoted by $P(G, \lambda)$, is the number of λ-colorings of G. For any positive integer k, let $\alpha(G, k)$ denote the number of partitions of $V(G)$ into exactly k non-empty independent sets. Then

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)(\lambda)_k, \tag{1}$$

where $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ and $n = v(G)$.

Let G be a graph of order n. It is obvious that there exist n-colorings of G. For any n-coloring Γ of G, let $l(\Gamma)$ be the actual number of colors used. The mean color number $\mu(G)$ of G, defined by Bartels and Welsh [1], is the average of $l(\Gamma)$'s over all n- colorings Γ. The number of n-colorings Γ of G with $l(\Gamma) = k$ is $\alpha(G,k)(n)_k$. Therefore by the definition of $\mu(G)$, we have

$$\mu(G) = \frac{\sum_{k=1}^{n} k(n)_k \alpha(G,k)}{\sum_{k=1}^{n} (n)_k \alpha(G,k)}.$$

Bartels and Welsh also presented an expression of $\mu(G)$ in terms of the chromatic polynomials.

Theorem 1.1 ([1]) *If* $v(G) = n$, *then*

$$\mu(G) = n\left(1 - \frac{P(G,n-1)}{P(G,n)}\right). \tag{2}$$

Theorem 1.1 shows that $\mu(G) \leq n$ where equality holds iff G is complete. For the empty graph O_n of order n, we have

$$\mu(O_n) = n\left(1 - \left(1 - \frac{1}{n}\right)^n\right).$$

Bartels and Welsh conjectured that $\mu(O_n)$ is a lower bound of $\mu(G)$ for any graph G of order n, and their conjecture was proved by Dong [4]. They also proposed a more general conjecture that if H is a spanning subgraph of G, then $\mu(G) \geq \mu(H)$. But counterexamples have been discovered by Mosca [8].

Thus, in general the following equality is not true:

$$\mu(G) \geq \mu(H), \tag{3}$$

where H is a subgraph of G. But it is true for some special cases. It is clear that (3) holds if G is complete. And Dong proved that (3) holds if H is a spanning subgraph of G and H is either a tree or an empty graph [4]. Several years later, he also proved that $\mu(G) \geq \mu(H)$ if G is a chordal graph and H is a spanning subgraph of G, and the equality holds iff $H \cong G$ [5].

In this paper, we are concerned with two conjectures proposed by Dong. The first is the following:

Conjecture 1 ([5]) *For any graph G and a vertex w in G with $d(w) \geq 1$, if H is a graph obtained from G by deleting all but one of the edges which are incident to w, then $\mu(G) \geq \mu(H)$.*

In Sect. 2 we shall show that Conjecture 1 holds under the condition that w is a simplicial vertex in G with $d(w) \geq 1$. And Conjecture 1 also holds for the wheel of order n.

The second conjecture is as follows:

Conjecture 2 ([5]) *For any graph G and a vertex w in G, $\mu(G) \geq \mu((G - w) \cup K_1)$.*

In Sect. 3 we shall show that Conjecture 2 also holds under the condition that w is a simplicial vertex in G. And when G is a connected (n, m)-graph and w is not a cut vertex in G with $d(w) = n - 1$, if $m \leq (\frac{\sqrt{2}}{2} + 2)n - 4.5 - \sqrt{2}$, then $\mu(G) \geq \mu((G - w) \cup K_1)$. For the wheel of order n, conjecture 2 holds too.

For some special cases, for example, chordal graph and 2-tree, the two conjectures are true [5].

2 The First Conjecture

For any graphs G, H and for any real λ, define

$$\tau(G, H, \lambda) = P(G, \lambda)P(H, \lambda - 1) - P(G, \lambda - 1)P(H, \lambda). \tag{4}$$

By Theorem 1.1, one may deduce that

Lemma 2.1 ([5]) *For any graphs G and H with $v(G) = v(H) = n$, the inequality $\mu(G) \geq \mu(H)$ is equivalent to $\tau(G, H, n) \geq 0$.*

Now we present a well known result on chromatic polynomial used in this paper.

Lemma 2.2 ([9]) *For any graph G, if $\lambda \geq v(G) - 1$, then $P(G, \lambda) \geq 0$ where equality holds iff G is complete and $\lambda = v(G) - 1$.*

For any vertex x in G, let $N_G(x)$ (or simply $N(x)$) denote the set of vertices in G which are adjacent to x, and let $d_G(x)$ (or simply $d(x)$) be the degree of x in G. The vertex x in G is called a simplicial vertex if either $d(x) = 0$ or $G[N(x)]$ is a clique.

Theorem 2.1 *For any graph G of order n and a simplicial vertex w in G with* $d(w) \geq 1$, *if* $\lambda \geq n - 1$ *and H is a graph obtained from G by deleting all but one of the edges which are incident to w, then* $\tau(G, H, \lambda) \geq 0$ *where equality holds iff* $d(w) = 1$ *or* $\lambda = n - 1$ *and* $G - w$ *is complete.*

Proof Let $G^* = G - w$. For any positive integer $\lambda \geq d$, since w is a simplicial vertex in G, we have

$$P(G, \lambda) = (\lambda - d)P(G^*, \lambda), \quad P(H, \lambda) = (\lambda - 1)P(G^*, \lambda), \tag{5}$$

where $d = d(w)$.

Thus, by (5) and the definition of $\tau(G, H, \lambda)$, it follows that

$$\begin{aligned} \tau(G, H, \lambda) =& (\lambda - d)P(G^*, \lambda)(\lambda - 2)P(G^*, \lambda - 1) \\ & - (\lambda - d - 1)P(G^*, \lambda - 1)(\lambda - 1)P(G^*, \lambda) \\ =& (d - 1)P(G^*, \lambda)P(G^*, \lambda - 1). \end{aligned} \tag{6}$$

In addition, for $\lambda \geq n - 1$, by Lemma 2.2, we get

$$P(G^*, \lambda) > 0, \quad P(G^*, \lambda - 1) \geq 0. \tag{7}$$

Observe that $d(w) \geq 1$. Therefore (6) and (7) imply the theorem holds. □

By Theorem 2.1 and Lemma 2.1, we have the first result on mean color number.

Theorem 2.2 *For any graph G and a simplicial vertex w in G with* $d(w) \geq 1$, *if H is a graph obtained from G by deleting all but one of the edges which are incident to w, then* $\mu(G) \geq \mu(H)$, *where equality holds iff* $d(w) = 1$.

Now let us show that Conjecture 1 holds for the wheel of order n: for that let us introduce some general results.

Lemma 2.3 *Let G be a graph of order n, let* $w \in V(G)$ *with* $d(w) = n - 1$, *and let us write* $G^* = G - w$. *If H is a graph obtained from G by deleting all but one of the edges which are incident to w, then* $\tau(G, H, \lambda) \geq 0$ *is equivalent to* $\lambda(\lambda - 2)(P(G^*, \lambda - 1))^2 \geq (\lambda - 1)^2 P(G^*, \lambda)P(G^*, \lambda - 2)$.

Proof By the definition of $\tau(G, H, \lambda)$, we have

$$\tau(G, H, \lambda) = P(G, \lambda)P(H, \lambda - 1) - P(G, \lambda - 1)P(H, \lambda). \tag{8}$$

By the equality $d(w) = n - 1$, one has that

$$P(G, \lambda) = \lambda P(G^*, \lambda - 1). \tag{9}$$

And it is evident that

$$P(H, \lambda) = (\lambda - 1)P(G^*, \lambda). \tag{10}$$

Combining (9) and (10) with (8), one may find that

$$\tau(G,H,\lambda) = \lambda(\lambda-2)(P(G^*,\lambda-1))^2 - (\lambda-1)^2 P(G^*,\lambda)P(G^*,\lambda-2). \quad (11)$$

Hence $\tau(G,H,\lambda) \geq 0$ iff $\lambda(\lambda-2)(P(G^*,\lambda-1))^2 \geq (\lambda-1)^2 P(G^*,\lambda)P(G^*,\lambda-2)$. This completes the proof of the theorem. □

By Lemma 2.3 and Lemma 2.1, we have the following

Corollary 2.1 *Let G be a graph of order n* $(n \geq 2)$ *and let* $w \in V(G)$ *with* $d(w) = n-1$. *If H is a graph obtained from G by deleting all but one of the edges which are incident to w, then* $\mu(G) \geq \mu(H)$ *iff* $n(n-2)(P(G^*,n-1))^2 \geq (n-1)^2 P(G^*,n)P(G^*,n-2)$, *where* $G^* = G - w$.

Remark 1 In the early 1970's Welsh and later, independently, Brenti [3] proposed a conjecture that for all $\lambda \in N$ and all graphs G, $(P(G,\lambda))^2 \geq P(G,\lambda+1)P(G,\lambda-1)$. But a counterexample was found by Seymour [10]. Although in general the conjecture is not true, Dong et al. [6] proposed another conjecture as follows :

Let G be a graph of order n. For $\lambda \in \mathbb{R}$ with $\lambda \geq n-1$,

$$(P(G,\lambda))^2 \geq P(G,\lambda+1)P(G,\lambda-1).$$

This conjecture remains open. Obviously, Corollary 2.1 is closely related to it. If this conjecture is not true, then $\mu(G) < \mu(H)$. And this leads to Conjecture 1 not being established.

The wheel of order n, denoted by W_n, is defined as $W_n = C_{n-1} + K_1$ (W_n is the join of C_{n-1} and K_1). For any vertex x in W_n $(n \geq 4)$, it is clear that $d(x) \geq 3$. The following result shows that Conjecture 1 is true for W_n.

Theorem 2.3 *For any wheel graph* W_n $(n \geq 4)$ *and a vertex w in* W_n, *if H is a graph obtained from* W_n *by deleting all but one of the edges which are incident to w, then* $\mu(W_n) \geq \mu(H)$.

Proof Let W_n be the wheel of order n $(n \geq 4)$ and w be a vertex in W_n. Now assume that H is a graph obtained from W_n by deleting all but one of the edges which are incident to w. The vertex w may be divided into the following two cases.

Case 1. $d(w) = n-1$, namely, w lies in the center of W_n.

Let us write $W_n^* = W_n - w$. By Corollary 2.1, we only need to check that $n(n-2)(P(W_n^*,n-1))^2 - (n-1)^2 P(W_n^*,n)P(W_n^*,n-2) \geq 0$.

According to the definition of chromatic polynomial of a graph, we have

$$P\left(W_n^*,\lambda\right) = P(C_{n-1},\lambda) = (\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1).$$

Thus,

$$\begin{aligned}&n(n-2)\left(P\left(W_n^*,n-1\right)\right)^2-(n-1)^2P\left(W_n^*,n\right)P\left(W_n^*,n-2\right)\\&\quad=n(n-2)\left[(n-2)^{n-1}+(-1)^{n-1}(n-2)\right]^2-(n-1)^2\left[(n-1)^{n-1}+(-1)^{n-1}(n-1)\right]\\&\qquad\times\left[(n-3)^{n-1}+(-1)^{n-1}(n-3)\right].\end{aligned}\tag{12}$$

By the parity of n, we divide into the following two subcases.

Case 1.1. n is even.

By (12), it is easy to verify that $n(n-2)(P(W_n^*,n-1))^2-(n-1)^2P(W_n^*,n)P(W_n^*,n-2)>0$ for $n=4$.

Now suppose that $n\geq 6$.

By (12) we have

$$\begin{aligned}&n(n-2)\left(P\left(W_n^*,n-1\right)\right)^2-(n-1)^2P\left(W_n^*,n\right)P\left(W_n^*,n-2\right)\\&\quad=n(n-2)\left[(n-2)^{n-1}-(n-2)\right]^2-(n-1)^2\left[(n-1)^{n-1}-(n-1)\right]\\&\qquad\times\left[(n-3)^{n-1}-(n-3)\right]\\&\quad=n(n-2)(n-2)^{2(n-1)}-2n(n-2)^{n+1}-(n-1)^2[(n-1)(n-3)]^{n-1}\\&\qquad+(n-3)(n-1)^{n+1}+(n-1)^3(n-3)^{n-1}+2n-3\\&\quad=n(n-2)[(n-1)(n-3)+1]^{n-1}-2n(n-2)^{n+1}-(n-1)^2[(n-1)(n-3)]^{n-1}\\&\qquad+(n-3)(n-1)^{n+1}+(n-1)^3(n-3)^{n-1}+2n-3\\&\quad>(n-1)^n(n-3)^{n-1}-2n(n-2)^{n+1}+(n-3)(n-1)^{n+1}\\&\quad>(n-1)^n(n-3)^{n-1}-2n(n-2)^{n+1}+(n-3)(2n+1)(n-2)^n\\&\quad>(n-2)^n(n-3)^{n-1}-(n+3)(n-2)^n\\&\quad>2(n-3)^2(n-2)^n-(n+3)(n-2)^n=(n-2)^n\left(2n^2-13n+15\right)>0,\end{aligned}$$

where the second inequality holds, as

$$\begin{aligned}(n-1)^{n+1}&=(n-2)^{n+1}+\binom{n+1}{1}(n-2)^n+\binom{n+1}{2}(n-2)^{n-1}+\cdots\\&>(n-2)^{n+1}+(n+1)(n-2)^n+2(n-2)^n.\end{aligned}$$

Case 1.2. n is odd.

By (12) we obtain

$$
\begin{aligned}
&n(n-2)\left(P\left(W_n^*,n-1\right)\right)^2-(n-1)^2P\left(W_n^*,n\right)P\left(W_n^*,n-2\right)\\
&=n(n-2)\left[(n-2)^{n-1}+(n-2)\right]^2-(n-1)^2\left[(n-1)^{n-1}+(n-1)\right]\\
&\quad\times\left[(n-3)^{n-1}+(n-3)\right]\\
&=n(n-2)(n-2)^{2(n-1)}+2n(n-2)^{n+1}-(n-1)^2[(n-1)(n-3)]^{n-1}\\
&\quad-(n-3)(n-1)^{n+1}-(n-1)^3(n-3)^{n-1}+2n-3\\
&>n(n-2)[(n-1)(n-3)+1]^{n-1}+4n(n-3)^{n+1}-(n-1)^2[(n-1)(n-3)]^{n-1}\\
&\quad-(n-3)(n-1)^{n+1}-(n-1)^3(n-3)^{n-1}\\
&>n(n-2)[(n-1)(n-3)+1]^{n-1}+\left[4(n-3)^2-(n-1)^2\right](n-1)(n-3)^{n-1}\\
&\quad-(n-1)^2[(n-1)(n-3)]^{n-1}-(n-3)(n-1)^{n+1}\\
&>n(n-2)(n-1)^{n-1}(n-3)^{n-2}-[(n-1)(n-3)]^{n-1}-(n-3)(n-1)^{n+1}\\
&>(n-1)^n(n-3)^{n-1}-(n-3)(n-1)^{n+1}=(n-1)^n(n-3)\left[(n-3)^{n-2}-(n-1)\right]\\
&>(n-1)^n(n-3)\left[(n-3)^2-(n-1)\right]=(n-1)^n(n-3)\left(n^2-7n+10\right)\geq 0,
\end{aligned}
$$

where the third inequality holds, as

$$
\begin{aligned}
[(n-1)(n-3)+1]^{n-1}&=[(n-1)(n-3)]^{n-1}+\binom{n-1}{1}[(n-1)(n-3)]^{n-2}+\cdots\\
&>[(n-1)(n-3)]^{n-1}+(n-1)^{n-1}(n-3)^{n-2}
\end{aligned}
$$

and $4(n-3)^2-(n-1)^2=3n^2-22n+35\geq 0$.

Case 2. $d(w)=3$, namely, w lies in the rim of W_n.

Since $d(w)=3$, it follows that

$$
P(H,\lambda)=(\lambda-1)P(W_n-w,\lambda)=\lambda(\lambda-1)^2(\lambda-2)^{n-3}. \tag{13}
$$

And it is clear that

$$
P(W_n,\lambda)=\lambda P(C_{n-1},\lambda-1)=\lambda\left[(\lambda-2)^{n-1}+(-1)^{n-1}(\lambda-2)\right]. \tag{14}
$$

Thus, by (4), (13) and (14), we have

$$
\begin{aligned}
\tau(W_n,H,\lambda)=&P(W_n,\lambda)P(H,\lambda-1)-P(W_n,\lambda-1)P(H,\lambda)\\
=&\lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^{n-3}\left[(\lambda-2)^{n-1}+(-1)^{n-1}(\lambda-2)\right]\\
&-\lambda(\lambda-1)^3(\lambda-2)^{n-3}\left[(\lambda-3)^{n-1}+(-1)^{n-1}(\lambda-3)\right].
\end{aligned} \tag{15}
$$

According to the parity of n, we can divide into the following two subcases.

Case 2.1. n is even.

By (15) we get

$$\begin{aligned}
\tau(W_n,H,n) &= n(n-1)(n-2)^2(n-3)^{n-3}\left[(n-2)^{n-1}-(n-2)\right]\\
&\quad -n(n-1)^3(n-2)^{n-3}\left[(n-3)^{n-1}-(n-3)\right]\\
&= n(n-1)(n-2)^{n-3}(n-3)^{n-3}\left[(n-2)^4-(n-1)^2(n-3)^2\right]\\
&\quad -n(n-1)(n-2)^3(n-3)^{n-3}+n(n-3)(n-1)^3(n-2)^{n-3}\\
&= n(n-1)(n-2)^{n-3}(n-3)^{n-3}\left(2n^2-8n+7\right)\\
&\quad -n(n-1)(n-2)^3(n-3)^{n-3}+n(n-3)(n-1)^3(n-2)^{n-3}\\
&\geq n(n-1)(n-2)(n-3)^{n-3}\left(2n^2-8n+7\right)\\
&\quad -n(n-1)(n-2)^3(n-3)^{n-3}+n(n-3)(n-1)^3(n-2)^{n-3}\\
&= n(n-1)(n-2)(n-3)^{n-3}\left(n^2-4n+3\right)+n(n-3)(n-1)^3(n-2)^{n-3}\\
&\quad >0.
\end{aligned}$$

Case 2.2. n is odd.

By (15) we have

$$\begin{aligned}
\tau(W_n,H,n) &= n(n-1)(n-2)^2(n-3)^{n-3}\left[(n-2)^{n-1}+(n-2)\right]\\
&\quad -n(n-1)^3(n-2)^{n-3}\left[(n-3)^{n-1}+(n-3)\right]\\
&= n(n-1)(n-2)^{n-3}(n-3)^{n-3}\left[(n-2)^4-(n-1)^2(n-3)^2\right]\\
&\quad +n(n-1)(n-2)^3(n-3)^{n-3}-n(n-3)(n-1)^3(n-2)^{n-3}\\
&= n(n-1)(n-2)^{n-3}(n-3)^{n-3}\left(2n^2-8n+7\right)\\
&\quad +n(n-1)(n-2)^3(n-3)^{n-3}-n(n-3)(n-1)^3(n-2)^{n-3}\\
&> n(n-1)(n-3)(n-2)^{n-3}\left(2n^2-8n+7\right)\\
&\quad +n(n-1)(n-2)^3(n-3)^{n-3}-n(n-3)(n-1)^3(n-2)^{n-3}\\
&= n(n-1)(n-3)(n-2)^{n-3}\left(n^2-6n+6\right)+n(n-1)(n-2)^3(n-3)^{n-3}\\
&> 0.
\end{aligned}$$

Thus, for $d(w)=3$, by Lemma 2.1, we have

$$\mu(W_n) > \mu(H).$$

This completes the proof of the theorem. □

3 The Second Conjecture

For two disjoint graphs G and H, let $G \cup H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For any graph H and positive integer m, let $H \cup mK_1$ be the graph obtained from H by adding m new vertices and no new edges.

Now we present the first result of this section.

Theorem 3.1 *For any graph G and a simplicial vertex w in G, $\mu(G) \geq \mu((G-w) \cup K_1)$.*

Proof If $d(w) = 0$, it is clear that the inequality holds.

Now assume that $d(w) \geq 1$. As w is a simplicial vertex in G, we have

$$P(G, \lambda) = (\lambda - d(w))P(G-w, \lambda). \tag{16}$$

Let $H = (G-w) \cup K_1$. Then

$$P(H, \lambda) = \lambda P(G-w, \lambda). \tag{17}$$

For $\lambda \geq n-1$, by (4), (16) and (17), one has that

$$\begin{aligned}\tau(G, H, \lambda) &= (\lambda-1)(\lambda-d(w))P(G-w,\lambda)P(G-w,\lambda-1) \\ &\quad - \lambda(\lambda-d(w)-1)P(G-w,\lambda)P(G-w,\lambda-1) \\ &= d(w)P(G-w,\lambda)P(G-w,\lambda-1) \geq 0.\end{aligned}$$

Thus, by Lemma 2.1, $\mu(G) \geq \mu(H) = \mu((G-w) \cup K_1)$. □

By the proof of Theorem 3.1, one may find that if w is a simplicial vertex in G with $d(w) \geq 1$ then $\mu(G) > \mu((G-w) \cup K_1)$.

Corollary 3.1 *Let G be a graph and w be a simplicial vertex in G with $d(w) \geq 1$. If H is a subgraph of G which is obtained from G by deleting all edges adjacent to w. Then $\mu(G) > \mu(H)$.*

Proof Let H be a subgraph of G which is obtained from G by deleting all edges adjacent to w. It is obvious that H is a spanning subgraph of G and $H \cong (G-w) \cup K_1$. By the discussion above, this corollary follows immediately. □

On the basis of Theorem 3.1, we can obtain a more general result as follows:

Corollary 3.2 *Let G be any graph and $w_1, w_2, \ldots, w_l$ be all simplicial vertices in G. Then $\mu(G) \geq \mu((G - \cup_{i=1}^{t} w_i) \cup tK_1)$ $(1 \leq t \leq l)$.*

Proof As w_t $(2 \leq t \leq l)$ is a simplicial vertex in G, w_t is a simplicial vertex in $(G - \cup_{i=1}^{t-1} w_i)$. Moreover, it is also a simplicial vertex in $(G - \cup_{i=1}^{t-1} w_i) \cup (t-1)K_1$.

By Theorem 3.1, we have $\mu((G - \cup_{i=1}^{t-1} w_i) \cup (t-1)K_1) \geq \mu(G - w_t - \cup_{i=1}^{t-1} w_i) \cup (t-1)K_1 \cup K_1) = \mu((G - \cup_{i=1}^{t} w_i) \cup tK_1)$. It follows that $\mu((G-w_1) \cup K_1) \geq \mu((G-(w_1 \cup w_2)) \cup 2K_1) \geq \cdots \geq \mu((G - \cup_{i=1}^{t} w_i) \cup tK_1)$. Observe that w_1 is a simplicial vertex in G too, by Theorem 3.1, we have $\mu(G) \geq \mu((G-w_1) \cup K_1)$. This implies the theorem holds. □

Similarly, we have the following

Corollary 3.3 *Let G be any graph and $w_1, w_2, \ldots, w_s$ be all simplicial vertices in G with $d(w_i) \geq 1$ $(1 \leq i \leq s)$. If H_j $(1 \leq j \leq s)$ is a subgraph of G which is obtained* *from G by sequentially deleting all edges adjacent to* w_i $(1 \le i \le j)$, *then* $\mu(G) > \mu(H_j)$.

Theorem 3.2 *Let G be a graph of order* $n\ (n \ge 2)$ *and* $w \in V(G)$ *with* $d(w) = n-1$. *Assume that* $H = (G-w) \cup K_1$ *and write* $G^* = G - w$. *Then for* $\lambda \ge 1, \tau(G,H,\lambda) \ge 0$ *iff* $(P(G^*, \lambda-1))^2 \ge P(G^*,\lambda)P(G^*,\lambda-2)$.

Proof By the definition of $\tau(G,H,\lambda)$, we have

$$\tau(G,H,\lambda) = P(G,\lambda)P(H,\lambda-1) - P(G,\lambda-1)P(H,\lambda). \tag{18}$$

Since $d(w) = n-1$, one may deduce that

$$P(G,\lambda) = \lambda P(G^*, \lambda-1). \tag{19}$$

And it is clear that

$$P(H,\lambda) = \lambda P(G^*,\lambda). \tag{20}$$

By substituting (19) and (20) into (18), one may find that

$$\tau(G,H,\lambda) = \lambda(\lambda-1)\left[(P(G^*,\lambda-1))^2 - P(G^*,\lambda)P(G^*,\lambda-2)\right].$$

Thus, for $\lambda \ge 1$, $\tau(G,H,\lambda) \ge 0$ iff $(P(G^*,\lambda-1))^2 - P(G^*,\lambda)P(G^*,\lambda-2) \ge 0$, namely, $(P(G^*,\lambda-1))^2 \ge P(G^*,\lambda)P(G^*,\lambda-2)$. This completes the proof of the theorem. □

By Theorem 3.2 and Lemma 2.1, we get another result on mean color numbers.

Theorem 3.3 *Let G be a graph of order n* $(n \ge 2)$, $w \in V(G)$ *with* $d(w) = n-1$ *and* $G^* = G - w$. *Then* $\mu(G) \ge \mu((G-w) \cup K_1)$ *iff* $(P(G^*, n-1))^2 \ge P(G^*,n)P(G^*,n-2)$.

Remark 2 It is evident that Theorem 3.3 is also related to Dong's conjecture in Remark 1. Thus, if the conjecture is true, then $\mu(G) \ge \mu((G-w) \cup K_1)$; otherwise, $\mu(G) < \mu((G-w) \cup K_1)$. This leads to Conjecture 2 not being established.

In what follows we introduce an known inequality on chromatic polynomials of graphs.

Lemma 3.1 ([7]) *Let G be a connected* (n, m)*-graph. If* $\lambda \in \mathbb{R}$ *and* $\lambda \ge \max\{n-1, \sqrt{2}(m-n+2.5)\}$, *then*

$$(P(G,\lambda))^2 \ge P(G,\lambda+1)P(G,\lambda-1). \tag{21}$$

Theorem 3.4 *Suppose that G is a connected* (n, m)*-graph and that w is a vertex such that* $d(w) = n-1$ *and w is not a cut vertex of G. If* $m \le (\frac{\sqrt{2}}{2}+2)n - 4.5 - \sqrt{2}$, *then*

$$\mu(G) \geq \mu((G-w) \cup K_1). \tag{22}$$

Proof Let $G^* = G - w$. As G is a connected graph and w is not a cut vertex in G, G^* is a connected graph too. It is clear that $|V(G^*)| = n-1$ and $|E(G^*)| = m-n+1$. Hence G^* is a connected $(n-1, m-n+1)$-graph.

By the inequality $m \leq (\frac{\sqrt{2}}{2}+2)n - 4.5 - \sqrt{2}$, we have

$$(n-1) - 1 = n-2 \geq \sqrt{2}(m-2n+4.5) = \sqrt{2}[(m-n+1)-(n-1)+2.5]. \tag{23}$$

Then, by (23) and Lemma 3.1, it follows that

$$(P(G^*, n-1))^2 \geq P(G^*, n)P(G^*, n-2). \tag{24}$$

Thus, by (24) and Theorem 3.3, the theorem holds. □

By Theorem 3.4, we have the following

Corollary 3.4 *Suppose that G is a 2-connected (n, m)-graph and w is any vertex in G with $d(w) = n-1$. If $m \leq (\frac{\sqrt{2}}{2}+2)n - 4.5 - \sqrt{2}$, then*

$$\mu(G) \geq \mu((G-w) \cup K_1). \tag{25}$$

Theorem 3.5 *For any wheel graph W_n $(n \geq 4)$ and any vertex x in W_n, one has $\mu(W_n) \geq \mu((W_n - x) \cup K_1)$.*

Proof Let W_n $(n \geq 4)$ be the wheel of order n and x be a vertex in W_n. The vertex x may be divided into the following two cases.

Case 1. $d(x) = n-1$, namely, x lies in the center of W_n.

It is clear that W_n is a connected $(n, 2n-2)$ graph and x is not a cut vertex in W_n. By Theorem 3.4, one may deduce that $\mu(W_n) \geq \mu((W_n - x) \cup K_1)$ for $n \geq 6$. And it is easy to verify that the inequality also holds for $n = 4, 5$.

Case 2. $d(x) = 3$, namely, x lies in the rim of W_n.

Let $H = (W_n - x) \cup K_1$. Since $d(x) = 3$, it follows that

$$P(H, \lambda) = \lambda P(W_n - x, \lambda) = \lambda^2(\lambda-1)(\lambda-2)^{n-3}. \tag{26}$$

And it is clear that

$$P(W_n, \lambda) = \lambda P(C_{n-1}, \lambda - 1) = \lambda\left[(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\right]. \tag{27}$$

Thus, by (4), (26) and (27), we have

$$\begin{aligned}\tau(W_n,H,\lambda) &= P(W_n,\lambda)P(H,\lambda-1)-P(W_n,\lambda-1)P(H,\lambda)\\ &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-3)^{n-3}\Big[(\lambda-2)^{n-1}+(-1)^{n-1}(\lambda-2)\Big]\\ &\quad -\lambda(\lambda-2)^{n-4}\Big[(\lambda-3)^{n-1}+(-1)^{n-1}(\lambda-3)\Big]\Big\}.\end{aligned} \tag{28}$$

According to the parity of n, we can divide into the following two subcases.

Case 2.1. n is even.

For $\lambda \geq 4$, by (28), we obtain

$$\begin{aligned}\tau(W_n,H,\lambda) &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-3)^{n-3}\Big[(\lambda-2)^{n-1}-(\lambda-2)\Big]\\ &\quad -\lambda(\lambda-2)^{n-4}\Big[(\lambda-3)^{n-1}-(\lambda-3)\Big]\Big\}\\ &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-2)^{n-4}(\lambda-3)^{n-3}\Big[(\lambda-2)^3-\lambda(\lambda-3)^2\Big]\\ &\quad +\lambda(\lambda-3)(\lambda-2)^{n-4}-(\lambda-2)(\lambda-3)^{n-3}\Big\}\\ &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(3\lambda-8)(\lambda-2)^{n-4}(\lambda-3)^{n-3}\\ &\quad +\lambda(\lambda-3)(\lambda-2)^{n-4}-(\lambda-2)(\lambda-3)^{n-3}\Big\}\\ &\geq \lambda(\lambda-2)(\lambda-1)^2\Big\{(3\lambda-8)(\lambda-2)^{n-4}(\lambda-3)^{n-3}\Big\}\\ &> 0.\end{aligned} \tag{29}$$

Case 2.2. n is odd.

For $\lambda \geq 4$, by (28), we have

$$\begin{aligned}\tau(W_n,H,\lambda) &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-3)^{n-3}\Big[(\lambda-2)^{n-1}+(\lambda-2)\Big]\\ &\quad -\lambda(\lambda-2)^{n-4}\Big[(\lambda-3)^{n-1}+(\lambda-3)\Big]\Big\}\\ &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-2)^{n-4}(\lambda-3)^{n-3}\Big[(\lambda-2)^3-\lambda(\lambda-3)^2\Big]\\ &\quad +(\lambda-2)(\lambda-3)^{n-3}-\lambda(\lambda-3)(\lambda-2)^{n-4}\Big\}\\ &= \lambda(\lambda-2)(\lambda-1)^2\Big\{(3\lambda-8)(\lambda-2)^{n-4}(\lambda-3)^{n-3}\\ &\quad +(\lambda-2)(\lambda-3)^{n-3}-\lambda(\lambda-3)(\lambda-2)^{n-4}\Big\}\\ &\geq \lambda(\lambda-2)(\lambda-1)^2\Big\{(\lambda-2)(\lambda-3)^{n-3}\Big\}\\ &> 0.\end{aligned} \tag{30}$$

Combining (29) with (30), we have

$$\tau(W_n, H, \lambda) > 0$$

for $\lambda \geq 4$. Hence $\tau(W_n, H, n) > 0$ $(n \geq 4)$. By Lemma 2.1, we have

$$\mu(W_n) > \mu(H) = \mu((W_n - x) \cup K_1).$$

Thus, the theorem holds. □

Remark 3 Let G be a chordal graph or 2-tree and H be a subgraph of G. By the results in [4, 5], $\mu(G) \geq \mu(H)$. It is clear that Conjecture 1 holds for a chordal graph or 2-tree G. In addition, for a vertex w in G with $d(w) \geq 1$, if H is a subgraph obtained from G by deleting all the edges which are incident to w, then $H \cong (G - w) \cup K_1$. Therefore $\mu(G) \geq \mu(H) = \mu((G - w) \cup K_1)$. It means that Conjecture 2 also holds for a chordal graph or 2-tree G.

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