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The Hitting Times of Random Walks on Bicyclic Graphs

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Abstract

Let $H_G(x, y)$ be the expected hitting time from vertex x to vertex y for the first time on a simple connected graph G and $\varphi(G) := \max\{H_G(x, y) : x, y \in V(G)\}$ be called the hitting time of G. In this paper, sharp upper and lower bounds for $\varphi(G)$ among all n-vertex bicyclic graphs are presented and the extremal graphs are determined.

Keywords Hitting time · Bicyclic graph · Random walk

1 Introduction

Random walks on graphs have been extensively studied. The theory of random walk is closely related to other branches of graph theory such as spectra of graphs and electrical resistances of graphs. The hitting time, cover time, mixing time, commute time are key parameters of random walks on graphs [1, 7, 15, 16, 20, 22, 23]. Lovász [21] published an excellent survey about the connection between random walks on undirected graphs and eigenvalues, electrical networks. Doyle and Snell [12] wrote a comprehensive book on electrical networks, which is the basis for studying random walks on graphs and electrical networks. Effective resistance is a significant parameter of electrical networks, Tetali [26], Klein and Randić [18] established a intertwined relationship between effective resistance and random

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walks on graphs. Georgakopoulos and Wagner [13] exhibited a subtle connection among the hitting time, the cover cost and the Wiener index on trees. Kirkland and Neumann [19] provided an explicit formula for the hitting times on graphs while there is a cut vertex in the graph from the perspective of matrix theory [17, 25]. Cogill and Peng [11] gave a method for computing an upper bound on hitting time from spanning tree of the graph. Moreover, spanning tree method is a crucial measure to study the hitting times on graphs. Chung and Yau [10] proved the explicit formula of hitting time on the basis of the discrete Green's function. Chang and Xu [6, 27, 28] derived some new explicit formulas of hitting times for random walks on graphs in terms of enumerations of spanning trees and the discrete Green's function. From the perspective of graph structure, Chen and Zhang [8, 9] and Huang and Li [14] gave explicit formulas of the hitting times for simple random walks on special graphs such as trees, unicyclic graphs, subdivision and triangulation graphs, quadrilateral graphs.

Furthermore, Brightwell and Winkler [5] proved that the n-vertex lollipop graph G and vertices x and y for which $H_G(x, y)$ is maximized among all n-vertex graphs. The extremal graph consists of a clique on $\left\lfloor \frac{(2n+1)}{3} \right\rfloor$ $\left(\text{or} \left\lfloor \frac{(2n-2)}{3} \right\rfloor \right)$ vertices, including x, to which a path on the remaining vertices, ending in y, has been attached. Then $\varphi(G)$ in this graph is approximately $\frac{4n^3}{27}$. Thus, if vertex number of graph is fixed, then the extremal graphs are determined. The results motivate us to propose the following general problem considering the fixed vertex number and edge number.

Question 1.1 Let $\mathcal{G}_{n,m}$ be set of all simple connected graphs with n vertices and m edges. What are the sharp upper and lower bounds for hitting times of graphs in $\mathcal{G}_{n,m}$. Furthermore, how to characterise all extremal graphs which attain these values?

Zhang and Li [29] determined the sharp upper and lower bounds on $\varphi(G)$ for $G \in \mathcal{G}_{n,n-1}$, and they showed that the n-vertex star S_n attains the lower bound, and the n-vertex path P_n attains the upper bound. Moreover, Zhu and Zhang [30] gave the sharp upper and lower bounds for hitting times of graphs in $\mathcal{G}_{n,n}$ and determined the extremal graphs.

In this paper, we continue to make some contributions to this problem. First we study the relationship between the hitting times of graphs and the hitting times of subgraphs. The results are used to present sharp upper and lower bounds for the hitting times of n-vertex bicyclic graphs. Moreover, we will characterize all extremal graphs which attain the upper and lower bounds. The main result is stated as follows:

Theorem 1.2 *Let G be any n-vertex bicyclic connected graph. If $n \geq 12$, then*

$$2n + 3 \leq \varphi(G) \leq n^2 + 2n - 19.$$

Moreover, the left equality holds if and only if G is a graph obtained by identifying the center of $K_{1,n-5}$ and the common vertex of two triangles without sharing edge, and $\varphi(G)$ is equal to the hitting time from a vertex with degree 2 in triangle to a pendant vertex. The right equality holds if and only if G is a graph obtained by

identifying an end of a path on $(n-3)$ vertices and a vertex with degree 2 in $K_4 - e$, where $K_4 - e$ is a graph obtained by deleting one edge from complete graph K_4, and $\varphi(G)$ is equal to the hitting time from the furthest vertex from the pendant vertex to the pendant vertex.

The rest of this paper is arranged as follows. In Sect. 2, we present symbols and some key results which will be used to prove the main results. In Sect. 3, we focus on the extremal problem of hitting times of all n-vertex bicyclic graphs which composed of two cycles joined by a path and in which a (possibly empty) tree is attached to each vertex. In Sect. 4, we concentrate on the extremal problem of hitting times of all n-vertex bicyclic graphs which composed of three internally disjoint paths with common end vertices and in which a (possibly empty) tree is attached to each vertex. In Sect. 5, we give the proof of Theorem 1.2 and some remarks.

2 Preliminary

First of all, we introduce some symbols and notations. A simple random walk on a graph is a walk from any vertex of a graph to its adjacent vertices with equal probability. Let G be a simple (without loops and multiedges) connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between vertices u and v in G is denoted by $dist_G(u,v)$, or $dist(u,v)$ for short. The *expected hitting time* $H_G(x,y)$ is the expected number of steps it takes a simple random walk on a graph G from a vertex x to vertex y for the first time. If $M \subseteq V(G)$, then $H_G(x,M)$ is the expected hitting time from vertex x to reach a vertex in M for the first time. It does not matter which the vertex is, but it must be the first vertex in M to be hit. If $x \in M$, then $H_G(x,M) = 0$. The *first return time* $H_G^+(y,y)$ is the expected number of steps it takes a simple random walk on a graph G starts from vertex y and then first return to y, which is different from $H_G(y,y) = 0$. For a given graph G, the *hitting time of G* is defined to be

$$\varphi(G) = \max\{H_G(x,y) : x,y \in V(G)\}.$$

Let $\mathcal{G}_n$ be a subset of the set of all n-vertex simple connected graphs. Let $\phi(n):= \max\{\varphi(G) : G \in \mathcal{G}_n\}$ and $\psi(n):= \min\{\varphi(G) : G \in \mathcal{G}_n\}$. If there is a graph G such that $\varphi(G) = \phi(n)$ (resp. $\varphi(G) = \psi(n)$), then G is called an n-maximal (resp. n-minimal) graph for $\mathcal{G}_n$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs such that $V_1 \cap V_2 = \{x,y\}$ and $E_1 \cap E_2 = \emptyset$. The graph $G = G_1 \cup G_2$ is *union* of two graphs G_1 and G_2, i.e., defined by $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. In that case, G is said to be *decomposed into G_1 and G_2 through x and y*. G_1, G_2 are regarded as a decomposition of G through x and y. The effective resistance $R(x,y)$ between x and y can be defined in terms of the potential difference between x and y when a unit current from x to y is maintained. While G is treated as an electrical network and each edge of G is replaced with a unit resistor, $R_G(x,y)$ is the effective resistance

between x and y in G by Ohm's law. Let $N[v]$ be the induced subgraph of G induced by v and all vertices adjacent to v. Let $N(v)$ be the neighbour set of vertex v.

A bicyclic graph is a connected graph where edge number equals one plus the vertex number, its cyclomatic number is 2. It is easy to see that any bicyclic graph contains two cycles, which have either no common edges or at least one common edge. We denote the set of all connected n-vertex bicyclic graphs by $\mathcal{B}(n)$ and $\mathcal{B}(n) = \mathcal{B}_1(n) \cup \mathcal{B}_2(n)$, where $\mathcal{B}_1(n)$ is the set of all n-vertex bicyclic graphs whose cycles have no common edges and $\mathcal{B}_2(n)$ is the set of all n-vertex bicyclic graphs whose cycles have at least one common edge (see Fig. 1).

If $G \in \mathcal{B}_1(n)$, then G has two edge disjoint cycles C_p and C_q. Without loss of generality, we may assume that G can be obtained from two cycles $C_p = u_1 u_2 \ldots u_p$ and $C_q = v_1 v_2 \ldots v_q$ joined by a $u_1 v_1$-path (possibly the length of path is 0) $P = u_1 w_0 w_1 \ldots w_{m-1} v_1$ and in which trees T_{u_i} $(1 \le i \le p)$, T_{v_j} $(1 \le j \le q)$ and T_{w_k} $(0 \le k \le m-1)$ are attached to u_i, v_j and w_k, respectively. G can be denoted by $G := B(T_{u_1}, \ldots, T_{u_p}, T_{w_0}, \ldots, T_{w_{m-1}}, T_{v_1}, \ldots, T_{v_q})$ (see Fig. 1). In particular, if u_1, w_0 and v_1 are coincident, i.e., the length of P is 0, then we denote the common vertex of two cycles by u_1; if the length of P is 1, then $P = u_1 v_1$. Moreover, let $P_n^{p,q}$ be the graph consisting of two cycles C_p and C_q with a common vertex u_1 (i.e., $V(C_p) \cap V(C_q) = u_1$) and the path of length $(n + 1 - p - q)$ is attached to the vertex v_i of C_q, where the path is rooted at v_i and the distance between u_1 and v_i is $\lfloor \frac{q}{2} \rfloor$ (see Fig. 2). Let $S_n^{p,q}$ be the graph obtained by identifying the unique common vertex of C_p and C_q and center of the $(n + 2 - p - q)$-vertex star $K_{1,n+1-p-q}$ (see Fig. 2). In particular, if $p = q = 3$, then we denote $P_n^{3,3}$ by $B_1^*(n)$ and denote $S_n^{3,3}$ by $B_1^\sharp(n)$.

If $G \in \mathcal{B}_2(n)$, then there are two cycles having common edges in G, i.e., there are three internally disjoint paths with common end vertices in G. Without loss of generality, we assume that G has three internally disjoint paths $P_p := x u_1 u_2 \ldots u_p y$, $P_q := x v_1 \ldots v_q y$ and $P_m := x w_1 \ldots w_m y$ with common end vertices x, y, where $p \ge q \ge m$ and $m \ge 0$. Moreover, if $m = 0$, then $P_m = xy$. Hence, G can be obtained from P_p, P_q, P_m by attaching the trees T_{u_i} $(1 \le i \le p)$, T_{v_j} $(1 \le j \le q)$, T_{w_k} $(1 \le k \le m)$, T_x and T_y to u_i, v_j, w_k, x, y respectively (see Fig. 1). Moreover, let $S_n^{p,q,m}$ be the graph obtained from three internally disjointed paths P_p, P_q and P_m by identifying one
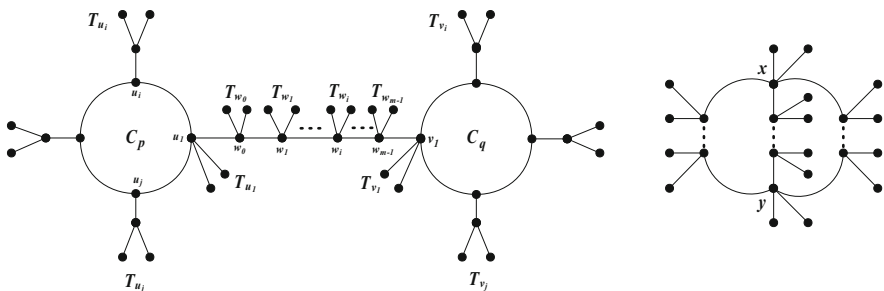

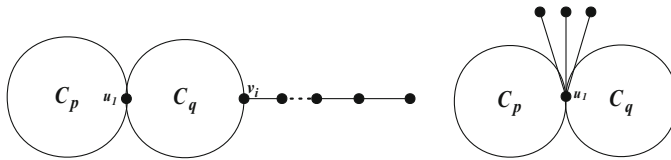**Fig. 1** Illustration for graphs in $\mathcal{B}_1(n)$ and $\mathcal{B}_2(n)$

Fig. 2 $P_n^{p,q}$ and $S_n^{p,q}$

common vertex x (or y) and center of the star $K_{1,n-2-p-q-m}$. In particular, if $p = q = 1$ and $m = 0$, then we denote $S_n^{1,1,0}$ by $B_2^{\sharp}(n)$. In addition, let $P_n^{p,q,m}$ be the graph obtained from three internally disjointed paths P_p, P_q, P_m and attached an $(n - 1 - p - q - m)$-vertex path rooted at some vertex of $V(P_p) \cup V(P_q) \cup V(P_m)$ (see Fig. 3). In particular, if the bicyclic graph obtained from three internally disjoint paths $P_p = xu_1y, P_q = xv_1y, P_m = xy$ and attached an $(n - 3)$-vertex path rooted at v_1, then we denote the graph by $B_2^*(n)$. In other words, $B_2^*(n)$ is the graph obtained by identifying a vertex with degree 2 in $K_4 - e$ and an end vertex of an $(n - 3)$-vertex path, where $K_4 - e$ is a graph obtained by deleting one edge from complete graph K_4 (see Fig. 3).

Next, we will present several important results which are useful in our proofs.

Theorem 2.1 [29] *Let T be an n-vertex tree for $n \geq 3$. Then*

$$2n - 2 \leq \varphi(T) \leq (n - 1)^2, \tag{1}$$

The left equality holds if and only if T is S_n, and the right equality holds if and only if T is P_n.

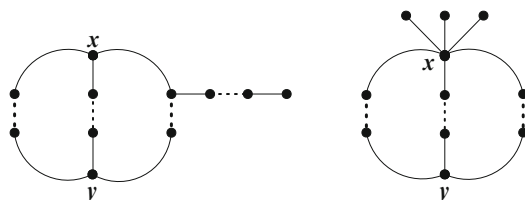**Theorem 2.2** [30] *Let G be any n-vertex unicyclic graph. Then*

$$\min\left\{\left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right), 2n + 1\right\} \leq \varphi(G) \leq n^2 - 7. \tag{2}$$

Theorem 2.3 [18] *Let G be a connected graph and $x, y \in V(G)$. If there exists a cut vertex z such that x and y are not in a same component of $G - z$, then*

$$H_G(x, y) = H_G(x, z) + H_G(z, y). \tag{3}$$

Moreover, if there exists a unique path $P = xv_1 \ldots v_{k-1}y$ in G, then

Fig. 3 $P_n^{p,q,m}$ and $S_n^{p,q,m}$

$$H_G(x, y) = H_G(x, v_1) + H_G(v_1, v_2) + \cdots + H_G(v_{k-1}, y). \tag{4}$$

Theorem 2.4 [4] *Let G be a simple connected graph on n vertices and $x, y \in V(G)$. If there exists a unique path $P = v_0 v_1 \ldots v_k$ with $v_0 = x$ and $v_k = y$, and m_i is the number of edges of subgraph G_i, where G_i is the component of $G - \{v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k\}$ containing v_i for $i = 0, \ldots, k$ (see Fig. 4), then*

$$H_G(x, y) = k^2 + 2 \sum_{i=0}^{k-1} m_i(k - i). \tag{5}$$

Theorem 2.5 [24] *Let G_1 and G_2 be decomposition of G through x and y with $V(G_1) \cap V(G_2) = \{x, y\}$ and $E(G_1) \cap E(G_2) = \emptyset$. Then*

$$H_G(x, y) = \frac{R^2(x, y)}{R^1(x, y) + R^2(x, y)} H_{G_1}(x, y) + \frac{R^1(x, y)}{R^1(x, y) + R^2(x, y)} H_{G_2}(x, y), \tag{6}$$

where $R^1(x, y)$ (resp. $R^2(x, y)$) is the effective resistance between x and y computed in graph G_1 (resp. G_2).

Lemma 2.6 [3] *Let C be a cycle of length g. Then for any $x, y \in V(C)$, we have*

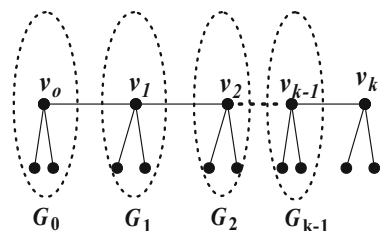$$H_C(x, y) = dist(x, y)(g - dist(x, y)). \tag{7}$$

Theorem 2.7 [26] *Let $d(z)$ be the degree of vertex $z \in V(G)$ and $e(G)$ be the edge number of G. Then*

$$H_G(x, y) = \frac{1}{2} \sum_{z \in V(G)} d(z)(R_G(x, y) + R_G(y, z) - R_G(x, z)), \tag{8}$$

and

$$H_G(x, y) + H_G(y, x) = 2e(G)R_G(x, y). \tag{9}$$

Fig. 4 The components of $G - \{v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k\}$

Lemma 2.8 [3] *Let x be a vertex in graph G. Then the first return time is*

$$H_G^+(x,x) = \frac{2e(G)}{d(x)}, \tag{10}$$

where $e(G)$ is the edge number of graph G.

Lemma 2.9 *Let G be a simple connected graph. If there exist two vertices $x, y \in V(G)$ such that $\varphi(G) = H_G(x,y)$, then x and y are not cut vertices.*

Proof Suppose that y is a cut vertex. By (3) of Theorem 2.3, there exists a vertex $z \in N(y)$ such that $H_G(x,z) = H_G(x,y) + H_G(y,z) > H_G(x,y)$. Therefore, $H_G(x,y)$ can not be maximum which leads to a contradiction. Similarly, we can prove that x is not a cut vertex. ☐

Lemma 2.10 *Let x and y be the vertices of graph G and w be a vertex in N[y] such that $\max_{v \in N(y)} H_G(v,y) = H_G(w,y)$ and $d_{N[y]}(w) = t+1$, where $d_{N[y]}(w)$ denotes the degree of vertex w in N[y]. Let N be the vertex set of N[y] and G/N be the quotient graph obtained from G by contracting N to a single vertex (also denoted N) and identifying any resulting multiple edges.*

1. *If $N(y) \cap N(w) = \emptyset$, then $H_G(x,y) \leq H_{G/N}(x,N) + 1 + 2(e(G) - e(N[y]))$;*
2. *If $N(y) \cap N(w) \neq \emptyset$, then $H_G(x,y) \leq H_{G/N}(x,N) + \frac{4e(G) - 2e(N[y]) - d(y) + t + 1}{t+2}$,*

where $e(G)$ (resp. $e(N[y])$) denotes the edge number of G (resp. $N[y]$).

Proof First of all, if $u \in N$, then we give definitions of $H(x,N)$, $H'(w,N)$, $p(x,u)$, $p(w,w)$ and $p(w,u)$.

- $H(x,N)$: hitting time for a random walk from x to first hit N.
- $H'(w,N)$: hitting time for a random walk from vertex w to first hit N, given that no edge inside N is used (i.e., the walk starts out by leaving N).
- $p(x,u)$: probability that the random walk starting from vertex x to u before any other vertex of N, given that no edge inside N is used.
- $p(w,w)$: probability that the random walk starting from vertex w to w before any other vertex of N, given that no edge inside N is used.
- $p(w,u)$: probability that the random walk starting from vertex w to u before any other vertex of N, given that no edge inside N is used.

By above definitions, we can get

$$H(x,y) = H(x,N) + \sum_{u \in N(y)} p(x,u)H(u,y). \tag{11}$$

Suppose that w sends k edges out of $N[y]$. Let $E(N[y])$ be the edge set of all edges in $N[y]$ and $G - E(N[y])$ be the graph removed all the edges inside $N[y]$ from G. Let G' be the component of $G - E(N[y])$ containing vertex w.

(1) If $N(y) \cap N(w) = \emptyset$, then

$$H_G(w,y) = \frac{1}{k+1} + \frac{k}{k+1}\left(H'_{G'}(w,N) + \sum_{u\in N(y)} p(w,u)H_G(u,y)\right).$$

Since $\quad H'_{G'}(w,N) + \max_{v\in N} H_G(v,y) \le H^+_{G'}(w,w) + H_G(w,y)\quad$ and $\sum_{u\in N(y)} p(w,u) \le 1$, by Lemma 2.8,

$$H^+_{G'}(w,w) = \frac{2e(G')}{k} \le \frac{2(e(G) - e(N[y]))}{k},$$

we have

$$\begin{aligned}
H_G(w,y) &\le \frac{1}{k+1} + \frac{k}{k+1}\left(H'_{G'}(w,N) + \sum_{u\in N(y)} p(w,u)H_G(w,y)\right) \\
&\le \frac{1}{k+1} + \frac{k}{k+1}\left(H^+_{G'}(w,w) + H_G(w,y)\right) \\
&= \frac{1}{k+1}\left(1 + kH^+_{G'}(w,w) + kH_G(w,y)\right) \\
&= \frac{1}{k+1}\left(1 + 2(e(G) - e(N[y])) + kH_G(w,y)\right).
\end{aligned}$$

Thus,

$$H_G(w,y) \le 1 + 2(e(G) - e(N[y])).$$

Further, $H_G(x,N) \le H_{G/N}(x,N)$, since that if multiple edges were kept when collapsing $N[y]$ to a single vertex, the two expected hitting times would be the same. Removing duplicated edges can only make the collapsed N harder to hit. Therefore, by Eq. (11) and $\sum_{u\in N(y)} p(x,u) \le 1$, we have

$$\begin{aligned}
H_G(x,y) &\le H_G(x,N) + \sum_{u\in N(y)} p(x,u)H(w,y) \\
&\le H_{G/N}(x,N) + H_G(w,y) \\
&\le H_{G/N}(x,N) + 1 + 2(e(G) - e(N[y])).
\end{aligned}$$

(2) If $N(y) = \{w, u_1, \ldots, u_t, v_1, \ldots, v_s\}$ and $N(w) \cap N(y) = \{u_1, \ldots, u_t\} \ne \emptyset$, then

$$\begin{aligned}
H_G(w,y) = {}&\frac{1}{k+t+1} + \frac{1}{k+t+1}\left(\sum_{i=1}^{t}(1 + H_G(u_i,y))\right) \\
&+ \frac{k}{k+t+1}\left(H'_{G'}(w,N) + \sum_{v\in N(y)} p(w,v)H_G(v,y)\right).
\end{aligned}$$

Since $H^+_G(y,y) = \frac{2e(G)}{d(y)}$ and

$$H_G^+(y,y) = 1 + \frac{1}{d(y)} \sum_{u \in N(y)} H_G(u,y)$$

$$= 1 + \frac{1}{d(y)} H_G(w,y) + \frac{1}{d(y)} \sum_{i=1}^{t} H_G(u_i,y) + \frac{1}{d(y)} \sum_{j=1}^{s} H_G(v_j,y),$$

we have

$$\sum_{i=1}^{t}(1 + H_G(u_i,y)) = 2e(G) - d(y) - \sum_{j=1}^{s} H_G(v_j,y) - H_G(w,y) + t$$

$$\leq 2e(G) - d(y) - H_G(w,y) + t.$$

Moreover, $H'_{G'}(w,N) + \max_{v \in N} H_G(v,y) \leq H_{G'}^+(w,w) + H_G(w,y)$, by Lemma 2.8,

$$H_{G'}^+(w,w) = \frac{2e(G')}{k} \leq \frac{2(e(G) - e(N[y]))}{k},$$

we have

$$H_G(w,y) \leq \frac{1}{k+t+1} + \frac{1}{k+t+1}\left(\sum_{i=1}^{t}(1 + H_G(u_i,y))\right)$$

$$+ \frac{k}{k+t+1}(H_{G'}^+(w,w) + H_G(w,y))$$

$$\leq \frac{1}{k+t+1}(1 + 2e(G) - d(y) - H_G(w,y) + t + kH_{G'}^+(w,w) + kH_G(w,y))$$

$$= \frac{1}{k+t+1}(4e(G) - 2e(N[y]) - d(y) + t + 1 + (k-1)H_G(w,y)).$$

Therefore,

$$H_G(w,y) \leq \frac{4e(G) - 2e(N[y]) - d(y) + t + 1}{t+2}.$$

Since $H_G(x,y) \leq H_{G/N}(x,N) + H_G(w,y)$, we have

$$H_G(x,y) \leq H_{G/N}(x,N) + \frac{4e(G) - 2e(N[y]) - d(y) + t + 1}{t+2}.$$

$\square$

3 The Extremal Hitting Times on Bicyclic Graphs in $\mathcal{B}_1(n)$

In this section, we will prove that the graph $B_1^*(n)$ is the n-maximal graph and $B_1^\sharp(n)$ (see Fig. 5) is the n-minimal graph among $\mathcal{B}_1(n)$.

Theorem 3.1 *Let $G \in \mathcal{B}_1(n)$ and $f(n) = n^2 + 2n - 27$. Then*

Fig. 5 $B_1^*(n)$ and $B_1^\sharp(n)$

$$\varphi(G) \leq f(n).$$

Moreover, equality holds if and only if $G = B_1^*(n)$ and $\varphi(B_1^*(n)) = H_{B_1^*(n)}(x,y)$ is the hitting time from the vertex x in cycle with degree 2 to the pendant vertex y with $dist(x,y) = n - 3$.

Proof   Let $\phi_1(n) = \max\{\varphi(G) : G \in \mathcal{B}_1(n)\}$. We need to prove that $\phi_1(n) = f(n)$, and if G is any graph in $\mathcal{B}_1(n)$ with $\varphi(G) = f(n)$, then $G = B_1^*(n)$ and $H_{B_1^*(n)}(x,y) = \varphi(G)$, where x is a vertex in cycle with degree 2 and y is the pendant vertex such that $dist(x,y) = n - 3$. We use the induction on n to prove it.

If $n = 5$, then $\mathcal{B}_1(5)$ consists of only one graph G_5^1 which is the 5-vertex bicyclic graph obtained from two triangles sharing one common vertex (see Fig. 6). By simple computations, $\phi_1(5) = \varphi(G_5^1) = H_{G_5^1}(v_1, v_5) = 8 = f(5)$ and $dist(v_1, v_5) = 2$.

If $n = 6$, then $\mathcal{B}_1(6)$ consists of four graphs $G_6^1, G_6^2, G_6^3, G_6^4$ (see Fig. 7). By simple computations, we see that $\varphi(G_6^1) = H_{G_6^1}(v_1, v_6) = 21$, $\varphi(G_6^2) = H_{G_6^2}(v_1, v_6) = 15$, $\varphi(G_6^3) = H_{G_6^3}(v_1, v_6) = 12$, $\varphi(G_6^4) = H_{G_6^4}(v_1, v_6) = \frac{49}{3}$. Hence, $\phi_1(6) = \max \{\varphi(G_6^1), \ldots, \varphi(G_6^4)\} = 21 = f(6)$. Thus, G_6^1 is the only 6-maximal graph in $\mathcal{B}_1(6)$ with $dist_{G_6^1}(v_1, v_6) = 6 - 3 = 3$. Hence, $\phi_1(n) = f(n)$ for $n = 5, 6$. Moreover, when $n = 5, 6$, if G is any graph in $\mathcal{B}_1(n)$ with $\varphi(G) = f(n)$, then $G = B_1^*(n)$ and $H_{B_1^*(n)}(x,y) = \varphi(G)$, where x is a vertex in cycle with degree 2 and y is the pendant vertex such that $dist(x,y) = n - 3$.

We assume that the assertion holds for $n - 1$. In other words, $\phi_1(n-1) = f(n-1)$. Further, if G is any graph in $\mathcal{B}_1(n-1)$ with $\varphi(G) = f(n-1)$, then $G = B_1^*(n-1)$ and $H_{B_1^*(n-1)}(x,y) = \varphi(G)$, where x is a
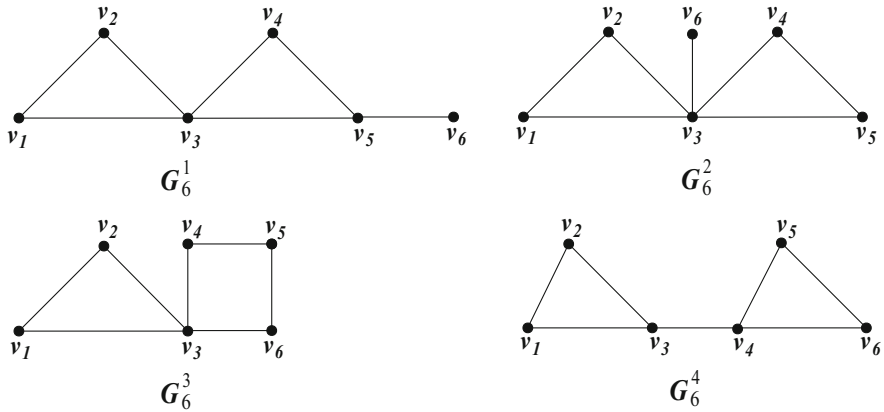
Fig. 6 G_5^1

$$G_5^1$$
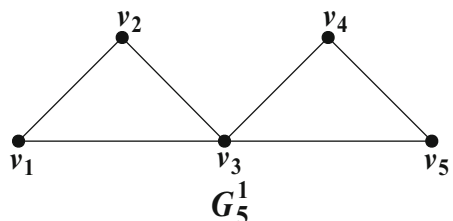
Fig. 7 G_6^1, G_6^2, G_6^3, G_6^4

vertex in cycle with degree 2 and y is the pendant vertex such that $dist(x, y) = (n - 1) - 3$. Next, we will prove that the assertion holds for n. Without loss of generality, we assume that G is an n-maximal graph in $\mathcal{B}_1(n)$, i.e., $\varphi(G) = \phi_1(n)$.

Then by the definition of $\varphi(G)$, there exist two vertices x and y in G such that $H_G(x, y) = \varphi(G) = \phi_1(n)$. By Lemma 2.9, y is not a cut vertex of G. Now we consider the following two cases.

Case 1. y is a vertex in cycle of G with $d(y) = 2$. Then there exists a vertex w in $N[y]$ such that $\max_{v \in N(y)} H_G(v, y) = H_G(w, y)$ and $d_{N[y]}(w) = t + 1$. Furthermore, we consider the following two subcases.

Subcase 1.1. $t = 0$. Then $e(N[y]) = 2$. By Lemma 2.10, $N(y) \cap N(w) = \emptyset$, $H_G(x, y) \leq H_{G/N}(x, N) + 2n - 1$. Moreover, G/N can be an $(n - 2)$-vertex unicyclic graph or $G/N \in \mathcal{B}_1(n - 2)$.

If G/N is an $(n - 2)$-vertex unicyclic graph, by Theorem 2.2, then $H_{G/N}(x, N) \leq (n - 2)^2 - 7$ and $H_G(x, y) \leq n^2 - 4n - 3 + 2n - 1 = n^2 - 2n - 4 < f(n)$.

If $G/N \in \mathcal{B}_1(n - 2)$, then $H_{G/N}(x, N) \leq (n - 2)^2 + 2(n - 2) - 27$ and $H_G(x, y) \leq n^2 - 2n - 27 + 2n - 1 = n^2 - 28 < f(n)$.

Subcase 1.2. $t = 1$. Then $e(N[y]) = 3$. By Lemma 2.10, $N(y) \cap N(w) \neq \emptyset$, $H_G(x, y) \leq H_{G/N}(x, N) + \frac{4n-2}{3}$. Moreover, G/N is an $(n - 2)$-vertex unicyclic graph, by Theorem 2.2, $H_{G/N}(x, N) \leq (n - 2)^2 - 7$ and $H_G(x, y) \leq n^2 - 4n - 3 + \frac{4n-2}{3} = \frac{3n^2 - 8n - 11}{3} < f(n)$ for $n \geq 6$.

Therefore, if y is a vertex in cycle of G with $d(y) = 2$, then $\phi_1(n) = \varphi(G) < f(n)$.

Case 2. y is a pendant vertex of G, there exists a unique neighbor v of y in G. Then v must be a cut vertex in G and $G^- := G - y$ is an $(n - 1)$-vertex connected bicyclic graph. By Theorem 2.3, we have $H_G(x, y) = H_G(x, v) + H_G(v, y) = H_{G^-}(x, v) + H_G(v, y)$. Moreover, by Theorem 2.4, $H_G(v, y) = 2n + 1$.

On the one hand, by the induction hypothesis for G^-, $H_{G^-}(x, v) \leq \phi_1(n-1) = f(n-1) = \varphi(B_1^*(n-1))$. Hence,

$$\phi_1(n) = \varphi(G) = H_G(x, y) = H_{G^-}(x, v) + H_G(v, y) \leq f(n-1) + 2n + 1 = f(n).$$

On the other hand, let x be a vertex of degree 2 in cycle and y be the pendant vertex in $B_1^*(n)$ such that $dist(x, y) = n - 3$. We denote a unique neighbor of y by v. By the induction hypothesis, we have

$$\phi_1(n) \geq \varphi(B_1^*(n)) \geq H_{B_1^*(n)}(x, y)$$
$$= H_{B_1^*(n)}(x, v) + H_{B_1^*(n)}(v, y)$$
$$= H_{B_1^*(n-1)}(x, v) + 2n + 1$$
$$= f(n-1) + 2n + 1 = f(n).$$

Hence, $\phi_1(n) = f(n)$. Furthermore, if G is any n-vertex graph in $\mathcal{B}_1(n)$ with $\varphi(G) = \phi_1(n) = f(n)$. Then there exist two vertices x and y such that $H_G(x, y) = \varphi(G) = f(n)$. Then y must be a pendant vertex. We denote a unique neighbor of y by v. Then $G - y$ is an $(n-1)$-vertex bicyclic graph in $\mathcal{B}_1(n-1)$. By the induction hypothesis,

$$f(n) = H_G(x, y) = H_G(x, v) + H_G(v, y)$$
$$= H_{G-y}(x, v) + 2n + 1$$
$$\leq \varphi(G - y) + 2n + 1$$
$$\leq f(n-1) + 2n + 1 = f(n),$$

which implies that $\varphi(G - y) = H_{G-y}(x, v) = f(n-1)$. By the induction hypothesis, $G - y$ must be $B_1^*(n-1)$ and v is the pendant vertex in $B_1^*(n-1)$. Thus, $G = B_1^*(n)$ and $f(n) = \varphi(B_1^*(n)) = H_{B_1^*(n)}(x, y)$, where x is a vertex in cycle and y is a pendant vertex with $dist(x, y) = n - 3$. Therefore, the assertion holds for all n. □

Next, we will consider the n-minimal graph among $\mathcal{B}_1(n)$.

Theorem 3.2 *Let $G := B(T_{u_1}, \ldots, T_{u_p}, T_{w_0}, \ldots, T_{w_{m-1}}, T_{v_1}, \ldots, T_{v_q})$ be a connected bicyclic graph with at least one pendant vertex in $\mathcal{B}_1(n)$. Then $\varphi(G) \geq 2n + 3$ with equality holds if and only if $G = B_1^\sharp(n)$.*

Proof Let v be the pendant vertex in G. Then there exists a vertex u_i (resp. v_j or w_k) such that $v \in V(T_{u_i})$ (resp. $v \in V(T_{v_j})$ or $v \in V(T_{w_k})$), where $1 \leq i \leq p$. (resp. $1 \leq j \leq q$, $0 \leq k \leq m-1$). If $v \in V(T_{u_i})$, by Theorem 2.4 and $dist(u_i, v) \geq 1$, then $H_G(u_i, v) \geq 1 + 2n$, with equality holds if and only if $dist(u_i, v) = 1$. Moreover, by Theorem 2.5 and Lemma 2.6, there exists a vertex u_j in cycle C_p such that $H_G(u_j, u_i) \geq 2$, with equality holding for $C_p = C_3$ and $T_{u_j} = u_j$, $T_{u_t} = u_t$ for $u_j, u_t \in V(C_3)$. Hence,

$$\varphi(G) \geq H_G(u_j, v) = H_G(u_j, u_i) + H_G(u_i, v) \geq 2 + 1 + 2n = 2n + 3.$$

Similarly, if $v \in V(T_{v_j})$, by a similar argument, $\varphi(G) \geq H_G(v_i, v) \geq 2n + 3$. If $v \in$

$V(T_{w_k})$ and $w_k \neq u_1, v_1$,   by   Theorem 2.4   and $dist(w_k, v) \geq 2$,   then
$\varphi(G) > H_G(w_k, v) > 2n + 3$.   Moreover, $\varphi(B_1^\sharp(n)) = 2n + 3$.   Therefore,
$\varphi(G) \geq 2n + 3$, with equality holds if and only if $G = B_1^\sharp(n)$. So the assertion holds.
□

4 The Extremal Hitting Times on Bicyclic Graphs in $\mathcal{B}_2(n)$

In this section, we will prove that the graph $B_2^*(n)$ is the n-maximal graph and
$B_2^\sharp(n)$ is the n-minimal graph among $\mathcal{B}_2(n)$ (see Fig. 8).

Theorem 4.1 *Let $G \in \mathcal{B}_2(n)$ and $g(n) = n^2 + 2n - 19$. Then*

$$\varphi(G) \leq g(n).$$

Moreover, equality holds if and only if $G = B_2^*(n)$ and $\varphi(B_2^*(n)) = H_{B_2^*(n)}(x, y)$ is
the hitting time from the vertex x in cycle with degree 2 to the pendant vertex y with
$dist(x, y) = n - 2$.

Proof Let $\phi_2(n) = \max\{\varphi(G) : G \in \mathcal{B}_2(n)\}$.   We   need   to   prove   that
$\phi_2(n) = g(n) = n^2 + 2n - 19$, and prove that for any graph G with $\varphi(G) = g(n)$,
then $G = B_2^*(n)$ and $g(n) = H_{B_2^*(n)}(x, y)$ with $dist(x, y) = n - 2$, where x is vertex of
degree 2 in cycle and y is the pendant vertex. We use the induction on n to prove the
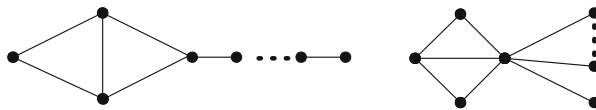assertion. If $n = 5$, then $\mathcal{B}_2(5):=\{G_5^1, G_5^2, G_5^3, G_5^4\}$ (see Fig. 9). By a simple

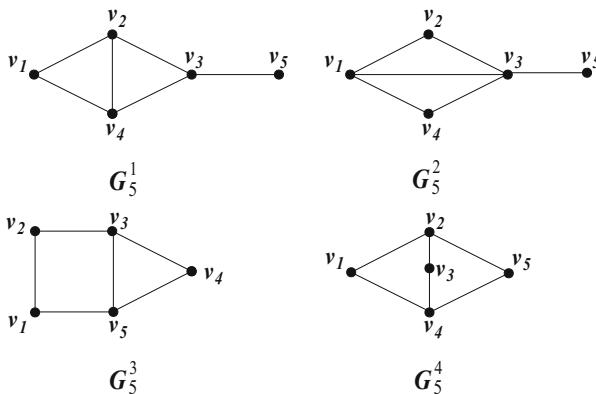

Fig. 8 $B_2^*(n)$ and $B_2^\sharp(n)$

Fig. 9 $G_5^1, G_5^2, G_5^3, G_5^4$

computation, $\varphi(G_5^1) = H_{G_5^1}(v_1, v_5) = 16$, $\varphi(G_5^2) = H_{G_5^2}(v_1, v_5) = 13.5$, $\varphi(G_5^3) = H_{G_5^3}(v_4, v_1) = \frac{79}{11}$, $\varphi(G_5^4) = H_{G_5^4}(v_1, v_5) = 6$. Hence, $\phi_2(5) = \max\{\varphi(G) : G \in \mathcal{B}_2(5)\} = g(5) = \varphi(G_5^1) = H_{G_5^1}(v_1, v_5)$ with $dist(v_1, v_5) = 5 - 2$. So the assertion holds for $n = 5$.

Suppose that the assertion holds for $n - 1$, i.e., $\phi_2(n-1) = g(n-1)$. Moreover, if G is any $(n-1)$-vertex bicyclic graph G with $\varphi(G) = \phi_2(n-1)$, then $G = B_2^*(n-1)$ and there exists a vertex x with degree 2 and a pendant vertex y in $B_2^*(n-1)$ such that $H_{B_2^*(n-1)}(x, y) = \varphi(B_2^*(n-1)) = \phi_2(n-1) = g(n-1)$ and $dist(x, y) = n - 3$.

Let G be any n-vertex bicyclic graph in $\mathcal{B}_2(n)$ such that $\varphi(G) = \phi_2(n)$. Next, we will prove that $\phi_2(n) = g(n)$ with equality holds if and only if $G = B_2^*(n)$.

In addition, by the definition of $\varphi(G)$, there exist two vertices x and y in G such that $H_G(x, y) = \varphi(G) = \phi_2(n)$. By Lemma 2.9, y is not a cut vertex. Let w be a vertex in $N[y]$ such that $\max_{v \in N(y)} H_G(v, y) = H_G(w, y)$ and $d_{N[y]}(w) = t + 1$. We consider the following three cases.

Case 1. y is a vertex in cycle with $d(y) = 2$. Then we need to consider the following two subcases.

Subcase 1.1. $t = 0$. By Lemma 2.10, we have $N(y) \cap N(w) = \emptyset$, then $e(N[y]) = 2$ and $H_G(x, y) \leq H_{G/N}(x, N) + 2n - 1$. In addition, G/N has four cases, $G/N \in \mathcal{B}_2(n-2)$, $G/N \in \mathcal{B}_1(n-2)$, G/N is an $(n-2)$-vertex unicyclic graph, or G/N is an $(n-2)$-vertex tree.

If $G/N \in \mathcal{B}_2(n-2)$, then $H_G(x, y) \leq (n-2)^2 + 2(n-2) - 19 + 2n - 1 = n^2 - 20 < g(n)$.

If $G/N \in \mathcal{B}_1(n-2)$, by Theorem 3.1, then $H_G(x, y) \leq (n-2)^2 + 2(n-2) - 27 + 2n - 1 = n^2 - 28 < g(n)$.

If G/N is an $(n-2)$-vertex unicyclic graph, by Theorem 2.2 and $n \geq 5$, then $H_G(x, y) \leq (n-2)^2 - 7 + 2n - 1 < g(n)$.

If G/N is an $(n-2)$-vertex tree, then by Theorem 2.1 and $n \geq 5$, we have $H_G(x, y) \leq (n-3)^2 + 2n - 1 < g(n)$.

Subcase 1.2. $t = 1$. By Lemma 2.10, we have $N(y) \cap N(w) \neq \emptyset$. Then $e(N[y]) = 3$ and $H_G(x, y) \leq H_{G/N}(x, N) + \frac{4n-2}{3}$. In addition, G/N is either an $(n-2)$-vertex tree or an $(n-2)$-vertex unicyclic graph.

If G/N is an $(n-2)$-vertex tree, then $H_G(x, y) \leq (n-3)^2 + \frac{4n-2}{3} < g(n)$.

If G/N is an $(n-2)$-vertex unicyclic graph, then $H_G(x, y) \leq (n-2)^2 - 7 + \frac{4n-2}{3} < g(n)$.

Case 2. y is a vertex in cycle with $d(y) = 3$. Then we need to consider the following three subcases.

Subcase 2.1. $t = 0$. By Lemma 2.10, we have $N(y) \cap N(w) = \emptyset$. Then $e(N[y]) = 3$ (resp. $e(N[y]) = 4$) and $H_G(x, y) \leq H_{G/N}(x, N) + 2n - 3$ (resp. $H_G(x, y) \leq H_{G/N}(x, N) + 2n - 5$).

On the one hand, if $e(N[y]) = 3$, then G/N has four cases, $G/N \in \mathcal{B}_2(n-3)$, $G/N \in \mathcal{B}_1(n-3)$, G/N is an $(n-3)$-vertex unicyclic graph, or G/N is an $(n-3)$-vertex tree.

If $G/N \in \mathcal{B}_2(n-3)$, then $H_G(x,y) \le (n-3)^2 + 2(n-3) - 19 + 2n - 3 < g(n)$.

If $G/N \in \mathcal{B}_1(n-3)$, then by Theorem 3.1, $H_G(x,y) \le (n-3)^2 + 2(n-3) - 27 + 2n - 3 < g(n)$.

If G/N is an $(n-3)$-vertex unicyclic graph, then by Theorem 2.2, $H_G(x,y) \le (n-3)^2 - 7 + 2n - 3 < g(n)$.

If G/N is an $(n-3)$-vertex tree, then $H_G(x,y) \le (n-4)^2 + 2n - 3 < g(n)$.

On the other hand, if $e(N[y]) = 4$, then G/N is either an $(n-3)$-vertex tree or an $(n-3)$-vertex unicyclic graph.

If G/N is an $(n-3)$-vertex tree, then $H_G(x,y) \le (n-4)^2 + 2n - 5 < g(n)$.

If G/N is an $(n-3)$-vertex unicyclic graph, then $H_G(x,y) \le (n-3)^2 - 7 + 2n - 5 < g(n)$.

Subcase 2.2. $t = 1$. By Lemma 2.10, we have $N(y) \cap N(w) \ne \emptyset$. Then $e(N[y]) = 4$ (resp. $e(N[y]) = 5$) and $H_G(x,y) \le H_{G/N}(x,N) + \frac{4n-5}{3}$ (resp. $H_G(x,y) \le H_{G/N}(x,N) + \frac{4n-7}{3}$).

On the one hand, if $e(N[y]) = 4$, then G/N is either an $(n-3)$-vertex tree or an $(n-3)$-vertex unicyclic graph.

If G/N is an $(n-3)$-vertex tree, then $H_G(x,y) \le (n-4)^2 + \frac{4n-5}{3} < g(n)$.

If G/N is an $(n-3)$-vertex unicyclic graph, then $H_G(x,y) \le (n-3)^2 - 7 + \frac{4n-5}{3} < g(n)$.

On the other hand, if $e(N[y]) = 5$, then G/N is an $(n-3)$-vertex tree. Thus, $H_G(x,y) \le (n-4)^2 + \frac{4n-7}{3} < g(n)$.

Subcase 2.3. $t = 2$. By Lemma 2.10, we have $N(y) \cap N(w) \ne \emptyset$. Then $e(N[y]) = 5$ and $H_G(x,y) \le H_{G/N}(x,N) + \frac{2n-3}{2}$. Moreover, G/N is an $(n-3)$-vertex tree, by Theorem 2.1, $H_G(x,y) \le (n-4)^2 + \frac{2n-3}{2} < g(n)$.

Case 3. y is a pendant vertex of G. Then there exists a unique neighbor v of y in G. Let $G^- = G - y$. Then v must be a cut vertex. By Theorem 2.3, we have $H_G(x,y) = H_G(x,v) + H_G(v,y) = H_{G^-}(x,v) + H_G(v,y)$.   By   Theorem 2.4, $H_G(v,y) = 2n + 1$.

On the one hand, by the induction hypothesis, we have

$$H_{G^-}(x,v) \le \varphi(G^-) \le \phi_2(n-1) = g(n-1).$$

Hence,

$$\phi_2(n) = \varphi(G) \le \phi_2(n-1) + 2n + 1 = g(n-1) + 2n + 1 = g(n).$$

On the other hand, let u_1, u_2 be two vertices in $B_2^*(n)$ with $d(u_1) = 2$, $d(u_2) = 1$ and $dist(u_1, u_2) = n - 2$. We denote the unique neighbor of u_2 by w. By the definition of $\phi_2(n)$ and the induction hypothesis, we have

$$\phi_2(n) \geq \varphi(B_2^*(n)) \geq H_{B_2^*(n)}(u_1, u_2)$$
$$= H_{B_2^*(n)}(u_1, w) + H_{B_2^*(n)}(w, u_2)$$
$$= H_{B_2^*(n-1)}(u_1, w) + 2n + 1$$
$$= g(n-1) + 2n + 1 = g(n).$$

Hence, $\phi_2(n) = g(n)$. Moreover, $\phi_2(n) = \varphi(B_2^*(n)) = H_{B_2^*(n)}(u_1, u_2) = g(n)$. For any n-maximal graph G with $\varphi(G) = \phi_2(n) = g(n)$, there exist two vertices x and y such that $H_G(x, y) = \varphi(G) = g(n)$. By above proof, y must be a pendant vertex in G. In addition, let v be a unique neighbor of y. Then $g(n) = H_G(x, y) \leq \varphi(G - y) + 2n + 1 \leq g(n-1) + 2n + 1 = g(n),$ which implies that $\varphi(G - y) = g(n-1)$. By the induction hypothesis, we have $G - y = B_2^*(n-1)$. Therefore, $G = B_2^*(n)$. We finish our proof. $\square$

Next, we will consider the n-minimal graph among $\mathcal{B}_2(n)$.

Lemma 4.2 *Let $G \in \mathcal{B}_2(n)$ without pendant vertices, i.e., G consists of three internally disjoint paths $P_p := xu_1u_2\ldots u_py$, $P_q := xv_1\ldots v_qy$ and $P_m := xw_1\ldots w_my$. If G' is obtained from G by adding a pendant edge e to one root vertex, then $H_{G'}(x, y) \geq H_G(x, y)$ for any two vertices $x, y \in V(G)$.*

Proof By Ohm's law, we have known that $R_G(x, y) = R_{G'}(x, y)$ for $x, y \in V(G)$ and the new added pendant edge has no effect on the effective resistance of any two vertices in graph G . Suppose that the new added vertex is u_0 and v is the unique neighbour of u_0. By Lemma 2.7 and triangle inequality $R_{G'}(x, y) + R_{G'}(y, z) - R_{G'}(x, z) \geq 0$, we have

$$H_{G'}(x, y) = \frac{1}{2} \sum_{z \in V(G')} d_{G'}(z)(R_{G'}(x, y) + R_{G'}(y, z) - R_{G'}(x, z))$$
$$= \frac{1}{2} \sum_{z \in V(G)} d_G(z)(R_{G'}(x, y) + R_{G'}(y, z) - R_{G'}(x, z))$$
$$+ \frac{1}{2}(R_{G'}(x, y) + R_{G'}(y, v) - R_{G'}(x, v))$$
$$+ \frac{1}{2} \sum_{z = u_0} d(z)(R_{G'}(x, y) + R_{G'}(y, z) - R_{G'}(x, z))$$
$$= H_G(x, y) + \frac{1}{2}(R_{G'}(x, y) + R_{G'}(y, v) - R_{G'}(x, v))$$
$$+ \frac{1}{2}(R_{G'}(x, y) + R_{G'}(y, u_0) - R_{G'}(x, u_0))$$
$$\geq H_G(x, y).$$

$\square$

Lemma 4.3 *Let $G \in \mathcal{B}_2(n)$ without pendant vertices, i.e., G consists of three internally disjoint paths $P_p := xu_1u_2\ldots u_py$, $P_q := xv_1 \cdots v_qy$ and $P_m := xw_1\ldots w_my$. Let*

u and v be two vertices in $V(P_p)$ with $dist_{P_p}(u,v) = d_1$, $dist_{P_p}(x,u) = d_2$ and $dist_{P_p}(v,y) = d_3$ (see Fig. 10). Then

$$H_G(u,v) = \frac{R_2}{d_1 + R_2}d_1^2 + \frac{d_1}{d_1 + R_2}\left(d_2^2 + qm + q + m + 1\right)$$
$$+ \frac{d_1}{d_1 + R_2}\left(\frac{2d_2(q+1)(m+1)}{q+m+2} + d_3^2 + 2d_3(d_2 + q + m + 2)\right),$$

where $R_2 = d_2 + d_3 + \frac{(q+1)(m+1)}{q+m+2}$.

Proof We can decompose graph G into G_1 and G_2 through vertices u and v, where G_1 is a uv-path and G_2 is a unicyclic graph (see Fig. 10). Let R_1 be the effective resistance between u and v in G_1 and $R_1 = d_1$. Let R_2 be the effective resistance between u and v in G_2 and $R_2 = d_2 + d_3 + \frac{(q+1)(m+1)}{q+m+2}$. By Theorems 2.4 and 2.5, we have

$$H_G(u,v) = \frac{R_2}{R_1 + R_2}H_{G_1}(u,v) + \frac{R_1}{R_1 + R_2}H_{G_2}(u,v)$$
$$= \frac{R_2}{d_1 + R_2}d_1^2 + \frac{d_1}{d_1 + R_2}\left(d_2^2 + qm + q + m + 1\right)$$
$$+ \frac{d_1}{d_1 + R_2}\left(\frac{2d_2(q+1)(m+1)}{q+m+2} + d_3^2 + 2d_3(d_2 + q + m + 2)\right).$$

$\square$

By Lemma 4.3, we can get the following Corollary.

Corollary 4.4 *Let $G \in \mathcal{B}_2(n)$ without pendant vertices, i.e., G consists of three internally disjoint paths $P_p := xu_1u_2\ldots u_py, P_q := xv_1\ldots v_qy$ and $P_m := xw_1\ldots w_my$. Then*

$$H_G(x,y) = \frac{(n+1)(p+1)(q+1)(m+1)}{(p+1)(q+1) + (p+1)(m+1) + (q+1)(m+1)}.$$

Lemma 4.5 *Let $G \in \mathcal{B}_2(n)$ without pendant vertices, i.e., G consists of three internally disjoint paths $P_p := xu_1u_2\ldots u_py, P_q := xv_1\ldots v_qy$ and $P_m := xw_1\ldots w_my$. Let $u \in V(P_p)$ with $dist_{P_p}(u,x) = d_1$ and $v \in V(P_q)$ with $dist_{P_q}(v,x) = d_2$. Then*

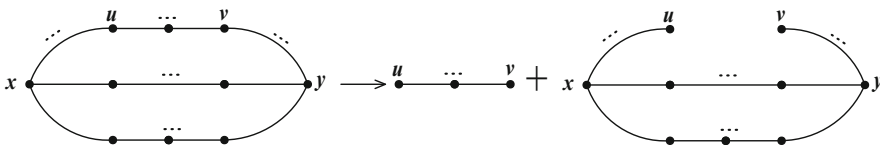**Fig. 10** A decomposition of G

$$R_G(u,v) = d_1 + d_2 - \frac{d_1^2}{p+m+2}$$
$$- \frac{[d_2(p+m+2) + d_1(m+1)]^2}{(p+m+2)(pq+pm+qm+2p+2q+2m+3)}.$$

Proof By the method of calculating the effective resistance for bridge circuit (see Fig. 11), we can get $R_G(u,v)$. More details about bridge circuit can be referred to [2].

Let $R_1 = d_1$, $R_2 = d_2$, $R_3 = m+1$, $R_4 = p+1-d_1$ and $R_5 = q+1-d_2$. Then

$$R_G(u,v) = d_1 + d_2 - \frac{d_1^2}{p+m+2}$$
$$- \frac{[d_2(p+m+2) + d_1(m+1)]^2}{(p+m+2)(pq+pm+qm+2p+2q+2m+3)}.$$

$\square$

Theorem 4.6 *Let $G \in \mathcal{B}_2(n)$ with at least one pendant vertex. Then $\varphi(G) \geq 2n + 3.5$ with equality holds if and only if $G = B_2^{\sharp}(n)$.*

Proof $G \in \mathcal{B}_2(n)$ has at least one pendant vertex, i.e., G can be obtained from three internally disjointed paths P_p, P_q, P_m with common end vertices x, y by attaching the trees rooted at each vertex in P_p, P_q and P_m. Without loss of generality, we assume that v is a pendant vertex of G, i.e., there exists one vertex z for $z \in V(P_p) \cup V(P_q) \cup V(P_m)$ such that $v \in V(T_z)$ and $|V(T_z)| \geq 2$. By Theorem 2.4 and $dist(z,v) \geq 1$, we have $H_G(z,v) \geq 1 + 2n$, with equality holds if and only if $dist(z,v) = 1$. By Lemma 4.2, 4.3 and Corollary 4.4, if $p+q+m \geq 3$, then there must exist a vertex u' in cycle such that $H_G(u',z) > 2.5$. Since z is a cut vertex, by Theorem 2.3, we have $\varphi(G) \geq H_G(u',v) = H_G(u',z) + H_G(z,v) > 2n + 3.5$. If
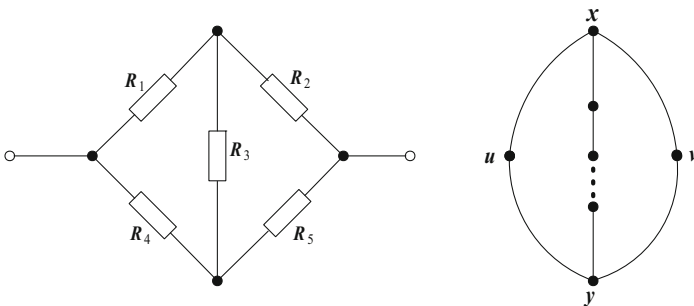

Fig. 11 Bridge circuit and G

$p + q + m = 2$, i.e., $p = q = 1$ and $m = 0$, then there must exist a vertex u' in cycle such that $H_G(u', z) \geq 2.5$, the equality holds if and only if $u', z \in V(K_4 - e)$ and u', z are common end vertices of three internally disjoint paths in $K_4 - e$, $T_{u'} = u'$, $T_{u_1} = u_1$ and $T_{v_1} = v_1$ for $u_1, v_1 \in V(K_4 - e)$. Since z is a cut vertex, by Theorem 2.3, we have $\varphi(G) \geq H_G(u', v) = H_G(u', z) + H_G(z, v) \geq 2n + 3.5$. Therefore, by Theorems 2.4, 2.5 and Lemma 4.2, we have $\varphi(B_2^\sharp(n)) = 2n + 3.5$ and $\varphi(G) \geq 2n + 3.5$, with equality holds if and only if $G = B_2^\sharp(n)$.   □

5 Proof of Theorem 1.2

In this section, we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2 For the upper bound, by Theorems 3.1 and 4.1, then

$$\varphi(G) \leq n^2 + 2n - 19.$$

For the lower bound, we consider the following two cases.

Case 1. $G \in \mathcal{B}_1(n)$, then G has two cycles of length p and q with no common edges and $p \geq q$. Furthermore, we consider the following three subcases.

Subcase 1.1. G has at least one pendant vertex. Then by Theorem 3.2, $\varphi(G) \geq 2n + 3$ with equality holding if and only if $G = B_1^\sharp(n)$.

Subcase 1.2. G has no pendant vertices and two cycles in G has no common vertex. Then $n + 1 > p + q$ and $|V(T_{u_i})| = 1$ for $i = 1, \ldots, p$, $|V(T_{w_j})| = 1$ for $j = 0, \ldots, m - 1$, $|V(T_{v_r})| = 1$ for $r = 1, \ldots q$ (see Fig. 12). Moreover, there exist two vertices $u, v \in V(G)$ such that $\varphi(G) = H_G(u, v)$. Then by Lemma 2.9, either u, v are vertices in different cycles, or u, v are vertices in the same cycle.
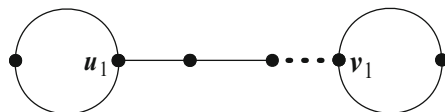
Let

$$M = \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil \frac{2(n+1) - q}{q} + (n + 1 - p - q)(n + 1 + p - q).$$

By $n + 1 - p - q \geq 1$, $p \geq q$ and $p \geq 3$, we have $M > 2n + 3$.

If $u \in V(C_p)$, $v \in V(C_q)$, $dist(u, u_1) = g_1$, and $dist(v_1, v) = g_2$, by Theorems 2.3, 2.4 and 2.5, then

Fig. 12 Illustration for graph in $\mathcal{B}_1(n)$ without pendant trees

$$H_G(u,v) = H_G(u,u_1) + H_G(u_1,v_1) + H_G(v_1,v)$$
$$= g_1(p - g_1) + (n + 1 - p - q)(n + 1 + p - q)$$
$$+ g_2(q - g_2) + \frac{2g_2(q - g_2)(n + 1 - q)}{q}$$
$$\leq \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil \frac{2(n + 1) - q}{q}$$
$$+ (n + 1 - p - q)(n + 1 + p - q),$$

with equality holds if and only if $g_1 = \left\lfloor \frac{p}{2} \right\rfloor$ and $g_2 = \left\lfloor \frac{q}{2} \right\rfloor$.

If $u \in V(C_q)$ and $v \in V(C_p)$, then by similar arguments,

$$H_G(u,v) \leq \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil \frac{2(n + 1) - p}{p}$$
$$+ (n + 1 - p - q)(n + 1 + q - p) \leq M.$$

If $u, v \in V(C_p)$, $dist(u,v) = g_1$, and $dist(u_1,v) = t_1$, by Theorems 2.3, 2.4 and 2.5, then $H_G(u,v) = g_1(p - g_1) + \frac{2g_1(n + 1 - p)t_1}{p}$. Since $t_1 \leq p - g_1$, we have

$$H_G(u,v) \leq g_1(p - g_1) + \frac{2g_1(p - g_1)(n + 1 - p)}{p}$$
$$\leq g_1(p - g_1) \frac{2n + 2 - p}{p}$$
$$\leq \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil \frac{2n + 2 - p}{p} \leq M.$$

If $u, v \in V(C_q)$, The by similar arguments, $H_G(u,v) \leq \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil \frac{2n+2-q}{q} \leq M$.

Hence, $\varphi(G) = M > 2n + 3$.

Subcase 1.3. G has no pendant vertices and two cycles C_p and C_q in G have one common vertex. Then $p + q = n + 1$. Let

$$M_0 = \max\left\{ \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil \frac{2(n + 1) - q}{q}, \quad \left\lfloor \frac{q}{2} \right\rfloor \left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil \frac{2(n + 1) - p}{p} \right\}.$$

From Subcase 1.2, we can get $\varphi(G) = M_0 > 2n + 3$ for $n \geq 8$.

Case 2. $G \in \mathcal{B}_2(n)$. We consider the following two subcases.

Subcase 2.1. G has at least one pendant vertex. By Theorem 4.6, we have $\varphi(G) \geq 2n + 3.5$.

Subcase 2.2. G has no pendant vertices. Then $p + q + m = n - 2$.

By Lemma 4.5 and $n \geq 12$, $p \geq q \geq m$, we have $p \geq 4$ and there exists $u \in V(P_p)$ and $v \in V(P_q)$ such that $R_G(u,v) > 2.2$. Furthermore, by (9) in Theorem 2.7, we have $H_G(u,v) + H_G(v,u) = 2(n + 1)R_G(u,v) > 4.4(n + 1)$, which implies that $2\varphi(G) \geq H_G(u,v) + H_G(v,u) > 4.4(n + 1)$. So $\varphi(G) \geq 2.2n + 2.2 > 2n + 3.5$ for $n \geq 12$. We complete the proof of Theorem 1.2. $\quad\square$
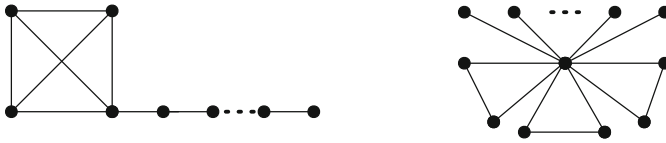
Fig. 13 The extremal graphs of n-vertex tricyclic graphs

Fig. 14 The extremal graphs G^* and $G^\sharp$ in $\mathcal{G}_{n,m}$

Remark If $n \leq 11$, then the left inequality in Theorem 1.2 does not hold. In fact, by a simple calculation and argument, $\varphi(G) \geq 5, 16, \frac{69}{8}, \frac{80}{7}, 13, 18, 22, 24$ for $n = 4, 5, 6, 7, 8, 9, 10, 11$ respectively; while $n \geq 5$, the right inequality holds in Theorem 1.2. It is natural to ask the analogous question for n-vertex tricyclic graphs (n-vertex connected graphs with $n + 2$ edges). We guess that there exists N such that the maximal and minimal extremal graphs in the set of all n-vertex tricyclic graphs with $n \geq N$ are the following two graphs (see Fig. 13).

Generally, we guess the extremal graphs in Question 1.1, that is, n-maximal graph is G^* and n-minimal graph is $G^\sharp$ (see Fig. 14). From Fig. 14, we can see that the extremal graph G^* is a graph obtained by identifying one vertex in G_0 and an end vertex of a path on remaining vertices, $G^\sharp$ is a graph obtained by identifying one vertex in G_1 and center of a star, where G_0 and G_1 are subgraphs of G^* and $G^\sharp$ respectively and G_0 is a clique or close to a clique probably. Moreover, if the edge number m of G is equal to $n + k$ with $k \geq 0$, then G_1 is a graph consists of $k + 1$ triangles sharing one common vertex. If $\phi(n) = \varphi(G) = H_G(x, y)$, then x is a vertex in $V(G_0)$ and y is a pendant vertex. If $\psi(n) = \varphi(G) = H_G(x, y)$, then x is a vertex with degree 2 in $V(G_1)$ and y is a pendant vertex.

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