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ORIGINAL PAPER

Extending Muirhead's Inequality

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Abstract
We present a method to prove an inequality concerning a linear combination of symmetric monomial functions. This is based on Muirhead's inequality combining with a graph theoretical setting. As an application we prove some interesting inequalities motivated from extremal combinatorics.

Keywords Symmetric monomial function · Muirhead's inequality · Maximum flow · Young diagram

1 Introduction

We start with the following conjecture due to the last author.

Conjecture 1 *Let $r \geq 3$ be an integer, let $a_1, \ldots, a_r \in (0,1)$ be real numbers, and let $a = a_1 \cdots a_r$. Then we have*

$$\prod_{i=1}^{r}\left(a_i + a_i^2 + \cdots + a_i^{r-1} + a_i^r\right) - \prod_{i=1}^{r}\left(a_i + a_i^2 + \cdots + a_i^{r-1} + a\right) \geq 0 \qquad (1)$$

with equality holding if and only if $a_1 = \cdots = a_r$.

This conjecture is motivated by study of multiply intersecting hypergraphs, where one of the main tools is the so-called random walk method, see Chapter 15 of [1]. Here we briefly explain how Conjecture 1 is related to a problem of random walk. Let $p \in (0, 1 - \frac{1}{r})$ be a real number, and let us define an infinite random walk

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W_p in the two-dimensional grid $\mathbb{Z}^2$. The walk W_p starts at the origin, and at each step it moves from (x, y) to $(x, y+1)$ (one step up) with probability p and from (x, y) to $(x+1, y)$ (one step right) with probability $1-p$. Then W_p hits the line $y = (r-1)x+1$ with probability α_p, where $\alpha_p \in (0,1)$ is a unique root of the equation

$$X = p + (1-p)X^r.$$

Let $p_1, \ldots, p_r \in (0, 1-\frac{1}{r})$ be real numbers. Let W' be another infinite random walk defined similarly to W_p, but this time, at step j ($j = 1, 2, \ldots$) the walk takes up with probability p_i and right with probability $1 - p_i$, where $i := j \bmod r$. This walk W' hits the line $y = (r-1)x + r$ with probability β, where $\beta \in (0,1)$ is a unique root of the equation

$$X = \prod_{i=1}^{r}(p_i + (1-p_i)X).$$

We are interested in β because we can use it to bound the measure of multiply intersecting hypergraphs. We also mention that computing (or approximating) β is more difficult than that of α_p. Thus it is desirable that β is bounded in terms of α_{p_i}. Indeed we conjecture that $\beta \le \alpha_{p_1}\alpha_{p_2}\cdots\alpha_{p_r}$, which follows from (1) (if true). See Appendix for the proof.

In this paper we prove Conjecture 1 for the cases $3 \le r \le 11$ with aid of computer search. For the proof we extend Muirhead's inequality, and propose an approach to prove a more general inequality concerning a linear combination of symmetric monomial functions.

Let r and d be fixed positive integers, and let

$$\Lambda := \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r : \lambda_1 \ge \cdots \ge \lambda_r \ge 0,\ |\lambda| = d\}, \tag{2}$$

where $|\lambda| := \lambda_1 + \cdots + \lambda_r$, that is, Λ is a set of non-increasing sequences (vectors) representing a partition of d into r parts. For $\lambda \in \Lambda$ we define the symmetric monomial functions $m_\lambda(x)$ of degree d in variables $x = (x_1, \ldots, x_r)$ by

$$m_\lambda(x) := \sum_\sigma x^\sigma = \sum_\sigma \prod_{i=1}^{r} x_i^{\sigma_i},$$

where the sums run over all distinct orderings (permutations) $\sigma = (\sigma_1, \ldots, \sigma_r)$ of the vector $\lambda = (\lambda_1, \ldots, \lambda_r)$. We also define the normalized symmetric monomial functions $\bar{m}_\lambda(x)$ by

$$\bar{m}_\lambda(x) := \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}.$$

For example, if $r = d = 4$ and $\lambda = (2,2,0,0)$ then

$$m_\lambda(x) = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2,$$

and $\bar{m}_\lambda(x) = m_\lambda(x)/6$. (For more about monomial symmetric functions, see e.g., Chapter 7 of [5].) For $\lambda, \mu \in \Lambda$ we write $\bar{m}_\lambda \ge \bar{m}_\mu$ if $\bar{m}_\lambda(x) \ge \bar{m}_\mu(x)$ holds for all

$x \geq 0$, where $x \geq 0$ means that $x = (x_1, \ldots, x_r)$ satisfies $x_i \geq 0$ for all $1 \leq i \leq r$. In [4] (see also [2, 3]) Muirhead proved that

$$\bar{m}_\lambda \geq \bar{m}_\mu \text{ if and only if } \lambda \succ \mu,$$

where $\lambda \succ \mu$ means

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } 1 \leq i \leq r. \tag{3}$$

Moreover $\bar{m}_\lambda(x) = \bar{m}_\mu(x)$ if and only if $\lambda = \mu$ or $x_1 = \cdots = x_r$.

Now we define a bipartite graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ corresponding to Λ as follows. For the vertex set let $V(\mathcal{G}) = U \sqcup U'$, where U and U' are distinct copies of Λ. Then two vertices $\lambda \in U$ and $\lambda' \in U'$ are adjacent in $\mathcal{G}$ if and only if $\lambda \succ \lambda'$. Let $c : V(\mathcal{G}) \to \mathbb{R}_{\geq 0}$ be a given capacity function. We say that a flow $\phi : E(\mathcal{G}) \to \mathbb{R}_{\geq 0}$ is optimal if

$$\sum_{\lambda \in e} \phi(e) \leq c(\lambda) \text{ for all } \lambda \in U, \text{ and} \sum_{\lambda' \in e} \phi(e) = c(\lambda') \text{ for all } \lambda' \in U'.$$

Let $\alpha : \Lambda \to \mathbb{R}_{\geq 0}$, which can be viewed as a coefficient vector $\alpha = (\alpha_\lambda : \lambda \in \Lambda) \in \mathbb{R}^\Lambda_{\geq 0}$. Then we define a linear combination of normalized symmetric monomial functions

$$\bar{m}(\alpha) := \sum_{\lambda \in \Lambda} \alpha_\lambda \bar{m}_\lambda. \tag{4}$$

Let $\alpha, \alpha' \in \mathbb{R}^\Lambda_{\geq 0}$ and let c be a capacity function on $V(\mathcal{G}) = U \sqcup U'$ defined by $c = \alpha$ on U and $c = \alpha'$ on U'. We say that α suppresses α', denoted by $\alpha \succ \alpha'$, if $\mathcal{G}$ admits an optimal flow. While the Muirhead's inequality says that if $\lambda \succ \mu$ then $\bar{m}_\lambda \succ \bar{m}_\mu$, our result stated below claims that if $\alpha \succ \alpha'$ then $\bar{m}(\alpha) \geq \bar{m}(\alpha')$. This can be seen as an extension from the inequality about normalized symmetric monomial functions to the inequality about linear combinations of them.

Theorem 1 *Let $\alpha, \alpha' \in \mathbb{R}^\Lambda_{\geq 0}$. If $\alpha \succ \alpha'$ then $\bar{m}(\alpha) \geq \bar{m}(\alpha')$ with equality holding if and only if the following two conditions are satisfied*:

- (i) $\sum_{\lambda \in e} \phi(e) = c(\lambda)$ for all $\lambda \in U$, and
- (ii) $\bar{m}_\lambda \geq \bar{m}_{\lambda'}$ for all adjacent $\lambda \in U$ and $\lambda' \in U'$.

Proof Let $\mathcal{G}$ be the bipartite graph on $V(\mathcal{G}) = U \sqcup U'$ defined above, and let ϕ be an optimal flow. If $\lambda \in U$ and $\lambda' \in U'$ are adjacent by an edge e in $\mathcal{G}$, then $\bar{m}_\lambda \geq \bar{m}_{\lambda'}$ and $\phi(e)\bar{m}_\lambda \geq \phi(e)\bar{m}_{\lambda'}$. Using this trivial fact we have

$$\begin{aligned}
\bar{m}(\alpha) &= \sum_{\lambda\in U} c(\lambda)\bar{m}_\lambda \\
&\geq \sum_{\lambda\in U}\left(\sum_{\lambda\in e}\phi(e)\right)\bar{m}_\lambda \\
&= \sum_{e\in E(\mathcal{G})}\sum_{\lambda\in e}\phi(e)\bar{m}_\lambda \\
&\geq \sum_{e\in E(\mathcal{G})}\sum_{\lambda'\in e}\phi(e)\bar{m}_{\lambda'} \\
&= \sum_{\lambda'\in U'}\left(\sum_{\lambda'\in e}\phi(e)\right)\bar{m}_{\lambda'} \\
&= \sum_{\lambda'\in U'} c(\lambda')\bar{m}_{\lambda'} \\
&= \bar{m}(\alpha').
\end{aligned}$$

One can readily verify the equality conditions. □

We note that $\bar{m}(\alpha)\geq\bar{m}(\alpha')$ does not necessarily imply $\alpha\succ\alpha'$, see Example 1 in the last section. Using Theorem 1 we were able to verify Conjecture 1 for $3\leq r\leq 11$.

Theorem 2 *Conjecture 1 is true for* $3\leq r\leq 11$.

Though our proof of Theorem 2 is based on Theorem 1, we need two more ideas. First, we actually prove a stronger inequality (see Conjecture 2) holding for all non-negative variables, from which we derive the inequality (1). This is needed because (1) holds only for variables in the unit interval, while Theorem 1 only applies to inequalities valid for all non-negative variables. Second, as a bipartite graph applied to Theorem 1 we do not use the graph whose vertex set is Λ because it is too large. Instead we construct a graph on a much smaller vertex set which has a nicer poset structure induced by (3) (see Theorem 3). This reduces the computation markedly.

2 Proof of Theorem 2

Throughout this section let $r\geq 3$ be a fixed integer. As mentioned in the end of the previous section the inequality (1) is not suitable for applying Theorem 1. So we should find an inequality which holds for all non-negative variables and implies (1). This inequality will be obtained by factorizing (1).

For $s=(s_1,\ldots,s_r)$ and $1\leq k\leq r$ we define the elementary symmetric functions $e_k(s)$ by

$$e_k(s) := \sum_{i_1<\cdots<i_k} s_{i_1}\cdots s_{i_k},$$

see, e.g., [5]. Let $e_0(s):=1$. Then we have

$$\prod_{i=1}^{r}(s_i+z)=\sum_{k=0}^{r}e_k(s)z^{r-k}.$$

To factorize the LHS of (1), we apply the above identity to $s=(s_1,\ldots,s_r)$, where

$$s_i=a_i+a_i^2+\cdots+a_i^{r-1}, \tag{5}$$

and $z=1$ or $z=a$. Then we have

$$\prod_{i=1}^{r}\left(a_i+a_i^2+\cdots+a_i^{r-1}+a_i^r\right)=\prod_{i=1}^{r}a_i(1+s_i)=a\prod_{i=1}^{r}(s_i+1)=\sum_{k=0}^{r}e_k(s)a,$$
$$\prod_{i=1}^{r}\left(a_i+a_i^2+\cdots+a_i^{r-1}+a\right)=\prod_{i=1}^{r}(s_i+a)=\sum_{k=0}^{r}e_k(s)a^{r-k}.$$

Thus, writing e_k instead of $e_k(s)$ for simplicity, the LHS of (1) is rewritten as

$$\begin{aligned}&\sum_{k=0}^{r}(a-a^{r-k})e_k\\&=\sum_{k=0}^{r-2}(a-a^{r-k})e_k-(1-a)e_r\\&=(a-a^r)e_0+(a-a^{r-1})e_1+\cdots+(a-a^2)e_{r-2}-(1-a)e_r\\&=(1-a)\left(\tilde{F}-\tilde{G}\right),\end{aligned}$$

where

$$\tilde{F}=(a+a^2+\cdots+a^{r-1})e_0+(a+\cdots+a^{r-2})e_1+\cdots+(a+a^2)e_{r-3}+ae_{r-2},$$
$$\tilde{G}=e_r.$$

Letting

$$f_i=e_0+e_1+\cdots+e_i, \tag{6}$$

we have

$$\begin{aligned}\tilde{F}&=a(e_0+e_1+\cdots+e_{r-2})+a^2(e_0+\cdots+e_{r-3})+\cdots+a^{r-2}(e_0+e_1)+a^{r-1}e_0,\\&=a(f_{r-2}+f_{r-3}a+\cdots+f_1a^{r-3}+f_0a^{r-2}),\end{aligned}$$

and also

$$\tilde{G}=e_r=\prod_{i=1}^{r}s_i=\prod_{i=1}^{r}a_i(s_i/a_i)=a\prod_{i=1}^{r}(1+a_i+\cdots+a_i^{r-2}).$$

Consequently we obtain the following expression:

$$\text{'the LHS of (1)'} = a(1-a)(F-G),$$

where

$$F = f_{r-2} + f_{r-3}a + \cdots + f_1 a^{r-3} + f_0 a^{r-2}, \tag{7}$$

$$G = \prod_{i=1}^{r}(1 + a_i + \cdots + a_i^{r-2}). \tag{8}$$

Finally to prove Conjecture 1 it suffices to show the following conjecture.

Conjecture 2 *Let $r \geq 3$ be an integer, and let $a_1, \ldots, a_r$ be non-negative real numbers. Then*

$$F - G \geq 0,$$

where F and G are defined by (7) and (8) with (5) and (6). Moreover, equality holds if and only if $a_1 = \cdots = a_r$.

Note that Conjecture 2 is slightly stronger than Conjecture 1 because the condition for a_i is weaker. Moreover this condition is precisely what we need to apply Theorem 1. Note also that $F - G \geq 0$ if

$$F^{(d)} - G^{(d)} \geq 0$$

for every d, where $F^{(d)}$ (resp. $G^{(d)}$) denote the degree d part of F (resp. G).

Now we translate the problem of showing $F^{(d)} - G^{(d)} \geq 0$ (for fixed r and d) to a problem of finding an optimal flow. Recall the definitions of Λ and $\bar{m}(\alpha)$ from (2) and (4). Let $\alpha, \alpha' \in \mathbb{R}^{\Lambda}_{\geq 0}$ be such that

$$\bar{m}(\alpha) = F^{(d)} \text{ and } \bar{m}(\alpha') = G^{(d)}.$$

Then, by Theorem 1, $F^{(d)} - G^{(d)} \geq 0$ follows from $\alpha \succ \alpha'$. Thus our problem is translated to show $\alpha \succ \alpha'$. However the number of non-zero entries in α and α' grows rapidly as r grows, and it is not so easy to verify $\alpha \succ \alpha'$ in this naive setting in practice. To overcome the difficulty we look at the posets derived from the polynomials $F^{(d)}$ and $G^{(d)}$ in detail, and we reduce the complexity using the structure of the posets.

Let $\Lambda(\alpha') = \{\mu \in \Lambda : \alpha'_\mu > 0\}$ be the set of partitions corresponding to $G^{(d)}$. For $\mu \in \Lambda(\alpha')$ it follows from (8) that $\mu_1 \leq r-2$. We partition $\Lambda(\alpha')$ by the value of μ_r. Let

$$\Lambda_h = \{\lambda \in \Lambda : \lambda_r = h\}.$$

Then we have $\Lambda(\alpha') = \bigsqcup_{h=0}^{r-2} \tilde{Q}_h$, where

$$\tilde{Q}_h = \{\mu \in \Lambda_h : \mu_1 \leq r-2\}. \tag{9}$$

Lemma 1 *Let $\Lambda(\alpha) = \{\lambda \in \Lambda : \alpha_\lambda > 0\}$. Then we have $\Lambda(\alpha) = \bigsqcup_{h=0}^{r-2} \tilde{P}_h$, where*

$$\tilde{P_h} = \{\lambda \in \Lambda_h : \lambda_1 \le r - 1 + h,\ \lambda_{r-1-h} = \lambda_r\}. \tag{10}$$

Proof In view of (7), $\lambda \in \tilde{P_h}$ comes from $f_{r-2-h}a^h$. Then, λ is decomposed into two parts $\mu = (\mu_1, \ldots, \mu_r)$ from f_{r-2-h} and $v = (v_1, \ldots, v_r)$ from a^h. By (5) and (6) we have

$$r - 1 \ge \mu_1 \ge \cdots \ge \mu_{r-h-2} \ge 0 = \mu_{r-1-h} = \cdots = \mu_r.$$

We also have $v_1 = \cdots = v_r = h$. Then the set of sequences $\lambda = \mu + v$ defines $\tilde{P_h}$. □

We note that $I := \Lambda(\alpha) \cap \Lambda(\alpha')$ is nonempty. For example we have

$$\tilde{P}_{r-2} = \tilde{Q}_{r-2} = \{(r-2, \ldots, r-2)\}.$$

We want to look at $\Lambda(\alpha) \setminus I$ and $\Lambda(\alpha') \setminus I$ rather than $\Lambda(\alpha)$ and $\Lambda(\alpha')$. To this end we need some preparation. Recall that Λ itself is a poset, and the bipartite graph $\mathcal{G}$ is defined on $V(\mathcal{G}) = U \sqcup U'$, where both U and U' are distinct copies of Λ. For $x \in \mathbb{R}^\Lambda$ let $p(x) := (y_\lambda : \lambda \in \Lambda)$, where

$$y_\lambda := \begin{cases} x_\lambda & \text{if } x_\lambda > 0, \\ 0 & \text{if } x_\lambda \le 0, \end{cases}$$

that is, $p(x)$ extracts positive entries from x. Let $\beta := p(\alpha - \alpha')$ and $\beta' := p(\alpha' - \alpha)$. Then $\alpha - \alpha' = \beta - \beta'$ and

$$\bar{m}(\alpha) - \bar{m}(\alpha') = \bar{m}(\alpha - \alpha') = \bar{m}(\beta - \beta').$$

So our aim is to show that $\bar{m}(\beta - \beta') \ge 0$. These coefficient vectors β and β' define two subposets of Λ:

$$\Lambda(\beta) := \{\lambda \in U : \beta_\lambda > 0\} \text{ and } \Lambda(\beta') := \{\mu \in U' : \beta'_\mu > 0\}.$$

These posets are equipped with a nice property as stated in the next theorem, and this is the reason why we can efficiently verify the existence of an optimal flow in $\mathcal{G}$ with the capacity function coming from β and β'.

Theorem 3 *There exist unique positive integer k and partitions*

$$\Lambda(\beta) = A_1 \sqcup \cdots \sqcup A_k,$$
$$\Lambda(\beta') = B_1 \sqcup \cdots \sqcup B_k,$$

with representatives $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k$ *and* $\tilde{\mu}_1, \ldots, \tilde{\mu}_k$ *satisfying* $\tilde{\lambda}_i = \min A_i$, $\tilde{\mu}_i = \max B_i$, *and* $\tilde{\lambda}_i \succ \tilde{\mu}_i$ *for all* $1 \le i \le k$.

For a concrete example of such partitions, see Example 2 in the next section. We remark that A_i (resp. B_i) does not necessarily have the maximum (resp. minimum) element, see Example 3.

We partition $\Lambda(\beta)$ and $\Lambda(\beta')$ by the value of the last element. It follows from (9) and (10) that $\Lambda(\beta) = \bigsqcup_{h=0}^{r-3} P_h$ and $\Lambda(\beta') = \bigsqcup_{h=0}^{r-3} Q_h$, where

$$P_h = \{\lambda \in \Lambda_h : r-1 \le \lambda_1 \le r-1+h,\ \lambda_{r-1-h} = \lambda_r\},$$
$$Q_h = \{\mu \in \Lambda_h : \mu_1 \le r-2,\ \mu_{r-1-h} > \mu_r\}.$$

Then the disjoint union $P_h \sqcup Q_h$ is a proper subset of Λ_h.

We have already fixed r and d, and now we fix h. We will define two maps $D^\infty : P_h \to Q_h$ and $U^\infty : Q_h \to P_h$, which will play a key role for the proof of Theorem 3. To this end we need two auxiliary maps D and U (down and up, respectively). Before going into the details of the proof we explain our plan. Let $\lambda \in P_h$ and $\mu \in Q_h$ with $\lambda \succ \mu$. We can draw the Young diagrams corresponding to λ and μ, and starting from λ we can get μ by moving a box at a right upper corner to a left lower corner one by one. (We will give a formal definition of such operations shortly.) In this process we can find $D^\infty(\lambda)$ and $U^\infty(\mu)$ such that $\lambda \succ U^\infty(\mu) \succ D^\infty(\lambda) \succ \mu$ with some additional nice properties. Actually we will get $D^\infty(\lambda)$ from λ by repeating the down map D finitely many times, and we will get $U^\infty(\mu)$ from μ by repeating the up map U finitely many times.

Definition 1 For $\lambda \in \Lambda_h \setminus Q_h$ define p and q as follows.
(Case I) If $\lambda_1 \ge r-1$ then

$$p = \max\{i : \lambda_i = \lambda_1\},$$
$$q = \min\{i : \lambda_i \le \lambda_1 - 2\}.$$

(Case II) If $\lambda_1 < r-1$ and $\lambda_{r-1-h} = \lambda_r$ then

$$p = \max\{i : \lambda_i \ge \lambda_r + 2\},$$
$$q = \min\{i : \lambda_i = \lambda_r\}.$$

Define a map $D : \Lambda_h \to \Lambda_h$ by $D(\lambda) := \lambda'$ if $\lambda \notin Q_h$, where

$$\lambda'_i = \begin{cases} \lambda_p - 1 & \text{if } i = p,\\ \lambda_q + 1 & \text{if } i = q,\\ \lambda_i & \text{otherwise,}\end{cases}$$

and by $D(\lambda) := \lambda$ if $\lambda \in Q_h$. (See Example 2.)

Definition 2 For $\mu \in \Lambda_h \setminus P_h$ define p and q as follows.
(Case III) If $\mu_{r-1-h} > \mu_r$ then

$$p = \min\{i : \mu_i = \mu_{r-2-h}\},$$
$$q = \max\{i : \mu_i > \mu_r\}.$$

(Case IV) If $\mu_{r-1-h} = \mu_r$ and $\mu_1 \le r-2$ then

$$p = 1,$$
$$q = \max\{i : \mu_i = \mu_2\}.$$

Define a map $U : \Lambda_h \to \Lambda_h$ by $U(\mu) := \mu'$ if $\mu \notin P_h$, where

$$\mu'_i = \begin{cases} \mu_p + 1 & \text{if } i = p, \\ \mu_q - 1 & \text{if } i = q, \\ \mu_i & \text{otherwise,} \end{cases}$$

and by $U(\mu) := \mu$ if $\mu \in P_h$.

Lemma 2 *Let $\lambda \in \Lambda_h \setminus Q_h$ and $\mu \in Q_h$. If $\lambda \succ \mu$ then $D(\lambda) \succ \mu$.*

Proof Let $\lambda' = D(\lambda)$. By definition of λ' we only need to check that

$$\lambda'_1 + \cdots + \lambda'_i \geq \mu_1 + \cdots + \mu_i \tag{11}$$

for all $p \leq i < q$ because if $i < p$ or $i \geq q$ then the sum for λ' and the sum for λ are the same.

For Case I we have $\lambda'_i \geq \lambda_i - 1 \geq r - 2 \geq \mu_i$ for each $p \leq i < q$, which yields (11).

For Case II we note that if $p < j < q$ then $h + 1 = \lambda_j \leq \mu_j$ and $h = \lambda_q < \mu_q$ (and if $j > q$ then $\lambda_j = \mu_j$). Hence, for $p \leq i < q$, we have

$$\lambda_{i+1} + \cdots + \lambda_r < \mu_{i+1} + \cdots + \mu_r.$$

Since $|\lambda| = |\mu| = d$ it follows that

$$\lambda_1 + \cdots + \lambda_i > \mu_1 + \cdots + \mu_i.$$

Thus we have

$$\lambda'_1 + \cdots + \lambda'_p + \cdots + \lambda'_i = \lambda_1 + \cdots + (\lambda_p - 1) + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i,$$

as needed. □

Lemma 3 *Let $\mu \in \Lambda_h \setminus P_h$ and $\lambda \in P_h$. If $\lambda \succ \mu$ then $\lambda \succ U(\mu)$.*

Proof Let $\mu' = U(\mu)$. We only need to show that

$$\lambda_1 + \cdots + \lambda_i \geq \mu'_1 + \cdots + \mu'_i$$

for all $p \leq i < q$.

For Case III suppose, to the contrary, that

$$\lambda_1 + \cdots + \lambda_i + 1 \leq \mu'_1 + \cdots + \mu'_i \tag{12}$$

for some $p \leq i < q$. The RHS is equal to $\mu_1 + \cdots + \mu_i + 1$, and at most $\lambda_1 + \cdots + \lambda_i + 1$ because $\lambda \succ \mu$. Thus we have

$$\lambda_1 + \cdots + \lambda_i = \mu_1 + \cdots + \mu_i. \tag{13}$$

Since $|\lambda| = |\mu|$ we also have $\lambda_{i+1} + \cdots + \lambda_r = \mu_{i+1} + \cdots + \mu_r$. Moreover, noting $\lambda_j < \mu_j$ for $r - 1 - h \leq j \leq q$, and $\lambda_j = \mu_j$ for $q < j \leq r$, we have that $p \leq i < r - 1 - h$ and

$$\lambda_{i+1} + \cdots + \lambda_{r-2-h} > \mu_{i+1} + \cdots + \mu_{r-2-h}.$$

This together with $\mu_{i+1} = \mu_{r-2-h}$ gives us $\lambda_{i+1} > \mu_{i+1}$ and

$$\lambda_i \geq \lambda_{i+1} > \mu_{i+1} = \mu_i.$$

On the other hand it follows from (13) and $\lambda_1 + \cdots + \lambda_{i-1} \geq \mu_1 + \cdots + \mu_{i-1}$ that $\lambda_i \leq \mu_i$. This is a contradiction.

For Case IV suppose (12). Since $\lambda_1 \geq r-1 \geq \mu_1 + 1 = \mu'_1$ we may assume that $2 \leq i < q$. Using $\lambda_1 > \mu_1$ and $\mu_2 = \mu_i$ with (13) we have $\lambda_i < \mu_i$. Then $\lambda_{i+1} \leq \lambda_i < \mu_i = \mu_{i+1}$ and

$$\lambda_1 + \cdots + \lambda_{i+1} = \mu_1 + \cdots + \mu_i + \lambda_{i+1} < \mu_1 + \cdots + \mu_i + \mu_{i+1},$$

which contradicts the assumption $\lambda \succ \mu$. □

Let $D^n = (D \circ \cdots \circ D)$ (n times), and define $D^\infty = \lim_{n\to\infty} D^n$. If $\lambda \in P_h$ then $D^\infty(\lambda) = D^n(\lambda)$ for some $n \leq r$. Indeed we first repeat Case I until $\lambda_1 < r-1$, and next repeat Case II until $\lambda_{r-1-h} > \lambda_r$, and then eventually the resulting λ comes into Q_h. Thus D^∞ is a map from P_h to Q_h. Similarly we define a map $U^\infty : Q_h \to P_h$ by $U^\infty = \lim_{n\to\infty} U^n$, which is actually obtained by applying U at most r times. By Lemma 2 and Lemma 3 we get the following results. (See Example 3.)

Lemma 4 *Let $\lambda \in P_h$ and $\mu \in Q_h$ with $\lambda \succ \mu$.*

(1) $D^\infty(\lambda) \succ \mu$ and $D^\infty(\lambda) = \max\{\mu' \in Q_h : \lambda \succ \mu'\}$.
(2) $\lambda \succ U^\infty(\mu)$ and $U^\infty(\mu) = \min\{\lambda' \in P_h : \lambda' \succ \mu\}$.

Lemma 5 *It follows that*

(i) $(U^\infty \circ D^\infty)(\lambda) = \lambda$ for all $\lambda \in U^\infty(Q_h)$, and
(ii) $(D^\infty \circ U^\infty)(\mu) = \mu$ for all $\mu \in D^\infty(P_h)$.

Proof Let $\mu \in Q_h$, and define $\lambda := U^\infty(\mu)$, $\mu' := D^\infty(\lambda)$, and $\lambda' := U^\infty(\mu')$.

Since $\lambda = U^\infty(\mu)$ we have $\lambda \succ \mu$. Then by (1) of Lemma 4 we have

$$\mu' = D^\infty(\lambda) = \max\{\mu'' \in Q_h : \lambda \succ \mu''\} \succ \mu.$$

Thus $\{\lambda'' \in P_h : \lambda'' \succ \mu'\} \subset \{\lambda'' \in P_h : \lambda'' \succ \mu\}$ and taking the minimum element of each set we get $\lambda' \succ \lambda$. On the other hand $\lambda \succ \mu'$ follows from $\mu' = D^\infty(\lambda)$. Applying (2) of Lemma 4 we obtain $\lambda \succ U^\infty(\mu') = \lambda'$. Consequently $\lambda = \lambda'$ and

$(U^\infty \circ D^\infty)(\lambda) = U^\infty(\mu') = \lambda' = \lambda$. This proves (i) of this lemma. One can show (ii) similarly. □

Proof of Theorem 3 By Lemma 5 there exists $k = k(h)$ such that $U^\infty(Q_h) = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k\}$ and $D^\infty(P_h) = \{\tilde{\mu}_1, \ldots, \tilde{\mu}_k\}$. For $1 \le i \le k$ let

$$A_i = \{\lambda \in P_h : (U^\infty \circ D^\infty)(\lambda) = \tilde{\lambda}_i\},$$
$$B_i = \{\mu \in Q_h : (D^\infty \circ U^\infty)(\mu) = \tilde{\mu}_i\}.$$

Then $P_h = A_1 \sqcup \cdots \sqcup A_k$ and $Q_h = B_1 \sqcup \cdots \sqcup B_k$. Moreover, by Lemma 4, we have $\tilde{\lambda}_i = \min A_i$ and $\tilde{\mu}_i = \max B_i$. Finally we get the desired partitions $\Lambda(\beta) = \bigsqcup_{h=0}^{r-3} \bigsqcup_{i=1}^{k(h)} A_i$ and $\Lambda(\beta') = \bigsqcup_{h=0}^{r-3} \bigsqcup_{i=1}^{k(h)} B_i$. □

Using Theorem 3 we can further reduce the problem and decrease the computation sharply. This reduction is based on the following simple observation. By Theorem 3 we have $\tilde{\lambda}_i = \min A_i$ implying

$$\sum_{\lambda \in A_i} \beta_\lambda \bar{m}_\lambda \ge \tilde{c}_i \, \bar{m}_{\tilde{\lambda}_i},$$

where $\tilde{c}_i := \sum_{\lambda \in A_i} \beta_\lambda$, and similarly

$$\sum_{\mu \in B_i} \beta'_\mu \bar{m}_\mu \ge \tilde{c}'_i \, \bar{m}_{\tilde{\mu}_i},$$

where $\tilde{c}'_i := \sum_{\mu \in B_i} \beta'_\mu$. So we define two coefficient vectors $s, s' \in \mathbb{R}^\Lambda_{\ge 0}$ as follows: for each $\lambda, \mu \in \Lambda$ let

$$s_\lambda := \begin{cases} \tilde{c}_i & \text{if } \lambda = \tilde{\lambda}_i \text{ for some } i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad s'_\mu := \begin{cases} \tilde{c}_i' & \text{if } \mu = \tilde{\mu}_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from the definition that $\bar{m}(\beta) \ge \bar{m}(s)$, $\bar{m}(s') \ge \bar{m}(\beta')$, and

$$\bar{m}(\beta - \beta') \ge \bar{m}(s - s').$$

Thus our aim is now reduced to showing $s \succ s'$, which yields $\bar{m}(s - s') \ge 0$ and so $\bar{m}(\beta - \beta') \ge 0$, as required. Finally, similarly as we changed from α, α' to β, β' we modify s, s' one last time to get $t, t' \in \mathbb{R}^\Lambda_{\ge 0}$ by $t := p(s - s')$ and $t' := p(s' - s)$. Since $t - t' = s - s'$ it suffices to show $t \succ t'$ to verify $s \succ s'$, see Example 4 in the next section. In summary we have shown that

$$\bar{m}(\alpha) - \bar{m}(\alpha') = \bar{m}(\beta) - \bar{m}(\beta') \geq \bar{m}(s) - \bar{m}(s') = \bar{m}(t) - \bar{m}(t').$$

The point is that the number of non-zero entries in t and t' is much smaller than those in α and α', for example, if $r = 11$ and $d = 52$ then the former is only 367 while the latter is 6594. This is the reason why such reductions make the computation much faster. Consequently we can complete the proof of Conjecture 2 by showing $t \succ t'$, and we have indeed verified this for $3 \leq r \leq 11$ (and all d) with aid of a computer.

Thus the inequality in Conjecture 2 holds for $3 \leq r \leq 11$. Moreover if $a_1 = \cdots = a_r$, then clearly $F - G = 0$. On the other hand if $F - G = 0$ then we need $\bar{m}(\alpha) - \bar{m}(\alpha') = 0$. This together with (ii) of Theorem 1 implies that $\bar{m}_\lambda = \bar{m}_{\lambda'}$ for all adjacent λ and λ' in the graph $\mathcal{G}$. Therefore all a_i are the same, which verifies the equality condition in Conjecture 2. This completes the proof of Theorem 2.

3 Some Examples

Example 1 We present an example showing that the converse of Theorem 1 does not hold, that is, an example satisfying $\bar{m}(\alpha) \geq \bar{m}(\alpha')$ but $\alpha \not\succ \alpha'$.

Let $r = 2$ and $d = 4$. Then $\Lambda = \{(4,0), (3,1), (2,2)\}$, and

$$\bar{m}_{(4,0)}(x) = \frac{1}{2}(x_1^4 + x_2^4), \quad \bar{m}_{(3,1)}(x) = \frac{1}{2}(x_1^3x_2 + x_1x_3^3), \quad \bar{m}_{(2,2)}(x) = x_1^2x_2^2.$$

Let $\alpha = (2,0,2)$ and $\alpha' = (0,4,0)$. Then we have

$$\begin{aligned}\bar{m}(\alpha) &= 2\bar{m}_{(4,0)}(x) + 2\bar{m}_{(2,2)}(x) = (x_1^4 + x_2^4) + 2x_1^2x_2^2,\\ \bar{m}(\alpha') &= 4\bar{m}_{(3,1)}(x) = 2(x_1^3x_2 + x_1x_2^3).\end{aligned}$$

A routine calculus shows that $\bar{m}(\alpha) \geq \bar{m}(\alpha')$ for all $x_1, x_2 \geq 0$.

Let $\mathcal{G}$ be the corresponding bipartite graph. Then, by writing only vertices with positive capacities, we have $V(\mathcal{G}) = \{(4,0),(2,2)\} \sqcup \{(3,1)\}$. There is only one edge joining (4, 0) and (3, 1), where the capacity of (4, 0) is 2 while the capacity of (3, 1) is 4. Thus there is no optimal flow, and $\alpha \not\succ \alpha'$.

Example 2 Let $r = 8$, $d = 29$, and $h = 0$. Let $\tilde{\lambda} = (7,7,5,4,4,2,0,0) \in P_0$ and $\tilde{\mu} = (6,6,6,5,4,1,1,0) \in Q_0$. We get the following sequence by applying D:

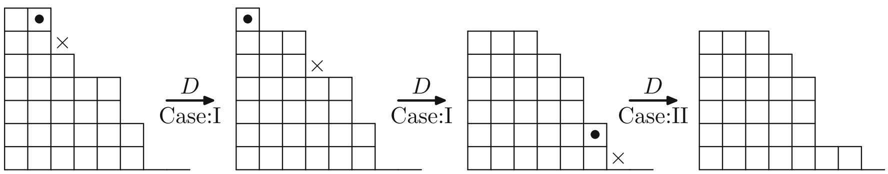

Fig. 1 The down map sending $\tilde{\lambda} = 77544200$ to $\tilde{\mu} = 66654110$

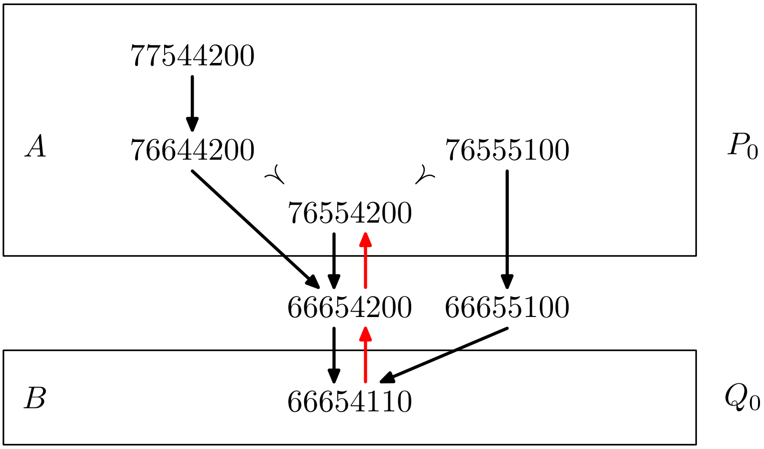

Fig. 2 Example for Lemma 4 and Lemma 5

$$\tilde{\lambda} \longrightarrow (7,6,6,4,4,2,0,0) \longrightarrow (6,6,6,5,4,2,0,0) \longrightarrow \tilde{\mu}.$$

This shows that $D^{\infty}(\tilde{\lambda}) = D^3(\tilde{\lambda}) = \tilde{\mu}$. See Figure 1, where the map D sends a box with '•' to the position marked by '×.'

Example 3 (Continued from Example 2) Let

$$A := \{\lambda \in P_0 : (U^{\infty} \circ D^{\infty})(\lambda) = \tilde{\lambda}\},$$
$$B := \{\mu \in Q_0 : (D^{\infty} \circ U^{\infty})(\mu) = \tilde{\mu}\}.$$

Then it follows that $A = \{76554200, 76555100, 76644200, \tilde{\lambda}\}$, and $B = \{\tilde{\mu}\}$, and moreover $\min A = \tilde{\lambda}$ (but A does not have the maximum element) and $\max B = \tilde{\mu}$, see Figure 2, where black arrows correspond to D and red arrows correspond to U.

Example 4 We describe the partitions in Theorem 3 for the case $r = 7$ and $d = 15$. Then $\Lambda(\beta) = \{\lambda_1, \ldots, \lambda_{19}\}$ and $\Lambda(\beta') = \{\mu_1, \ldots, \mu_{19}\}$ listed below with their capacities, e.g., $\lambda_1 = (6,2,2,2,1,1,1)$ and $c(\lambda_1) = 140$.

λ_1	6222111	140	μ_1	3322221	105
λ_2	6321111	210	μ_2	3332211	210
λ_3	6411111	42	μ_3	3332220	140
λ_4	7221111	105	μ_4	3333210	210
λ_5	7311111	42	μ_5	4222221	42
λ_6	6322200	420	μ_6	4322211	420
λ_7	6332100	1260	μ_7	4322220	210

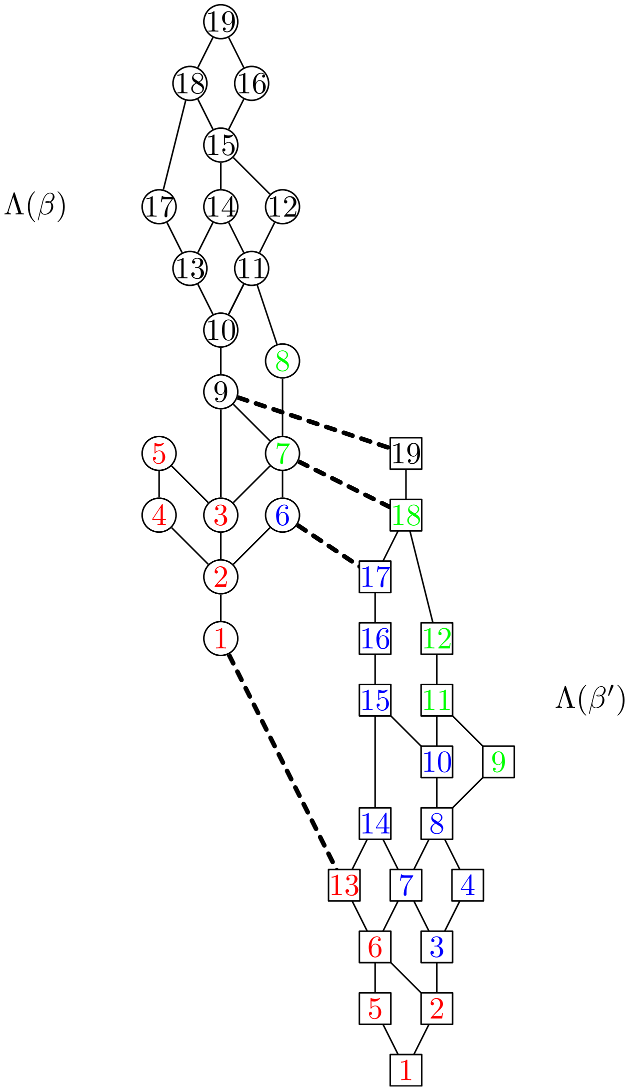

Fig. 3 Hasse diagram of $\Lambda(\beta)$ and $\Lambda(\beta')$ as posets for $r = 7$ and $d = 15$

λ_8	6333000	140	μ_8	4332210	1260
λ_9	6422100	1260	μ_9	4333110	420
λ_{10}	6431100	1260	μ_{10}	4422210	420
λ_{11}	6432000	840	μ_{11}	4432110	1260
λ_{12}	6441000	420	μ_{12}	4441110	140
λ_{13}	6521100	1260	μ_{13}	5222211	105
λ_{14}	6522000	420	μ_{14}	5222220	42
λ_{15}	6531000	840	μ_{15}	5322210	840
λ_{16}	6540000	210	μ_{16}	5332110	1260
λ_{17}	6611100	210	μ_{17}	5422110	1260
λ_{18}	6621000	420	μ_{18}	5431110	840
λ_{19}	6630000	105	μ_{19}	5521110	420

Then we can view $\Lambda(\beta)$ and $\Lambda(\beta')$ as posets, where the partial order is introduced by the majorization. Figure 3 shows the corresponding Hasse diagrams, where λ_i is denoted by ⓘ and μ_j is denoted by $\boxed{j}$.

The partitions in Theorem 3 in this case are

$$\Lambda(\beta) = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4,$$
$$\Lambda(\beta') = B_1 \sqcup B_2 \sqcup B_3 \sqcup B_4,$$

where

$$\begin{aligned} A_1 &= \{\lambda_i : 1 \le i \le 5\}, & B_1 &= \{\mu_1, \mu_2, \mu_5, \mu_6, \mu_{13}\}, \\ A_2 &= \{\lambda_6\}, & B_2 &= \{\mu_3, \mu_4, \mu_7, \mu_8, \mu_{10}, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{17}\}, \\ A_3 &= \{\lambda_7, \lambda_8\}, & B_3 &= \{\mu_9, \mu_{11}, \mu_{12}, \mu_{18}\}, \\ A_4 &= \{\lambda_i : 9 \le i \le 19\}, & B_4 &= \{\mu_{19}\}. \end{aligned}$$

The representatives $\tilde{\lambda}_i = \min A_i$ and $\tilde{\mu}_i = \max B_i$ are given by

$$\tilde{\lambda}_1 = \lambda_1, \quad \tilde{\lambda}_2 = \lambda_6, \quad \tilde{\lambda}_3 = \lambda_7, \quad \tilde{\lambda}_4 = \lambda_9,$$
$$\tilde{\mu}_1 = \mu_{13}, \quad \tilde{\mu}_2 = \mu_{17}, \quad \tilde{\mu}_3 = \mu_{18}, \quad \tilde{\mu}_4 = \mu_{19}.$$

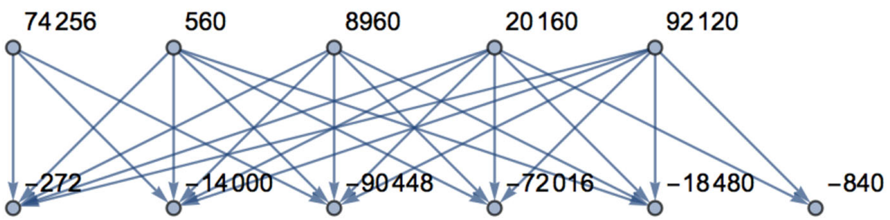

Fig. 4 The reduced graph for t and t' ($r = 8$ and $d = 29$)

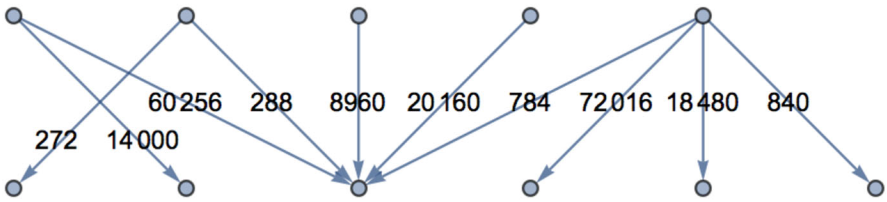

Fig. 5 An optimal flow in the reduced graph ($r = 8$ and $d = 29$)

Example 5 Here we present an example of the reduction after applying Theorem 3. Let $r = 8$, $d = 29$. In this case the number of partitions in Theorem 3 is $k = 12$, and the corresponding representatives are listed below.

$$\begin{array}{llll} \tilde{\lambda}_1 = 74333333, & \tilde{\lambda}_2 = 75542222, & \tilde{\lambda}_3 = 75544400, & \tilde{\lambda}_4 = 75554111, \\ \tilde{\lambda}_5 = 75554300, & \tilde{\lambda}_6 = 75555200, & \tilde{\lambda}_7 = 76544111, & \tilde{\lambda}_8 = 76544300, \\ \tilde{\lambda}_9 = 76553111, & \tilde{\lambda}_{10} = 76553300, & \tilde{\lambda}_{11} = 76554200, & \tilde{\lambda}_{12} = 76653200, \end{array}$$

$$\begin{array}{llll} \tilde{\mu}_1 = 54443333, & \tilde{\mu}_2 = 66533222, & \tilde{\mu}_3 = 66544310, & \tilde{\mu}_4 = 66553211, \\ \tilde{\mu}_5 = 66554210, & \tilde{\mu}_6 = 66555110, & \tilde{\mu}_7 = 66643211, & \tilde{\mu}_8 = 66644210, \\ \tilde{\mu}_9 = 66652211, & \tilde{\mu}_{10} = 66653210, & \tilde{\mu}_{11} = 66654110, & \tilde{\mu}_{12} = 66663110. \end{array}$$

The corresponding capacities are the following.

$$\begin{array}{llll} \tilde{c}_1 = 64, & \tilde{c}_2 = 13272, & \tilde{c}_3 = 2520, & \tilde{c}_4 = 1120, \\ \tilde{c}_5 = 3360, & \tilde{c}_6 = 840, & \tilde{c}_7 = 6720, & \tilde{c}_8 = 11760, \\ \tilde{c}_9 = 75936, & \tilde{c}_{10} = 16800, & \tilde{c}_{11} = 23520, & \tilde{c}_{12} = 93800, \end{array}$$

$$\begin{array}{llll} \tilde{c}'_1 = 336, & \tilde{c}'_2 = 27272, & \tilde{c}'_3 = 74536, & \tilde{c}'_4 = 91568, \\ \tilde{c}'_5 = 21840, & \tilde{c}'_6 = 1680, & \tilde{c}'_7 = 6720, & \tilde{c}'_8 = 11200, \\ \tilde{c}'_9 = 1680, & \tilde{c}'_{10} = 7840, & \tilde{c}'_{11} = 3360, & \tilde{c}'_{12} = 1680. \end{array}$$

Then to construct t and t' we compute

$$\begin{array}{llll} \tilde{c}_1 - \tilde{c}'_1 = -272, & \tilde{c}_2 - \tilde{c}'_2 = -14000, & \tilde{c}_3 - \tilde{c}'_3 = -72016, & \tilde{c}_4 - \tilde{c}'_4 = -90448, \\ \tilde{c}_5 - \tilde{c}'_5 = -18480, & \tilde{c}_6 - \tilde{c}'_6 = -840, & \tilde{c}_7 - \tilde{c}'_7 = 0, & \tilde{c}_8 - \tilde{c}'_8 = 560, \\ \tilde{c}_9 - \tilde{c}'_9 = 74256, & \tilde{c}_{10} - \tilde{c}'_{10} = 8960, & \tilde{c}_{11} - \tilde{c}'_{11} = 20160, & \tilde{c}_{12} - \tilde{c}'_{12} = 92120. \end{array}$$

Figure 4 shows the reduced graph for t and t', that is, the top five vertices are corresponding to $\bar{m}(t)$ and the bottom six vertices are corresponding to $\bar{m}(t')$, and $\tilde{\lambda}$ and $\tilde{\mu}$ are adjacent if $\tilde{\lambda} \succ \tilde{\mu}$. In this picture we label the vertices by $\tilde{c}_i - \tilde{c}'_i$ instead of the capacity $|\tilde{c}_i - \tilde{c}'_i|$. Then one of the optimal flows is shown in Fig. 5.

Appendix

Write a_i for α_{p_i}, and let $q_i = 1 - p_i$.

Fact 1 Let $t \geq 1$ be an integer, and let l_j be the line $y = (r-1)x + j$. Then the walk W_{p_i} hits the line l_t with probability a_i^t, and the walk W' hits the line l_{rt} with probability β^t.

Proof Suppose that the probability that the walk W_{p_i} hits the line l_t is given by X^t for some $X \in (0,1)$. After the first step, the walk is at (0, 1) with probability p_i, and at (1, 0) with probability q_i. From (0, 1) the probability for the walk hitting l_t is X^{t-1}, and from (1, 0) the probability is X^{t-1+r}. Then it follows

$$X^t = p_i X^{t-1} + q_i X^{t-1+r},$$

that is,

$$X = p_i + q_i X^r.$$

Thus $X = a_i$, and the walk hits the line l_t with probability a_i^t.

Next suppose that the probability that the walk W' hits the line l_{rt} is given by Y^t for some $Y \in (0,1)$. After the first r steps, it is at $(x, r-x)$ with probability

$$\sum_{I \in \binom{[r]}{x}} \prod_{i \in I} p_i \prod_{j \in [r] \setminus I} q_j,$$

where $[r] = \{1, 2, \ldots, r\}$. From $(x, r-x)$ the probability for the walk hitting l_{rt} is Y^{x+t-1}. This yields

$$Y^t = \sum_{x=0}^{r} Y^{x+t-1} \sum_{I \in \binom{[r]}{x}} \prod_{i \in I} p_i \prod_{j \in [r] \setminus I} q_j,$$

that is,

$$Y = \sum_{x=0}^{r} Y^{x} \sum_{I \in \binom{[r]}{x}} \prod_{i \in I} p_i \prod_{j \in [r] \setminus I} q_j = \prod_{i=1}^{r} (p_i + q_i Y).$$

Thus $Y = \beta$, and the walk hits the line l_{rt} with probability β^t. □

Define a polynomial $f(x)$ by

$$f(X) := -X + \prod_{i=1}^{r}(p_i + q_i X).$$

By definition $f(\beta) = 0$.

Fact 2 If $0 < y < 1$ and $f(y) \le 0$, then $\beta \le y$.

Proof This follows because $f(0) > 0, f(1) = 0, f'(1) = -1 + \sum_{i=1}^{r} q_i > -1 + r \cdot \frac{1}{r} = 0$ (here we used $p_i < 1 - \frac{1}{r}$), and $f''(x) > 0$ for $x > 0$. □

Fact 3 The inequality $a := a_1 \cdots a_r \le \beta$ follows from (1).

Proof By Fact 7 it suffices to show $f(a) \le 0$. Since $a_i = p_i + q_i a_i^r$ we have

$$a = a_1 \cdots a_r = \prod_{i=1}^{r}(p_i + q_i a_i^r),$$

and

$$f(a) = -a + \prod_{i=1}^{r}(p_i + q_i a) = -\prod_{i=1}^{r}(p_i + q_i a_i^r) + \prod_{i=1}^{r}(p_i + q_i a).$$

So we need to show

$$\prod_{i=1}^{r}(p_i + q_i a) \le \prod_{i=1}^{r}(p_i + q_i a_i^r). \tag{14}$$

Solving $a_i = p_i + (1 - p_i)a_i^r$ for p_i gives

$$p_i = \frac{a_i - a_i^r}{1 - a_i^r}.$$

Then

$$p_i + q_i a = \frac{a_i - a_i^r}{1 - a_i^r} + \left(1 - \frac{a_i - a_i^r}{1 - a_i^r}\right)a = \frac{1}{1 - a_i^r}\left(a_i - a_i^r + (1 - a_i)a\right),$$
$$p_i + q_i a_i^r = a_i = \frac{1}{1 - a_i^r}\left(a_i(1 - a_i^r)\right).$$

Noting that $0 < a_i < 1$ we see that (14) is equivalent to

$$\begin{aligned}
&\prod_{i=1}^{r}(a_i - a_i^r + (1-a_i)a) \le \prod_{i=1}^{r} a_i(1-a_i^r)\\
\iff\quad &\prod_{i=1}^{r}\left(1 - a_i^{r-1} + (1-a_i)\frac{a}{a_i}\right) \le \prod_{i=1}^{r}(1-a_i^r)\\
\iff\quad &\prod_{i=1}^{r}\left(1 + a_i + \cdots + a_i^{r-2} + \frac{a}{a_i}\right) \le \prod_{i=1}^{r}\left(1 + a_i + \cdots + a_i^{r-1}\right),
\end{aligned}$$

and multiplying both sides by $a = a_1 \cdots a_r$ we get (1). □

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