Graphs and Combinatorics (2021) 37:2287–2304
https://doi.org/10.1007/s00373-021-02351-4

ORIGINAL PAPER

On the Maximal Colorings of Complete Graphs Without Some Small Properly Colored Subgraphs

Chunqiu Fang[1,2,4] · Ervin Győri[2] · Jimeng Xiao[2,3,5]

Received: 14 December 2019 / Revised: 10 May 2021 / Accepted: 4 June 2021 /
Published online: 15 June 2021

Abstract

Let $\mathrm{pr}(K_n, G)$ be the maximum number of colors in an edge-coloring of K_n with no properly colored copy of G. For a family $\mathcal{F}$ of graphs, let $\mathrm{ex}(n, \mathcal{F})$ be the maximum number of edges in a graph G on n vertices which does not contain any graphs in $\mathcal{F}$ as subgraphs. In this paper, we show that $\mathrm{pr}(K_n, G) - \mathrm{ex}(n, \mathcal{G}') = o(n^2)$, where $\mathcal{G}' = \{G - M : M \text{ is a matching of } G\}$. Furthermore, we determine the value of $\mathrm{pr}(K_n, P_l)$ for sufficiently large n and the exact value of $\mathrm{pr}(K_n, G)$, where G is C_5, C_6 and K_4^-, respectively. Also, we give an upper bound and a lower bound of $\mathrm{pr}(K_n, K_{2,3})$.

Keywords Properly colored subgraphs · Turán numbers · Anti-Ramsey numbers

C. Fang: supported in part by CSC(No. 201806210164). E. Győri: supported in part by the National Research, Development and Innovation Office NKFIH, Grants K116769, K117879 and K126853. J. Xiao: supported in part by CSC (No. 201706290171).

✉ Ervin Győri
gyori.ervin@renyi.mta.hu

Chunqiu Fang
fcq15@tsinghua.org.cn

Jimeng Xiao
xiaojimeng@mail.nwpu.edu.cn

1 Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

2 Alfréd Rényi Institute of Mathematics, ELKH, Reáltanoda u.13-15, Budapest 1053, Hungary

3 Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China

4 Yau Mathematical Sciences Center, Tsinghua University, Beijing 10084, China

5 Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, China

1 Introduction

We call a subgraph of an edge-colored graph *rainbow*, if all of its edges have different colors. While a subgraph is called *properly colored* (also can be called *locally rainbow*), if any two adjacent edges receive different colors. The *anti-Ramsey number* of a graph G in a complete graph K_n, denoted by $\mathrm{ar}(K_n, G)$, is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of G. Namely, $\mathrm{ar}(K_n, G) + 1$ is the minimum number k of colors such that any k-edge-coloring of K_n contains a rainbow copy of G. In this paper, we let $\mathrm{pr}(K_n, G)$ be the maximum number of colors in an edge-coloring of K_n with no properly colored copy of G. Namely, $\mathrm{pr}(K_n, G) + 1$ is the minimum number k of colors such that any k-edge-coloring of K_n contains a properly colored copy of G.

Given a family $\mathcal{F}$ of graphs, we call a graph G an $\mathcal{F}$*-free* graph, if G contains no graph in $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in a graph G on n vertices which is $\mathcal{F}$-free. Such a graph G is called an *extremal* graph, and the set of extremal graphs is denoted by $\mathrm{EX}(n, \mathcal{F})$. The celebrated result of Erdős-Stone-Simonovits Theorem [7, 9] states that for any $\mathcal{F}$ we have

$$\mathrm{ex}(n, \mathcal{F}) = \left(\frac{p-1}{2p} + o(1)\right)n^2, \tag{1.1}$$

where $p = \Psi(\mathcal{F}) = \min\{\chi(F) : F \in \mathcal{F}\} - 1$, is the *subchromatic number*.

The anti-Ramsey number was introduced by Erdős, Simonovits and Sós in [8]. There they showed that $\mathrm{ar}(K_n, G) \geq \mathrm{ex}(n, \mathcal{G}) + 1$, where $\mathcal{G} = \{G - e : e \in E(G)\}$ and by (1.1), they showed that $\mathrm{ar}(K_n, G) = (\frac{d-1}{2d} + o(1))n^2$, where $d = \Psi(\mathcal{G})$. This determined $\mathrm{ar}(K_n, G)$ asymptotically when $\Psi(\mathcal{G}) \geq 2$. In the case $\Psi(\mathcal{G}) = 1$, the situation is more complex. Already the cases when G is a tree or a cycle are nontrival. For a path P_k on k vertices, Simonovits and Sós [20] proved $\mathrm{ar}(K_n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon$, for large n, where $\epsilon = 0$ or 1. Jiang [11] showed $\mathrm{ar}(K_n, K_{1,p}) = \lfloor \frac{n(p-2)}{2} \rfloor + \lfloor \frac{n}{n-p+2} \rfloor$ or possibly this value plus one if certain conditions hold. For a general tree T of k edges, Jiang and West [12] proved $\frac{n}{2}\lfloor \frac{k-2}{2} \rfloor + O(1) \leq \mathrm{ar}(K_n, T) \leq \mathrm{ex}(n, T)$ for $n \geq 2k$ and conjectured that $ar(K_n, T) \leq \frac{k-2}{2}n + O(1)$. For cycles, Erdős, Simonovits and Sós [8] conjectured that for every fixed $k \geq 3$, $\mathrm{ar}(K_n, C_k) = (\frac{k-2}{2} + \frac{1}{k-1})n + O(1)$, and proved that for $k = 3$. Alon [1] proved this conjecture for $k = 4$ and gave some upper bounds for $k \geq 5$. Finally, Montellano-Ballesteros and Neumann-Lara [18] completely proved this conjecture, that is, for $n \geq k \geq 3$ and $n \equiv r_k (\mathrm{mod\ (k-1)})$, where $0 \leq r_k \leq k-2$,

$$ar(K_n, C_k) = \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r_k}{2} + \left\lceil \frac{n}{k-1} \right\rceil - 1. \tag{1.2}$$

For cliques, Erdős, Simonovits and Sós [8] showed $\mathrm{ar}(K_n, K_{p+1}) = \mathrm{ex}(n, K_p) + 1$ for $p \geq 3$ and sufficiently large n. Montellano-Ballesteros and Neumann-Lara [17] and independently Schiermeyer [19] showed that $\mathrm{ar}(K_n, K_{p+1}) = \mathrm{ex}(n, K_p) + 1$ holds for every $n \geq p \geq 3$. For complete bipartite graphs, Axenovich and Jiang [2] showed that $\mathrm{ar}(K_n, K_{2,t}) = \mathrm{ex}(n, K_{2,t-1}) + O(n)$, where $t \geq 2$. Krop and York [13] showed that $\mathrm{ar}(K_n, K_{s,t}) = \mathrm{ex}(n, K_{s,t-1}) + O(n)$, where $t \geq s \geq 2$. Also, there are many other results about anti-Ramsey numbers. We mention the excellent survey by Fujita, Magnant and Ozeki [10] for more conclusions on this topic.

The maximum number of colors in an edge-colored complete graph without some properly colored subgraphs was first studied by Manoussakis, Spyratos, Tuza and Voigt in [15]. For cliques, they [15] obtained the approximate value of $\mathrm{pr}(K_n, K_t)$.

Theorem 1 [15] *For $t \geq 3$, let $b = \left\lfloor \frac{t-1}{2} \right\rfloor$, we have* $\mathrm{pr}(K_n, K_t) = \left(\frac{b-1}{2b} + o(1)\right)n^2$.

For paths and cycles, they [15] showed that $\mathrm{pr}(K_n, P_n) = \binom{n-3}{2} + 1$ for large n and $\mathrm{pr}(K_n, C_n) = \binom{n-1}{2} + 1$. Also, they gave a conjecture about cycles as follows.

Conjecture 1 [15] *Let $n > l \geq 4$. Assume that K_n is colored with at least k colors, where*

$$k = \begin{cases} \frac{1}{2}l(l+1) + n - l + 1, \text{ if } n < \dfrac{10l^2 - 6l - 18}{6(l-3)}; \\ \frac{1}{3}ln - \frac{1}{18}l(l+3) + 2, \text{ if } n \geq \dfrac{10l^2 - 6l - 18}{6(l-3)}, \end{cases}$$

then K_n admits a properly colored cycle of length $l+1$.

In this paper, we generalize Theorem 1 to an arbitrary graph G which shows that $\mathrm{pr}(K_n, G)$ is related to the Turán number like the anti-Ramsey number.

Theorem 2 *Let G be a graph and $\mathcal{G}' = \{G - M : M$ is a matching of $G\}$, then $\mathrm{pr}(K_n, G) \geq \mathrm{ex}(n, \mathcal{G}') + 1$ and $\mathrm{pr}(K_n, G) = \left(\frac{d-1}{2d} + o(1)\right)n^2$, where $d = \Psi(\mathcal{G}')$.*

We will prove Theorem 2 in Sect. 2 by the method used in the proof of Theorem 1 in [15]. Theorem 2 determines $pr(K_n, G)$ asymptotically when $\Psi(\mathcal{G}') \geq 2$. As the anti-Ramsey number, the case $\Psi(\mathcal{G}') = 1$ is more complex.

In Sect. 3, we will determine $\mathrm{pr}(K_n, P_l)$ for large n by proving the following theorem.

Theorem 3 *Let P_l be a path on l vertices and $l \equiv r_l (\mathrm{mod}\ 3)$, where $0 \leq r_l \leq 2$. For $n \geq 2l^3$, we have*

$$\mathrm{pr}(K_n, P_l) = \left(\left\lfloor \frac{l}{3} \right\rfloor - 1\right)n - \binom{\left\lfloor \frac{l}{3} \right\rfloor}{2} + 1 + r_l.$$

For cycles, we slightly improve the lower bound of Conjecture 1 (See Proposition 4). Also, We modify Conjecture 1 as follows.

Conjecture 2 *Let C_k be a cycle on k vertices and $(k-1) \equiv r_{k-1} (\mathrm{mod}\ 3)$, where $0 \le r_{k-1} \le 2$. For $n \ge k$,*

$$\mathrm{pr}(K_n, C_k) = \max\left\{ \binom{k-1}{2} + n - k + 1, \left\lfloor \frac{k-1}{3} \right\rfloor n - \binom{\left\lfloor \frac{k-1}{3} \right\rfloor + 1}{2} + 1 + r_{k-1} \right\}.$$

It is easy to see that $\mathrm{pr}(K_n, C_3) = \mathrm{ar}(K_n, C_3) = n - 1$. Also, by Proposition 4 and (1.2), one can check that for $n \ge 3$,

$$\mathrm{pr}(K_n, C_n) = \mathrm{ar}(K_n, C_n) = \binom{n-1}{2} + 1, \tag{1.3}$$

$$\mathrm{pr}(K_{n+1}, C_n) = \mathrm{ar}(K_{n+1}, C_n) = \binom{n-1}{2} + 2. \tag{1.4}$$

Li, Broersma and Zhang [14], and later Xu, Magnant and Zhang [21] showed that for $n \ge 4$, $\mathrm{pr}(K_n, C_4) = n$. We obtain the exact value of $\mathrm{pr}(K_n, C_5)$ and $\mathrm{pr}(K_n, C_6)$ in Sect. 4.

Theorem 4 *For $n \ge 5$, $\mathrm{pr}(K_n, C_5) = n + 2$.*

Theorem 5 *For $n \ge 6$, $\mathrm{pr}(K_n, C_6) = n + 5$.*

Let K_4^- be the diamond, the graph obtained from K_4 by deleting an edge. We obtain the exact value of $\mathrm{pr}(K_n, K_4^-)$ in Sect. 5.

Theorem 6 *For $n \ge 3$, $\mathrm{pr}(K_n, K_4^-) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$.*

We also give a lower bound and an upper bound of $\mathrm{pr}(K_n, K_{2,3})$ in Section 5.

Theorem 7 *For $n \ge 5$, $\frac{7}{4}n + O(1) \le \mathrm{pr}(K_n, K_{2,3}) \le 2n - 1$.*

Notations: Let G be a simple undirected graph. For $x \in V(G)$, we denote the *neighborhood* and the *degree* of x in G by $N_G(x)$ and $d_G(x)$, respectively. The *maximum degree* of G is denoted by $\Delta(G)$. The *common neighborhood* of $U \subset V(G)$ is the set of vertices in $V(G) \backslash U$ that are adjacent to each vertex of U. We will use $G - x$ to denote the graph that arises from G by deleting the vertex $x \in V(G)$. For a vertex set $X \subset V(G)$, $G[X]$ is the subgraph of G induced by X and $G - X$ is the subgraph of G induced by $V(G) \backslash X$. Given a graph $G = (V, E)$, for any (not necessarily disjoint) vertex sets $A, B \subset V$, we let $E_G(A, B) := \{uv \in E(G) | u \ne v, u \in A, v \in B\}$. We use $\overline{G}$ to denote the complement of G. Given two vertex disjoint graphs G_1 and G_2, we denote by $G_1 + G_2$ the *join* of graphs G_1 and G_2, that is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2.

Given an edge-coloring c of K_n, we denote the color of an edge uv by $c(uv)$. For any vetex $v \in V(G)$, let $C(v) := \{c(vw) : w \in V(K_n) \backslash \{v\}\}$ and $d_c(v) := |C(v)|$. A color a is *starred* (at x) if all the edges with color a induce a star $K_{1,r}$ (centered at the vertex x). We let $d^c(v) = |\{a \in C(v) : a \text{ is starred at } v\}|$. For a subgraph H of G, we denote $C(H) = \{c(uv) : uv \in E(H)\}$. A *representing subgraph* of an edge-colored K_n is a spanning subgraph containing exactly one edge of each color. The *weak representing subgraph* of an edge-colored K_n is consisting of all the edges whose color appears only once in K_n. Note that an edge xy is the unique edge with color a in K_n if and only if the color a is stared at both x and y. Thus, if G is the weak representing subgraph of an edge-colored K_n, then we have

$$|E(G)| \geq \sum_{v \in V(K_n)} d^c(v) - |C(K_n)|. \tag{1.5}$$

2 The Proof of Theorem 2

In this section, we will prove Theorem 2 by a similar argument used in the proof of Theorem 1 in [15].

Theorem 2 *Let G be a graph and $\mathcal{G}' = \{G - M : M$ is a matching of $G\}$, then $\mathrm{pr}(K_n, G) \geq \mathrm{ex}(n, \mathcal{G}') + 1$ and $\mathrm{pr}(K_n, G) = \left(\frac{d-1}{2d} + o(1)\right)n^2$, where $d = \Psi(\mathcal{G}')$.*

Proof Let F be a graph in $\mathrm{EX}(n, \mathcal{G}')$. We color the edges of K_n as follows. Take a subgraph F of K_n, and assign distinct colors to all of $E(F)$ and a new color c_0 to all the remaining edges. Suppose there is a properly colored G, then $M = \{e \in E(G), e \text{ is colored with } c_0\}$ is a matching of G, and $G - M \subset F$. By the definition of $\mathcal{G}'$, we have $G - M \in \mathcal{G}'$, and this is a contradiction with F being $\mathcal{G}'$-free. Thus we have $\mathrm{pr}(K_n, G) \geq \mathrm{ex}(n, \mathcal{G}') + 1 = (\frac{d-1}{2d} + o(1))n^2$ by (1.1).

Let $G_0 = G - M_p$, where M_p is a p-matching of G and $\chi(G_0) = d + 1$. We prove that for every fixed $\varepsilon > 0$, and for n large enough with respect to $n_0 = |V(G)|$ and ε, there is a properly colored copy of G in any $(\frac{d-1}{2d} + \varepsilon)n^2$-edge-coloring of K_n. In a representing subgraph of K_n with $(\frac{d-1}{2d} + \varepsilon)n^2$ edges, for an arbitrarily fixed s, and for n sufficiently large, by (1.1), there exists a complete $(d+1)$-partite subgraph $K_{s,s,\ldots,s}$ with s vertices in each class. We take $s = 2^{n_0+d+1}$.

Denote by V the vertex set of $K_{s,s,\ldots,s}$ and by $V_1, V_2, \ldots, V_{d+1}$ its vertex classes. We apply the following procedure.

For each $i = 1, 2, \ldots, d+1$ do sequentially the following:

(1) Select arbitrarily $2^{n_0+d+1-i}$ pairwise disjoint pairs $\{u_{ij}, v_{ij}\}$ in V_i, $j = 1, 2, \ldots, 2^{n_0+d+1-i}$.
(2) For $j = 1, 2, \ldots, 2^{n_0+d+1-i}$, delete from $K_{s,s,\ldots,s}$ the (at most one) vertex $z \in V \backslash V_i$ for which either $c(zu_{ij}) = c(u_{ij}v_{ij})$ or $c(zv_{ij}) = c(u_{ij}v_{ij})$, and if z has already been selected in a previous pair $\{u_{i'j'}, v_{i'j'}\}$, for some $i' < i$, then also delete the other member of its pair.

Claim 1 *The above procedure can be executed smoothly and there are at least 2^{n_0} pairs remains undeleted in each V_i at the end of the execution.*

The Proof of Claim 1 In the beginning, each V_i contains 2^{n_0+d+1} vertices, $i = 1, 2, \ldots, d+1$. In the first iteration, $i = 1$, we can carry out (1) and (2) easily. Suppose we have carried out up to the $(i-1)$-st iteration. Before executing the i-th iteration observe that at most $\sum_{1 \leq j \leq i-1} 2^{n_0+d+1-j} = 2^{n_0+d+1} - 2^{n_0+d+2-i}$ vertices have been deleted from V_i. Thus, V_i contains at least $2^{n_0+d+2-i}$ vertices and it is enough to execute instruction (1) in the ith iteration.

On the other hand, for any $i = 1, 2, \ldots, d$, from the $(i+1)$-st iteration up to the end, due to instructions of type (2), at most $\sum_{i+1 \leq j \leq d+1} 2^{n_0+d+1-j} = 2^{n_0+d+1-i} - 2^{n_0}$ pairs in V_i have been deleted and thus at least 2^{n_0} pairs in V_i remains undeleted. Note also that V_{d+1} contains 2^{n_0} pairs of vertices and there is no deletion of pair in V_{d+1}. □

For $1 \leq i \leq d+1$, let $\{x_{ij}y_{ij} : 1 \leq j \leq 2^{n_0}\}$ be the 2^{n_0} pairs in V_i which remain undeleted and $V'_i = \{x_{ij}, y_{ij} : 1 \leq j \leq 2^{n_0}\}$. Let H be the graph obtained by adding the edge set $\{x_{ij}y_{ij} : 1 \leq i \leq d+1, 1 \leq j \leq 2^{n_0}\}$ to the graph $K_{s,s,\ldots,s}[V'_1 \cup \cdots \cup V'_{d+1}]$. Then H is properly colored by Claim 1. Since $G_0 = G - M_p$ and $\chi(G_0) = d+1$, we have $H \supset G$. Thus $\mathrm{pr}(K_n, G) \leq (\frac{d-1}{2d} + o(1))n^2$. □

3 Paths

In this section, we study the maximum number of colors in an edge-colored complete graph without properly edge-colored paths, and prove Theorem 3. Before doing so, we determine $\mathrm{pr}(K_n, P_l)$ for some small values of l.

Proposition 1

(a) $\mathrm{pr}(K_n, P_3) = 1$, for $n \geq 3$.
(b) $\mathrm{pr}(K_n, P_4) = 2$, for $n \geq 4$.
(c) $\mathrm{pr}(K_n, P_5) = 3$, for $n \geq 5$.

Proof

(a) The conclusion holds trivially.
(b) Choose a vertex v of K_n, color all edges incident to v with color c_1 and color all the remaining edges with color c_2. We use two colors and there is no properly colored P_4. Hence $\mathrm{pr}(K_n, P_4) \geq 2$.

For $n \geq 5$, we have $\mathrm{pr}(K_n, P_4) \leq \mathrm{ar}(K_n, P_4) = 2$ (see [3]). For $n = 4$, let $V(K_4) = \{u, v, x, y\}$. Given a 3-edge-coloring of K_4, there exists at least one edge in $E(\{u, v\}, \{x, y\})$, we say ux, such that $c(ux) \neq c(uv)$ and $c(ux) \neq c(xy)$. Thus $vuxy$ is a properly colored P_4 and $\mathrm{pr}(K_n, P_4) \leq 2$.

(c) Choose two vertices u and v of K_n, assign one color c_1 to all edges incident with u, one new color c_2 to all edges incident with v (except the edge uv) and the other new color c_3 to all the remaining edges. We use three colors and there is no properly colored P_5. Hence $\mathrm{pr}(K_n, P_5) \geq 3$.

Let $n \geq 5$. Given a 4-edge-coloring of K_n, there is always a rainbow $P_4 = u_1u_2u_3u_4$ since $\mathrm{ar}(K_n, P_4) = 2$ (see [3]). Since $|C(P_4)| = |E(P_4)| = 3$, there is a color $c_0 \in C(K_n) \backslash C(P_4)$. Suppose there is no properly colored P_5 in the 4-edge-coloring of K_n. Then for all $u \in V(K_n) \backslash V(P_4)$, it must be $c(uu_1) = c(u_1u_2)$, $c(uu_4) = c(u_3u_4)$, $c(uu_2) \in \{c(u_1u_2), c(u_2u_3)\}$ and $c(uu_3) \in \{c(u_2u_3), c(u_3u_4)\}$. If $c(u_1u_4) = c_0$, then $uu_1u_4u_3u_2$ is a properly colored P_5, a contradiction. If $c(u_1u_3) = c_0$ or $c(u_2u_4) = c_0$, say $c(u_1u_3) = c_0$, then $u_4uu_1u_3u_2$ is a properly colored P_5, a contradiction. So we may assume that there are two vertices $x, y \in V(K_n) \backslash V(P_4)$ such that $c(xy) = c_0$. In this case, $u_4yxu_2u_1$ or $u_4yxu_2u_3$ is a properly colored P_5, a contradiction. Hence $\mathrm{pr}(K_n, P_5) \leq 3$. □

Here, we give the lower bound of $pr(K_n, P_l)$ by the following proposition.

Proposition 2 *Let P_l be a path on l vertices and $l \equiv r_l (\mathrm{mod}\ 3)$, where $0 \leq r_l \leq 2$. For $n \geq l$, we have*

$$\mathrm{pr}(K_n, P_l) \geq \max\left\{ \binom{l-3}{2} + 1, \left(\left\lfloor \frac{l}{3} \right\rfloor - 1 \right) n - \binom{\left\lfloor \frac{l}{3} \right\rfloor}{2} + 1 + r_l \right\}.$$

Proof We color the edges of K_n as follows. For the first lower bound, we choose a K_{l-3} and color it rainbow, and use one extra color for all the remaining edges. In such way, we use exactly $\binom{l-3}{2} + 1$ colors and do not obtain a properly colored P_l.

For the second lower bound, we partition K_n into two graphs $K_{\lfloor \frac{l}{3} \rfloor - 1} + \overline{K}_{n - \lfloor \frac{l}{3} \rfloor + 1}$ and $K_{n - \lfloor \frac{l}{3} \rfloor + 1}$. First we color $K_{\lfloor \frac{l}{3} \rfloor - 1} + \overline{K}_{n - \lfloor \frac{l}{3} \rfloor + 1}$ rainbow. Then we color $K_{n - \lfloor \frac{l}{3} \rfloor + 1}$ by $(1 + r_l)$ new colors without producing a properly colored P_{3+r_l} (See the proof of Proposition 3.1). In such way, we use exactly $\left(\lfloor \frac{l}{3} \rfloor - 1 \right) n - \binom{\left\lfloor \frac{l}{3} \right\rfloor}{2} + 1 + r_l$ colors and do not obtain a properly colored P_l. □

The proof of the following proposition is trivial. We will use it to prove Theorem 3.

Proposition 3 *Let P_l be a path with l vertices, and $l \equiv r_l (\mathrm{mod}\ 3)$, where $0 \leq r_l \leq 2$. If an edge-colored K_n contains a rainbow copy of $K_{\lfloor \frac{l}{3} \rfloor - 1, 2\lfloor \frac{l}{3} \rfloor + 3}$ but does not contain a properly colored P_l. We denote by Q the vertices of $K_n - K_{\lfloor \frac{l}{3} \rfloor - 1, 2\lfloor \frac{l}{3} \rfloor + 3}$, by X the smaller class of $K_{\lfloor \frac{l}{3} \rfloor - 1, 2\lfloor \frac{l}{3} \rfloor + 3}$ and by Y the other one. Then $|C(K_n[Y])| \leq 1 + r_l$. Furthermore, we have $|C(K_n[Y]) \cup C(E_{K_n}(Y, Q))| \leq 1 + r_l$ and $|C(K_n[Y \cup Q])| \leq 1 + r_l$. We get the most colors if the colors of all the edges between X and $Y \cup Q$ and all the edges in X are different, they differ from all the other edges and we use exactly $1 + r_l$ colors in $Y \cup Q$ such that there is no properly colored P_{3+r_l} in $Y \cup Q$. Then the number of colors is*

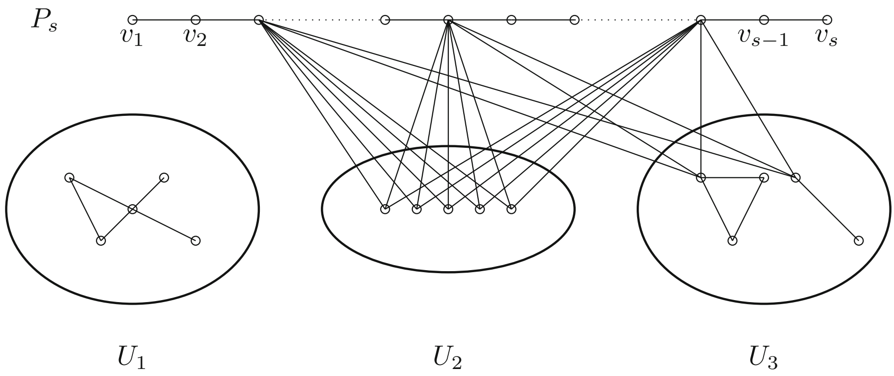

Fig. 1 The structure of graph G

$$\left(\left\lfloor\frac{l}{3}\right\rfloor - 1\right)n - \binom{\left\lfloor\frac{l}{3}\right\rfloor}{2} + 1 + r_l.$$

Now, we will prove Theorem 3, and the idea comes from [20] (Fig. 1).

Theorem 3 *Let P_l be a path on l vertices and $l \equiv r_l (\bmod 3)$, where $0 \le r_l \le 2$. For $n \ge 2l^3$, we have*

$$\mathrm{pr}(K_n, P_l) = \left(\left\lfloor\frac{l}{3}\right\rfloor - 1\right)n - \binom{\left\lfloor\frac{l}{3}\right\rfloor}{2} + 1 + r_l.$$

Proof We just need prove the upper bound for $l \ge 6$. We shall use the following famous results of Erdős and Gallai (see [5]): for $n \ge r \ge 2$,

$$\mathrm{ex}(n, P_r) \le \frac{r-2}{2}n, \tag{3.1}$$

$$\mathrm{ex}(n, \{C_{r+1}, C_{r+2}, \ldots\}) \le \frac{r(n-1)}{2}. \tag{3.2}$$

Let c be an edge-coloring of K_n using $\mathrm{pr}(K_n, P_l)$ colors without producing a properly colored P_l. Take a longest properly colored path $P_s = v_1v_2\cdots v_s$, where $s \le l-1$. Denote by G the graph obtained by choosing one edge from each remaining color such that the number of edges joining P_s to the remaining $n-s$ vertices is as large as possible. We would partition $V(G)\backslash V(P_s)$ into three sets U_1, U_2 and U_3 as follows:

(a) U_1 is the vertex set of $V(K_n)\backslash V(P_s)$ not jointed to P_s at all: neither by edges nor by paths;

(b) U_2 is the set of isolated vertices of $V(K_n)\backslash V(P_s)$ jointed to P_s by edges;
(c) $U_3 = V(K_n)\backslash(V(P_s) \cup U_1 \cup U_2)$.

Claim 1 *For any vertex $u \in U_1 \cup U_2 \cup U_3$, we have $c(uv_1) = c(v_1v_2)$ and $c(uv_s) = c(v_{s-1}v_s)$. Moreover, $E_G(U_2 \cup U_3, \{v_1, v_2, v_{s-1}, v_s\}) = \emptyset$.*

Proof of Claim 1 It is obvious that $c(uv_1) = c(v_1v_2)$ and $c(uv_s) = c(v_{s-1}v_s)$ for any vertex $u \in U_1 \cup U_2 \cup U_3$ by the maximality of P_s, thus we have $E_G(U_2 \cup U_3, \{v_1, v_s\}) = \emptyset$. Suppose that there is a vertex $u \in U_2 \cup U_3$ such that $uv_2 \in E(G)$ or $uv_{s-1} \in E(G)$, we say $uv_2 \in E(G)$. Notice that $c(uv_1) = c(v_1v_2) \neq c(uv_2)$ by the definition of G, it follows that $v_1uv_2 \cdots v_s$ is a properly colored path of order $s+1$, a contradiction to the maximality of P_s. $\square$

Claim 2 *$G[U_1]$ contains no $P_{\lfloor \frac{s}{2} \rfloor}$.*

Proof of Claim 2 Suppose $P_{\lfloor \frac{s}{2} \rfloor} = u_1u_2\ldots u_{\lfloor \frac{s}{2} \rfloor}$ is a path in $G[U_1]$. By the definition of G, the colors of $C(G[U_1])$ can not appear in any edges between U_1 and $V(P_s)$. Thus, $c(u_1v_{\lceil \frac{s}{2} \rceil}) \neq c(u_1u_2)$, $c(u_{\lfloor \frac{s}{2} \rfloor}v_1) \neq c(u_{\lfloor \frac{s}{2} \rfloor}u_{\lfloor \frac{s}{2} \rfloor-1})$ and $c(u_{\lfloor \frac{s}{2} \rfloor}v_s) \neq c(u_{\lfloor \frac{s}{2} \rfloor}u_{\lfloor \frac{s}{2} \rfloor-1})$. Since $c(v_{\lceil \frac{s}{2} \rceil-1}v_{\lceil \frac{s}{2} \rceil}) \neq c(v_{\lceil \frac{s}{2} \rceil}v_{\lceil \frac{s}{2} \rceil+1})$, at most one of $c(v_{\lceil \frac{s}{2} \rceil-1}v_{\lceil \frac{s}{2} \rceil})$ and $c(v_{\lceil \frac{s}{2} \rceil}v_{\lceil \frac{s}{2} \rceil+1})$ is the same as $c(u_{\lceil \frac{s+1}{2} \rceil}v_{\lceil \frac{s}{2} \rceil})$. So at least one of $v_1v_2\ldots v_{\lceil \frac{s}{2} \rceil}u_1u_2\ldots u_{\lfloor \frac{s}{2} \rfloor}v_s$ and $v_sv_{s-1}\ldots v_{\lceil \frac{s}{2} \rceil}u_1u_2\ldots u_{\lfloor \frac{s}{2} \rfloor}v_1$ is a properly colored path of order at least $s+1$, a contradiction to the maximality of P_s. Hence, $G[U_1]$ contains no $P_{\lfloor \frac{s}{2} \rfloor}$. $\square$

By Claim 2 and (3.1), we have

$$|E(G[U_1])| \leq \frac{1}{2}\left(\left\lfloor \frac{s}{2} \right\rfloor - 2\right)|U_1| \leq \left(\frac{1}{2}\left\lfloor \frac{l-1}{2} \right\rfloor - 1\right)|U_1|. \tag{3.3}$$

Claim 3 *For any vertex $u \in U_2 \cup U_3$ and any three consecutive vertices $v_i, v_{i+1}, v_{i+2} \in V(P_s)$, we have $|E_G(u, \{v_i, v_{i+1}, v_{i+2}\})| \leq 1$.*

Proof of Claim 3 Suppose there exist a vertex $u \in U_2 \cup U_3$ and three consecutive vertices $v_i, v_{i+1}, v_{i+2} \in V(P_s)$ such that $|E_G(u, \{v_i, v_{i+1}, v_{i+2}\})| \geq 2$, that is at least two of uv_i, uv_{i+1}, uv_{i+2} are edges of G, then whatever $c(vv_i)$ is, at least one of $v_1\ldots v_iuv_{i+1}v_{i+2}\ldots v_s$ and $v_1\ldots v_iv_{i+1}uv_{i+2}\ldots v_s$ is a properly colored path of order $s+1$, a contradiction to the maximality of P_s. $\square$

By Claims 1 and 3, we have $|E_G(u, P_s)| \leq \lceil \frac{s-4}{3} \rceil \leq \lceil \frac{l-5}{3} \rceil = \lfloor \frac{l}{3} \rfloor - 1$ for all $u \in U_2$. Thus, we have

$$|E_G(U_2, P_s)| \leq \left(\left\lfloor \frac{l}{3} \right\rfloor - 1\right)|U_2|. \tag{3.4}$$

Let H be any component of $G[U_3]$ and r be the length of the longest cycle in H. If H contains no cycles, then we write $r = 2$. By (3.2), we have

$$|E(H)| \leq \frac{r|V(H)| - r}{2}. \tag{3.5}$$

Now we will estimate the number of edges between $V(H)$ and $V(P_s)$ in G by the following two claims.

Claim 4 *For any vertex $u \in V(H)$, we have*

$$E_G(u, \{v_1, \ldots, v_{2r+1}, v_{s-2r}, \ldots, v_s\}) = \emptyset. \tag{3.6}$$

Proof of Claim 4 Since H is connected and the length of the longest cycle in H is r, we can always find a path $P_r \subset H$ starting from u in H. Let $P_r = u_1u_2\ldots u_r$ be such a path, where $u_1 = u$. By an argument very similar to the one in Claim 1, we have $E_G(u, \{v_1, \ldots, v_{r+1}, v_{s-r}, \ldots, v_s\}) = \emptyset$. By the symmetry, we just need to show that there is no edge between u and $\{v_{r+2}, \ldots, v_{2r+1}\}$. If there exists $v_i \in \{v_{r+2}, \ldots, v_{2r}\}$ such that $uv_i \in E(G)$, we have $i \geq r+2 \geq 4$. By the definition of G, we have $c(u_r v_{\lfloor \frac{i}{2} \rfloor}) \neq c(u_{r-1}u_r)$. Since $c(v_{\lfloor \frac{i}{2} \rfloor - 1} v_{\lfloor \frac{i}{2} \rfloor}) \neq c(v_{\lfloor \frac{i}{2} \rfloor} v_{\lfloor \frac{i}{2} \rfloor + 1})$, at most one of $c(v_{\lfloor \frac{i}{2} \rfloor - 1} v_{\lfloor \frac{i}{2} \rfloor})$ and $c(v_{\lfloor \frac{i}{2} \rfloor} v_{\lfloor \frac{i}{2} \rfloor + 1})$ is the same as $c(u_r v_{\lfloor \frac{i}{2} \rfloor})$. Thus at least one of $v_1v_2\ldots v_{\lfloor \frac{i}{2} \rfloor} u_s \ldots u_1 v_i v_{i+1} \ldots v_s$ and $v_{i-1}v_{i-2}\ldots v_{\lfloor \frac{i}{2} \rfloor} u_s \ldots u_1 v_i v_{i+1} \ldots v_s$ is a properly colored path of order at least $s+1$, a contradiction to the maximality of P_s. If $uv_{2r+1} \in E(G)$, then we have $c(uv_{2r}) \neq c(v_{2r}v_{2r+1})$, otherwise $v_1v_2 \cdots v_{2r}uv_{2r+1}v_{2r+2} \cdots v_s$ is a properly colored path of order $s+1$, a contradiction to the maximality of P_s. Also, we have $c(uv_{2r}) \neq c(uu_2)$. By an argument similar to the above, one can find a properly colored path of order at least $s+1$, a contradiction to the maximality of P_s. □

Claim 5 *For any six consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} \in V(P_s)$, all edges between $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ and $V(H)$ of G induce a star.*

Proof of Claim 5 If not, suppose xv_i and yv_j are two independent edges between $V(H)$ and $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ in G, where $x, y \in V(H)$ and $j \in \{i+1, i+2, i+3, i+4, i+5\}$. Let P_{xy} be a path of H which connect x and y. If $j \in \{i+1, i+2, i+3\}$, then whatever $c(xv_{i+1})$ is, at least one of $v_1\ldots v_i x v_{i+1} \cdots v_s$ and $v_1\ldots v_i v_{i+1} x P_{xy} y v_j \ldots v_s$ is a prorperly colored path of order at least $s+1$, a contradiction to the maximality of P_s. If $j = i+4$, then we have $c(xv_{i+3}) = c(v_{i+2}v_{i+3})$ and $c(yv_{i+1}) = c(v_{i+1}v_{i+2})$, otherwise, $v_1v_2\ldots v_{i+3} x P_{xy} y v_{i+4} \ldots v_s$ or $v_1v_2\ldots v_{i+1} y P_{yx} x v_{i+3} \ldots v_s$ is a properly colored path of order at least $s+2$, a contradiction to the maximality of P_s. It follows that $v_1\ldots v_{i+1} y P_{yx} x v_{i+3} \ldots v_s$ is a properly colored path of order at least $s+1$, a contradiction to the maximality of P_s. If $j = i+5$, by a similar argument of the former case, we have $c(xv_{i+3}) = c(yv_{i+2}) = c(v_{i+2}v_{i+3})$ and $v_1v_2\ldots v_{i+2} y P_{yx} x v_{i+3} \ldots v_s$ is a properly colored path of order at least $s+2$, a contradiction to the maximality of P_s. □

By Claim 3, for any six consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} \in V(P_s)$ and any vertex $u \in V(H)$, we have $|E_G(u, \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\})| \leq 2$. Thus, by Claim 5, we have

$$|E_G(V(H), \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\})| \leq \max\{2, |V(H)|\} \leq |V(H)|. \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$|E_G(V(H), V(P_s))| \leq \left\lceil \frac{s - 2(2r+1)}{6} \right\rceil |V(H)|. \tag{3.8}$$

Combining (3.5) and (3.8), we have

$$\begin{aligned} |E_G(V(H), V(P_s))| + |E(H)| &\leq \left\lceil \frac{s - 2(2r+1)}{6} \right\rceil |V(H)| + \frac{r|V(H)| - r}{2} \\ &\leq \left(\left\lceil \frac{s - 4r - 2}{6} \right\rceil + \frac{r}{2} \right) |V(H)| \\ &\leq \left(\left\lceil \frac{s - 4r - 2}{6} + \frac{r+1}{2} \right\rceil \right) |V(H)| \\ &\leq \left\lceil \frac{s-1}{6} \right\rceil |V(H)| \end{aligned}$$

The last inequality holds since $r \geq 2$. Note that $|E_G(V(H), V(P_s))| + |E(H)| \leq \lceil \frac{s-1}{6} \rceil |V(H)|$ holds for each component H of $G[U_3]$. Thus, we have

$$|E_G(U_3, P_s)| + |E(G[U_3])| \leq \left\lceil \frac{s-1}{6} \right\rceil |U_3| \leq \left\lceil \frac{l-2}{6} \right\rceil |U_3|. \tag{3.9}$$

By (3.3), (3.4) and (3.9), we have

$$\begin{aligned} \mathrm{pr}(K_n, P_l) = |C(K_n)| &\leq |C(P_s)| + |E(G)| \\ &\leq \binom{s}{2} + |E(G[U_1])| + |E_G(U_2, P_s)| + |E_G(U_3, P_s)| + |E(G[U_3])| \\ &\leq \binom{s}{2} + \left(\frac{1}{2} \left\lfloor \frac{l-1}{2} \right\rfloor - 1 \right) |U_1| + \left(\left\lfloor \frac{l}{3} \right\rfloor - 1 \right) |U_2| + \left\lceil \frac{l-2}{6} \right\rceil |U_3|. \end{aligned}$$

Note that $\frac{1}{2} \lfloor \frac{l-1}{2} \rfloor - 1 \leq \lfloor \frac{l}{3} \rfloor - 1 - \frac{1}{2}$ for $l \geq 6$ and $\lceil \frac{l-2}{6} \rceil \leq \lfloor \frac{l}{3} \rfloor - 1 - \frac{1}{2}$ for all $l \geq 12$. When $l \leq 11$, we have $s \leq 10$, by Claim 4, $U_3 = \emptyset$. Let $U^* = \{u \in U_2 : d_G(u) = \lfloor \frac{l}{3} \rfloor - 1\}$. Then we have

$$\mathrm{pr}(K_n, P_l) \leq \binom{s}{2} + \left(\left\lfloor \frac{l}{3} \right\rfloor - 1 - \frac{1}{2} \right) (n - s - |U^*|) + \left(\left\lfloor \frac{l}{3} \right\rfloor - 1 \right) |U^*|. \tag{3.10}$$

Since $n \geq 2l^3$, by Proposition 2, we have

$$\mathrm{pr}(K_n, P_l) \geq \left(\left\lfloor \frac{l}{3} \right\rfloor - 1\right)n - \binom{\left\lfloor \frac{l}{3} \right\rfloor}{2} + 1 + r_l. \tag{3.11}$$

Combining (3.10) and (3.11), since $n \geq 2l^3$, we have $|U^*| \geq l^3$. By Claims 1 and 3, there are at most $\binom{s-4-2(\lfloor \frac{l}{3} \rfloor - 1 - 1)}{\lfloor \frac{l}{3} \rfloor - 1}$ distinct $(\lfloor \frac{l}{3} \rfloor - 1)$-subset of $V(P_s)$ can be the neighborhood of some vertex in U^*. Since $s \leq l-1$ and $6 \leq l \leq 3\lfloor \frac{l}{3} \rfloor + 2$, we have

$$\binom{s-4-2(\lfloor \frac{l}{3} \rfloor - 1 - 1)}{\lfloor \frac{l}{3} \rfloor - 1} \leq \binom{l-1-2\lfloor \frac{l}{3} \rfloor}{\lfloor \frac{l}{3} \rfloor - 1} \leq \binom{\lfloor \frac{l}{3} \rfloor + 1}{\lfloor \frac{l}{3} \rfloor - 1} = \binom{\lfloor \frac{l}{3} \rfloor + 1}{2} \leq \frac{l^2}{9}.$$

Note that $|U^*| \geq l^3 > \frac{l^2}{9}(2\lfloor \frac{l}{3} \rfloor + 3)$, by Pigeonhole Principle, U^* contains at least $2\lfloor \frac{l}{3} \rfloor + 3$ vertices which have a common neighborhood of size $\lfloor \frac{l}{3} \rfloor - 1$ in G. That is, we find a rainbow $K_{\lfloor \frac{l}{3} \rfloor - 1, 2\lfloor \frac{l}{3} \rfloor + 3}$. By Proposition 3, the proof is complete.

□

4 Cycles

The lower bound of $\mathrm{pr}(K_n, C_k)$ was given roughly by Manoussakis, Spyratos, Tuza and Voigt in [15]. Here we prove the lower bound precisely again.

Proposition 4 *Let C_k be a cycle on k vertices and $(k-1) \equiv r_{k-1} (\mathrm{mod}\ 3)$, where $0 \leq r_{k-1} \leq 2$. For $n \geq k$,*

$$\mathrm{pr}(K_n, C_k) \geq \max\left\{\binom{k-1}{2} + n - k + 1, \left\lfloor \frac{k-1}{3} \right\rfloor n - \binom{\left\lfloor \frac{k-1}{3} \right\rfloor + 1}{2} + 1 + r_{k-1}\right\}.$$

Proof We color the edges of K_n as follows. For the first lower bound, we choose a K_{k-1} and color it rainbow, and use one extra color for all the remaining edges. In such way, we use exactly $\binom{k-1}{2} + 1$ colors and do not obtain a properly colored C_k.

For the second lower bound, we partition K_n into two graphs $K_{\lfloor \frac{k-1}{3} \rfloor} + \overline{K}_{n-\lfloor \frac{k-1}{3} \rfloor}$ and $K_{n-\lfloor \frac{k-1}{3} \rfloor}$. First we color $K_{\lfloor \frac{k-1}{3} \rfloor} + \overline{K}_{n-\lfloor \frac{k-1}{3} \rfloor}$ rainbow. Then we color $K_{n-\lfloor \frac{k-1}{3} \rfloor}$ by $(1 + r_{k-1})$ new colors without producing a properly colored $P_{3+r_{k-1}}$ (See the proof of

Proposition 3.1). In such way, we use exactly $\lfloor\frac{k-1}{3}\rfloor n - \binom{\lfloor\frac{k-1}{3}\rfloor+1}{2} + 1 + r_{k-1}$ colors and do not obtain a properly colored C_k. □

Conjecture 3 *Let C_k be a cycle on k vertices and $(k-1) \equiv r_{k-1} (\mathrm{mod}\ 3)$, where $0 \le r_{k-1} \le 2$. For $n \ge k$,*

$$\mathrm{pr}(K_n, C_k) = \max\left\{ \binom{k-1}{2} + n - k + 1, \left\lfloor\frac{k-1}{3}\right\rfloor n - \binom{\left\lfloor\frac{k-1}{3}\right\rfloor + 1}{2} + 1 + r_{k-1} \right\}.$$

Now we prove Conjecture 2 holds for C_5 and C_6, respectively.

Theorem 4 *For $n \ge 5$, $\mathrm{pr}(K_n, C_5) = n + 2$.*

Proof By Proposition 4, we have $\mathrm{pr}(K_n, C_5) \ge n + 2$ for $n \ge 5$. We will prove $\mathrm{pr}(K_n, C_5) \le n + 2$ by induction on n. The base cases $n = 5$ and $n = 6$ follow from (1.3) and (1.4), respectively. For $n \ge 7$, assume that the conclusion holds for order less than n. Let c be an $(n+3)$-edge-coloring of K_n. If there is a vertex v such that $d^c(v) \le 1$, then $|C(K_n - v)| \ge n + 3 - 1 = (n-1) + 3$ and there is a properly colored C_5 by the induction hypothesis. Thus we assume that $d^c(v) \ge 2$, for all $v \in V(K_n)$. Let G be the weak representing subgraph of K_n. By (1.5), we have $|E(G)| \ge 2n - (n+3) = n - 3 \ge 4$. Thus, G contains a 2-matching. Let $\{xy, zw\}$ be a 2-matching of G. Choose a vertex $u \in V(K_n) \backslash \{x, y, z, w\}$. We consider the following two cases.

Case 1. There are at least two edges of $\{ux, uy, uz, uw\}$ are colored with distinct colors.

In this case, there are at least one edge of $\{ux, uy\}$, we say ux, and at least one edge of $\{uz, uw\}$, we say uz, such that $c(ux) \ne c(uz)$. By the definition of G, we have $c(ux) \ne c(xy)$, $c(uz) \ne c(zw)$ and $c(xy) \ne c(yw) \ne c(zw)$. Thus, $uxywzu$ is a properly colored C_5.

Case 2. The four edges ux, uy, uz and uw are colored with the same color.

If $c(ux)$ is starred at u, since $d^c(u) \ge 2$, there exists a vertex $v \in V(K_n) \backslash \{x, y, z, w, u\}$ such that $c(uv)$ is starred at u and $c(uv) \ne c(ux)$. Also, we have $c(ux) \ne c(xz) \ne c(zw)$ and $c(zw) \ne c(vw) \ne c(uv)$. Thus, $uxzwvu$ is a properly colored C_5. If $c(ux)$ is not starred at u, since $d^c(u) \ge 2$, there exists two vertices $v_1, v_2 \in V(K_n) \backslash \{x, y, z, w, u\}$ such that $c(uv_1)$ and $c(uv_2)$ are starred at u and $c(uv_1) \ne c(uv_2)$. Also, we have $c(uv_1) \ne c(v_1x) \ne c(xy)$ and $c(uv_2) \ne c(v_2z) \ne c(xy)$. Thus, uv_1xyv_2u is a properly colored C_5. □

For C_6, we consider more cases to prove it.

Theorem 5 *For $n \ge 6$, $\mathrm{pr}(K_n, C_6) = n + 5$.*

Proof By Proposition 4, we have $\mathrm{pr}(K_n, C_6) \geq n+5$ for $n \geq 6$. We will prove $pr(K_n, C_6) \leq n+5$ by induction on n. The base cases $n=6$ and $n=7$ follow from (1.3) and (1.4), respectively. For $n \geq 8$, assume that the conclusion holds for order less than n. Let c be an $(n+6)$-edge-coloring of K_n. If there is a vertex v such that $d^c(v) \leq 1$, then $|C(K_n - v)| \geq n+6-1 = (n-1)+6$ and there is a properly colored C_6 by the induction hypothesis. Thus we assume that $d^c(v) \geq 2$ for all $v \in V(K_n)$. Let G be the weak representing subgraph of K_n. By (1.5), we have $|E(G)| \geq 2n - (n+6) = n-6 \geq 2$.

Case 1. $\Delta(G) \geq 2$.

In this case, G contains a path of order 3. Let $P_3 = xyz$ be such a path of G and $U = V(K_n) \backslash \{x, y, z\}$. Let H be a subgraph K_n obtained by choosing one edge from the colors which are starred at some vertex of U such that the number of edges between $\{x, y, z\}$ and U is as large as possible.

Case 1.1 $|E(H[U])| \geq 2$.

Let $u_1u_2, v_1v_2 \in E(H[U])$. If u_1u_2 and v_1v_2 have a common end vertex, we say $u_2 = v_1$, then $c(xu_1) \neq c(u_1v_1)$ and $c(zv_2) \neq c(v_1v_2)$ by the choice of H. Thus $xyzv_2v_1u_1x$ is a properly colored C_6. Now we may assume that $\{u_1u_2, v_1v_2\}$ is a 2-matching of H. Assume that $c(u_1u_2)$ and $c(v_1v_2)$ are starred at u_1 and v_1 respectively. Thus $c(u_2v_2) \neq c(u_1u_2)$ and $c(u_2v_2) \neq c(v_1v_2)$. By the choice of H, we have $c(xu_1) \neq c(u_1u_2)$ and $c(yv_1) \neq c(v_1v_2)$. Thus, $xyv_1v_2u_2u_1x$ is a properly colored C_6.

Case 1.2 $|E(H[U])| = 1$.

Assume $uv \in E(H[U])$ and $c(uv)$ is starred at u. Then we have $c(xu) \neq c(uv)$. Also, $c(vz) \neq c(uv)$. Take a vertex $w \in U \backslash \{u, v\}$. Since $d^c(w) \geq 2$, we have $|E_H(w, \{x, y, z\})| \geq 2$. There is at least one of $\{x, z\}$, say x, such that $c(wx)$ is starred at w and $c(wx) \neq c(wy)$. Also, we have $c(wx) \neq c(ux)$. Thus $wxuvzyw$ is a properly colored C_6.

Case 1.3 $E(H[U]) = \emptyset$.

For all $v \in U$, since $d^c(v) \geq 2$, we have $|E_H(v, \{x, y, z\})| \geq 2$. Notice that $|U| \geq n-3 \geq 5$. If there are three vertices in U, say $u_1, u_2, u_3 \in U$, such that they have a common neighborhood $\{x, z\}$ in H, then at least one of $\{u_1x, u_1z\}$, say u_1x, such that $c(u_1y) \neq c(u_1x)$. Also, at most one edge of $\{u_2x, u_2z, u_3x, u_3z\}$ has the same color as $c(u_2u_3)$. Thus, at least one of $xu_1yzu_3u_2x$ and $xu_1yzu_2u_3x$ is a properly colored C_6.

Now we may assume that there are at least two vertices in U, say u_1, u_2, such that they have a common neighborhood $\{x, y\}$ or $\{y, z\}$ in H, say $\{x, y\}$. If there is a vertex $u_3 \in U \backslash \{u_1, u_2\}$ such that $u_3y, u_3z \in E(H)$, we have $c(zx) \notin \{c(xu_1), c(xu_2), c(zu_3)\}$ and at most one edge of $\{u_1x, u_1y, u_2x, u_2y\}$ has the same color as $c(u_1u_2)$. Thus, at least one of $xu_1u_2yu_3zx$ and $xu_2u_1yu_3zx$ is a properly colored C_6. If there is a vertex $u_3 \in U \backslash \{u_1, u_2\}$ such that $u_3x, u_3z \in E(H)$, at least one of $xu_1u_2yzu_3x$ and $xu_2u_1yzu_3x$ is a properly colored C_6. Now we may assume that U has a common neighborhood $\{x, y\}$ in H. Take four distinct vertices $u_1, u_2, u_3, u_4 \in U$. At most one edge of $\{u_1x, u_1y, u_2x, u_2y\}$ has the same color as $c(u_1u_2)$ and at most one edge of $\{u_3x, u_3y, u_4x, u_4y\}$ has the same color as $c(u_3u_4)$. Thus the graph induced by the edges set $\{u_1u_2, u_3u_4, xu_i, yu_i : 1 \leq i \leq 4\}$ contains a a properly colored C_6.

Case 2. $\Delta(G)=1$.

Note that if G has three independent edges, then we can find a properly colored C_6. Recall that $|E(G)|\geq n-6\geq 2$. Now we may assume that $n=8$ and $|E(G)|=2$. Let $E(G)=\{xy,zw\}$ and $U=V(K_8)\backslash\{x,y,z,w\}=\{u_1,u_2,u_3,u_4\}$.

Case 2.1 There is an edge u_iu_j such that $c(u_iu_j)$ is starred at u_i, say $c(u_1u_2)$ is starred at u_1.

If there is one vertex in $\{x,y,z,w\}$, say x, such that $c(u_1x)\neq c(u_1u_2)$, then $u_1xyzwu_2u_1$ is a properly colored C_6. We assume that $c(u_1x)=c(u_1y)=c(u_1z)=c(u_1w)=c(u_1u_2)$. Since $d^c(u_1)\geq 2$, we can assume that $c(u_1u_3)$ is starred at u_1 and $c(u_1u_3)\neq c(u_1u_2)$. Thus $u_1xyzwu_3u_1$ is a properly colored C_6.

Case 2.2 For all edge u_iu_j, $c(u_iu_j)$ is not starred at u_i or u_j.

Since $d^c(u_1)\geq 2$ and $d^c(u_2)\geq 2$, we can find two distinct vertices $v_1,v_2\in\{x,y,z,w\}$ such that $c(u_1v_1)$ is starred at u_1 and $c(u_2v_2)$ is starred at u_2. If $v_1=x$ and $v_2=y$, then $u_1xzwyu_2u_1$ is a properly colored C_6. If $v_1=x$ and $v_2=z$, then $u_1xywzu_2u_1$ is a properly colored C_6. □

5 K_4^- and $K_{2,3}$

In this section, we will prove Theorems 6 and 7. First, we determine the exact value of $\mathrm{pr}(K_n,K_4^-)$.

Theorem 6 *For $n\geq 4$, $\mathrm{pr}(K_n,K_4^-)=\left\lfloor\frac{3(n-1)}{2}\right\rfloor$.*

Proof **The lower bound:** Consider an edge-coloring of K_n as follows. Take a triangle $C_3=xyz$ of K_n and a maximum matching $M=\{x_1y_1,x_2y_2,\ldots,x_{\lfloor\frac{n-3}{2}\rfloor}y_{\lfloor\frac{n-3}{2}\rfloor}\}$ of $K_n-\{x,y,z\}$. There is one vertex w in $V(K_n)\backslash(V(M)\cup\{x,y,z\})$ when n is even. For $1\leq i\leq\lfloor\frac{n-3}{2}\rfloor$, color all the edges of $\{ux_i:u\in\{x,y,z,x_1,y_1,x_2,y_2,\ldots,x_{i-1},y_{i-1}\}\}$ with color c_{1i} and all the edges of $\{uy_i:u\in\{x,y,z,x_1,y_1,x_2,y_2,\ldots,x_{i-1},y_{i-1}\}\}$ with color c_{2i}. If n is even, color all edges of $\{uw:u\in V(K_n-w)\}$ with a new color. Finally, assign distinct new colors to all edges of $C_3\cup M$. In such a coloring, there is no properly colored K_4^-, and the number of colors is $\lfloor\frac{3(n-1)}{2}\rfloor$.

The upper bound: We will prove that for any $\lfloor\frac{3n-1}{2}\rfloor$-edge-coloring of K_n, there is a properly colored K_4^- by induction on n. The base case $n=4$ is trivial. For $n\geq 5$, assume that the conclusion holds for order less than n. Let c be a $\left\lfloor\frac{3n-1}{2}\right\rfloor$-edge-coloring of K_n. If there is a vertex v such that $d^c(v)\leq 1$, then $|C(K_n-v)|\geq\lfloor\frac{3n-1}{2}\rfloor-1\geq\left\lfloor\frac{3(n-1)-1}{2}\right\rfloor$, and there is a properly colored K_4^- in K_n-v by the induction hypothesis. We may assume that $d^c(v)\geq 2$ for all $v\in V(K_n)$. Let G be the weak representing subgraph of K_n. By (1.5), we have $|E(G)|\geq 2n-\lfloor\frac{3n-1}{2}\rfloor=\lceil\frac{n+1}{2}\rceil$, which implies there is a path $P_3=xyz$ in G. By the construction of G, if $e=uv\in E(G)$, the $c(e)$ is starred at u and v. We consider the following two cases.

Case 1. $xz \notin E(G)$.

In this case, $c(xz)$ is not starred at x or z, say x. Since $d^c(x) \geq 2$, there is a vertex $w \notin \{x, y, z\}$ such that $c(xw)$ is starred at x. Then $c(xz), c(yw) \notin \{c(xy), c(yz), c(xw)\}$ and the edge set $\{xy, yz, xz, xw, yw\}$ induces a properly colored K_4^-.

Case 2. $xz \in E(G)$.

In this case, we can assume $c(ux) = c(uy) = c(uz)$ for all $u \in V(K_n) \backslash \{x, y, z\}$; otherwise we easily have a properly colored copy of K_4^- in $K_n[x, y, z, u]$. Thus we have

$$|C(K_n - \{x, y\})| \geq \left\lfloor \frac{3n-1}{2} \right\rfloor - 3 = \left\lfloor \frac{3(n-2)-1}{2} \right\rfloor.$$

If $n = 5$, then $3 = |E(K_5 - \{x, y\})| \geq |C(K_5 - \{x, y\})| \geq 4$, a contradiction. Thus we may assume that $n \geq 6$, there is a properly colored K_4^- in $K_n - \{x, y\}$ by the induction hypothesis. □

Now we prove the lower bound and upper bound of $\mathrm{pr}(K_n, K_{2,3})$. We conjecture that the exact value is closer to the lower bound.

Theorem 7 *For* $n \geq 5$, $\frac{7}{4}n + O(1) \leq \mathrm{pr}(K_n, K_{2,3}) \leq 2n - 1$.

Proof **The lower bound:** Let $n = 4k + r$, where $1 \leq r \leq 4$. Set $V(K_n) = V_1 \cup \cdots \cup V_k \cup V_{k+1}$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$, $|V_i| = 4$ for $1 \leq i \leq k$ and $|V_{k+1}| = r$. We color the edges with end-vertices in the same set with $6k + \binom{r}{2}$ distinct colors and color the remaining edges with k addition colors $c_1, c_2, \ldots, c_k$ such that all edges between V_i and V_j are colored with $c_{\min\{i,j\}}$, where $i \neq j$. The total number of colors is $\frac{7}{4}n + O(1)$ and there is no properly colored $K_{2,3}$.

The upper bound: We will prove that for any $2n$ edge-coloring of K_n, there is a properly colored $K_{2,3}$ by induction on n. The base case $n = 5$ is trivial. For $n \geq 6$, assume that the conclusion holds for order less than n. Let c be a $2n$-edge-coloring of K_n. If there is a vertex v such that $d^c(v) \leq 2$, then $|C(K_n - v)| \geq 2n - 2$ and there is a properly colored $K_{2,3}$ in $K_n - v$ by the induction hypothesis. We may assume that $d^c(v) \geq 3$ for all $v \in V(K_n)$. Let G be the weak representing subgraph of K_n. By (1.5), we have $|E(G)| \geq 3n - 2n = n$. Note that for $n \geq 4, \mathrm{ex}(n, P_4) \leq n$ and the equality holds for the graph of disjoint copies of C_3 (see [5]). So we will consider the following two cases.

Case 1. G contains a $P_4 = xyzw$.

If $G[V(P_4)] \cong K_4$, then we can assume $c(ux) = c(uy) = c(uz) = c(uw)$ for all $u \in V(K_n) \backslash \{x, y, z, w\}$; otherwise we easily have a properly colored copy of $K_{2,3}$. Therefore

$$|C(K_n - \{x, y, z\})| \geq 2n - 6 = 2(n - 3).$$

If $n = 6$, then $3 = |E(K_6 - \{x, y, z\})| \geq |C(K_6 - \{x, y, z\})| \geq 6$, a contradiction. If $n = 7$, then $6 = |E(K_6 - \{x, y, z\})| \geq |C(K_6 - \{x, y, z\})| \geq 8$, a contradiction. Thus we may assume that $n \geq 8$, there is a properly colored $K_{2,3}$ in $K_n - \{x, y, z\}$ by the induction hypothesis.

Now we consider the case $G[V(P_4)] \ncong K_4$. Since $d^c(x) \geq 3$ and $d^c(w) \geq 3$, there is a vertex $u \in V(K_n)\setminus\{x, y, z, w\}$ such that $c(xu)$ or $c(wu)$, say $c(xu)$ is starred at x and $c(xu) \notin \{c(xy), c(xw)\}$. Therefore, the edges between $\{x, z\}$ and $\{y, u, w\}$ induce a properly colored $K_{2,3}$.

Case 2. G is the graph of disjoint copies of C_3.

Let $T_1 = xyzx$ be a triangle of G. Since $d^c(x) \geq 3$, there is a vertex $u \in V(K_n)\setminus\{x, y, z\}$ such that $c(xu)$ is starred at x and $c(xu) \notin \{c(xy), c(xz)\}$. Suppose u belong to the triangle $T_2 = uvwu$ of G. Therefore, the edges between $\{y, u\}$ and $\{x, z, v\}$ induce a properly colored $K_{2,3}$.

□

6 Conclusion

In this paper, we obtain the relationship of $\mathrm{pr}(K_n, G)$ and $\mathrm{ex}(n, \mathcal{G}')$ by Theorem 2. We also determine the value of $\mathrm{pr}(K_n, G)$ for some small graphs. Since the lower bound of $\mathrm{pr}(K_n, C_k)$ is very similar to the paths, we expect that the idea of the proof of Theorem 3 would be helpful to prove Conjecture 2 for large n.

Another interesting open problem is determining the behavior of $\mathrm{pr}(K_n, K_4)$. Theorem 1 shows that $\mathrm{pr}(K_n, K_4) = o(n^2)$ and Theorem 2 shows that $\mathrm{pr}(K_n, K_4) \geq \mathrm{ex}(n, C_4) + 1$. Since $\mathrm{ex}(n, C_4) = \frac{1}{2}n^{3/2} + o(n^{3/2})$ (See [4, 6]), one can prove that $\mathrm{pr}(K_n, K_4) = O(n^{3/2})$. The main idea is that for an edge-coloring of K_n, if the weak representing subgraph contains a C_4, then there exists a properly colored K_4 in K_n.

Acknowledgements The authors are thankful to the anonymous referee for his/her useful comments.

Funding Open access funding provided by ELKH Alfréd Rényi Institute of Mathematics.

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