



On the Connectivity of Enhanced Power Graphs of Finite Groups

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Abstract

This paper deals with the vertex connectivity of enhanced power graphs of finite groups. We classify all abelian groups G such that the vertex connectivity of enhanced power graph of G is 1. We derive an upper bound for the vertex connectivity of the enhanced power graph of any general abelian group G . Also we completely characterize all abelian groups G , such that the proper enhanced power graph is connected. Moreover, we study some special class of non-abelian groups G such that the proper enhanced power graph is connected and we find their vertex connectivity.

Keywords Abelian group · Dominating vertex · Enhanced power graph · Vertex connectivity

Mathematics Subject Classification 05C25

1 Introduction

The exploration of graphs associated with algebraic structures is important, as graphs like these enrich both algebra and graph theory. Besides, they have important applications (see, for example, [2, 21]) and are related to automata theory [22]. During the last two decades, investigation of the interplay between the properties of

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an algebraic structure S and the graph-theoretic properties of $\Gamma(S)$, a graph associated with S , has been an exciting topic of research. Different types of graphs, specifically zero-divisor graph of a ring [3], semiring [5], semigroup [14], poset [20], power graph of semigroup [12, 24], group [23], normal subgroup based power graph of group [8], intersection power graph of group [6] etc. have been introduced to study algebraic structures using graph theory. The concept of a power graph was introduced in [23]. As explained in the survey [2], this definition also covered the undirected graphs. Accordingly, the present paper follows Chakrabarty et al. and uses the brief term “power graph” defined as follows.

Definition 1.1 ([2, 12, 23]) Let S be a semigroup, then the *power graph* $\mathcal{P}(S)$ of S , is a simple graph, whose vertex set is S and two distinct vertices u and v are edge connected if and only if either $u^m = v$ or $v^n = u$, where $m, n \in \mathbb{N}$.

Another well-studied graph, called *commuting graph* associated with a group G is studied in [10] as a part of the classification of finite simple groups. For more information about the commuting graph, see [4, 17].

Definition 1.2 ([10]) Let G be a group, then the *commuting graph* of G , denoted by $\mathcal{C}(G)$, is the simple graph whose vertex set is a set of non-central elements of G and two distinct vertices u and v are adjacent if and only if $uv = vu$.

Definition 1.3 ([1]) Given a group G , the *enhanced power graph* of G , denoted by $\mathcal{G}_e(G)$, is the graph with vertex set G , in which u and v are joined if and only if there exists an element $w \in G$ such that both u and v are powers of w .

The authors Aalipour et al. [1] measure how close the power graph is to the commuting graph by using the enhanced power graph. In fact, the enhanced power graph contains the power graph and is a subgraph of the commuting graph. They characterized the finite groups such that, for an arbitrary pair of these three graphs, this pair of graphs are equal. Besides, Zahirović et al. [29], the researchers proved that finite groups with isomorphic enhanced power graphs have isomorphic directed power graphs. They showed that any isomorphism between the undirected power graphs of finite groups is an isomorphism between enhanced power graphs of these group. Ma and She [25] derived the metric dimension of enhanced power graphs of finite groups where as Hamzeh et al. [19] derived the automorphism groups of enhanced power graphs of finite groups. Recently Panda et al. [26] have studied independence number, vertex covering number and some other graph invariants of enhanced power graphs.

1.1 Basic Definitions, Notations and Main Results

For the convenience of the reader and also for later use, we recall some basic definitions and notations about graphs. Let $\Gamma = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. Two elements u and v are said to be adjacent if $(u, v) \in E$. The standard distance between two vertices u and v in a connected graph Γ is denoted by $d(u, v)$. Clearly, if u and v are adjacent, then $d(u, v) = 1$. For a graph Γ , its *diameter* is defined as $\text{diam}(\Gamma) = \max_{u, v \in V} d(u, v)$. That is, the diameter

of graph is the largest possible distance between pair of vertices of a graph. A *path* of length l between two vertices v_0 and v_k is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, \dots, v_{k-1}, e_{k-1}, v_k$, where the v_i 's are distinct (except possibly the first and last vertices) and e_i is the edge (v_i, v_{i+1}) . A graph Γ is said to be *connected* if for any pair of vertices u and v , there exists a path between u and v . Γ is said to be *complete* if any two distinct vertices are adjacent. A vertex of a graph $\Gamma = (V, E)$ is called a *dominating vertex* if it is adjacent to every other vertex. For a graph Γ , let $Dom(\Gamma)$ denote the set of all dominating vertices in Γ . The *vertex connectivity* of a graph Γ , denoted by $\kappa(\Gamma)$ is the minimum number of vertices which need to be removed from the vertex set Γ so that the induced subgraph of Γ on the remaining vertices is disconnected. The complete graph with n vertices has connectivity $n - 1$. For more on graph theory we refer [9, 18, 28]. The enhanced power graph is called *dominatable* if it has a dominating vertex other than identity.

Throughout this paper we consider G as a finite group. $|G|$ denotes the cardinality of the set G . For a prime p , a group G is said to be a p -group if $|G| = p^r, r \in \mathbb{N}$. For any element $g \in G, o(g)$ denotes the order of the element $g \in G$. Let G be a group and $a \in G$, then $Gen(a)$ is the set of all generators of the cyclic group $\langle a \rangle$. Let m and n be any two positive integers, then the greatest common divisor of m and n is denoted by $gcd(m, n)$. The Euler's phi function $\phi(n)$ is the number of integers k in the range $1 \leq k \leq n$ for which the $gcd(n, k)$ is equal to 1. The set $\{1, 2, \dots, n\}$ is denoted by $[n]$. Throughout this paper, the group operation of any abelian group is taken to be additive.

In this paper, our focus is on the vertex connectivity of enhanced power graphs of finite abelian groups. If G is a non-cyclic non-generalized quaternion p -group, then we determine the exact value of the vertex connectivity of $\mathcal{G}_e(G)$.

Theorem 1.1 *Let G be a finite p -group such that G is neither cyclic nor generalized quaternion group. Then $\kappa(\mathcal{G}_e(G)) = 1$.*

Our next result classifies all non-cyclic abelian groups G such that $\kappa(\mathcal{G}_e(G)) = 1$.

Theorem 1.2 *Let G be a finite non-cyclic abelian group. Then $\kappa(\mathcal{G}_e(G))$ is equal to 1 if and only if G is a p -group.*

The authors Aalipour et al. [1][Question 40] asked about the connectivity of power graphs when all the dominating vertices are removed. Recently, Cameron and Jafari [11] answered this question for power graphs. In this paper, we investigate the same question for enhanced power graphs. To seek the answer of this question, define the following graph:

Definition 1.4 Given a group G , the *proper enhanced power graph* of G , denoted by $\mathcal{G}_e^{**}(G)$, is the graph obtained by deleting all the dominating vertices from the enhanced power graph $\mathcal{G}_e(G)$. Moreover, by $\mathcal{G}_e^*(G)$ we denote the graph obtained by deleting only the identity element of G and this is called *deleted enhanced power graph* of G . Note that if there is no such dominating vertex other than identity, then $\mathcal{G}_e^*(G) = \mathcal{G}_e^{**}(G)$.

Bera et al. [7] characterized all abelian groups G , such that $|\text{Dom}(\mathcal{G}_e(G))| > 1$. In fact, they proved the following:

Theorem 1.3 ([7]) *Let G be a finite abelian group. Then $\mathcal{G}_e(G)$ is dominatable if and only if G has a cyclic Sylow subgroup.*

So, from Theorem 1.3, $|\text{Dom}(\mathcal{G}_e(G))| > 1$ if and only if $G = G_1 \times \mathbb{Z}_n$, where $\text{gcd}(|G_1|, n) = 1$ and G_1 has no cyclic sylow subgroup. Then one natural question is which are the dominatable vertices of $\mathcal{G}_e(G)$. The next theorem gives the complete list of the dominating vertices of $\mathcal{G}_e(G)$.

Theorem 1.4 *Let G_1 be a non-cyclic abelian group such that G_1 has no cyclic sylow subgroup. If $n \in \mathbb{N}$, and $\text{gcd}(|G_1|, n) = 1$, then $\text{Dom}(\mathcal{G}_e(G_1 \times \mathbb{Z}_n)) = \{(e, x)$, where x is any element of \mathbb{Z}_n and e is the identity of $G_1\}$.*

Theorem 1.2 completely characterizes the connectivity of $\mathcal{G}_e^*(G)$ for any finite abelian p -group G . Now if G is a non-cyclic abelian non p -group such that G has no cyclic sylow subgroup, then by Theorem 1.3, G has no dominating vertex other than the identity. So, in this case we care about the connectivity of $\mathcal{G}_e^{**}(G) = \mathcal{G}_e^*(G)$ and by Theorem 1.2, $\mathcal{G}_e^*(G)$ is connected. Therefore when the graph $\mathcal{G}_e^*(G)$ has a dominating vertex other than identity, the connectivity of $\mathcal{G}_e(G)$ is a more interesting question. In this paper, we characterize for which finite abelian groups, the proper enhanced power graphs $\mathcal{G}_e^{**}(G)$ are connected and for which they are not. Our contributions on this paper in this theme is the following:

Theorem 1.5 *Let G be a non-cyclic abelian non p -group such that $G \cong G_1 \times \mathbb{Z}_n$, $\text{gcd}(|G_1|, n) = 1$ and G_1 has no cyclic sylow subgroup. Then $\mathcal{G}_e^{**}(G)$ is disconnected if and only if G_1 is a p -group.*

Therefore, from Theorem 1.5, when G_1 is not a p -group, $\mathcal{G}_e^{**}(G)$ remains connected. Thus, the number of additional vertices required to make it disconnected is an interesting question. On this theme, our next result is the following:

Theorem 1.6 *Let G be a non-cyclic abelian group such that*

$$G \cong \mathbb{Z}_{p_1^{t_{11}}} \times \mathbb{Z}_{p_1^{t_{12}}} \times \mathbb{Z}_{p_1^{t_{13}}} \times \cdots \times \mathbb{Z}_{p_1^{t_{1k_1}}} \times \mathbb{Z}_{p_2^{t_{21}}} \times \mathbb{Z}_{p_2^{t_{22}}} \times \cdots \\ \times \mathbb{Z}_{p_2^{t_{2k_2}}} \times \cdots \times \mathbb{Z}_{p_r^{t_{r1}}} \times \mathbb{Z}_{p_r^{t_{r2}}} \times \cdots \times \mathbb{Z}_{p_r^{t_{rk_r}}},$$

where $k_i \geq 1$ and $1 \leq t_{i1} \leq t_{i2} \leq \cdots \leq t_{ik_i}$, for all $i \in [r]$. Then

$$\kappa(\mathcal{G}_e(G)) \leq p_1^{t_{11}} p_2^{t_{21}} \cdots p_r^{t_{r1}} - \phi(p_1^{t_{11}} p_2^{t_{21}} \cdots p_r^{t_{r1}}).$$

When G_1 is a p -group, the following result gives the exact value of the vertex connectivity of $\mathcal{G}_e(G)$.

Theorem 1.7 *Let G be a non-cyclic abelian non- p -group such that $G \cong G_1 \times \mathbb{Z}_n$, $\text{gcd}(|G_1|, n) = 1$ and G_1 is a p -group with no cyclic sylow subgroup. Then $\kappa(\mathcal{G}_e(G)) = n$.*

2 Preliminaries

We first recall some earlier known results on enhanced power graphs which we need throughout the paper. Bera et al. [7], studied about the completeness, dominatability and many other properties of enhanced power graph of finite group. In fact, they proved the following:

Lemma 2.1 (Theorem 2.4, [7]) *The enhanced power graph $\mathcal{G}_e(G)$ of the group G is complete if and only if G is cyclic.*

Lemma 2.2 (Theorem 3.3, [7]) *Let G be a non-abelian 2-group. Then the enhanced power graph $\mathcal{G}_e(G)$ is dominatable if and only if G is generalized quaternion group.*

Lemma 2.3 (Theorem 3.1, [7]) *Let G be a finite group and $n \in \mathbb{N}$. If $\gcd(|G|, n) = 1$, then the enhanced power graph $\mathcal{G}_e(G \times \mathbb{Z}_n)$ is dominatable.*

Lemma 2.4 (Lemma 2.1, [7]) *Let $a, b \in G$ with $o(a) = o(b)$ and $\langle a \rangle \neq \langle b \rangle$. Then x is not adjacent with y for every $x \in \text{Gen}(a)$ and $y \in \text{Gen}(b)$.*

We next prove some important lemmas which are used to prove our main theorems.

Lemma 2.5 *Let G be a finite group and $x, y \in G \setminus \{e\}$ be such that $\gcd(o(x), o(y)) = 1$ and $xy = yx$. Then, $x \sim y$ in $\mathcal{G}_e^*(G)$.*

Proof Let $o(a) = m$ and $o(b) = n$. Now $\gcd(m, n) = 1$ implies that $n^{\phi(m)} = mk + 1, k \in \mathbb{N}$, (by Euler’s theorem). Again, $ab = ba$ implies that $(ab)^{n^{\phi(m)}} = a^{n^{\phi(m)}} b^{n^{\phi(m)}} = a^{n^{\phi(m)}} = a^{mk+1} = a$. As a result, $a \in \langle ab \rangle$. Similarly we can prove that $b \in \langle ab \rangle$. Consequently, $a \sim b$ in $\mathcal{G}_e^*(G)$. □

Lemma 2.6 *Let G be a p -group. Let a, b be two elements of G of order $p, p^i (i \geq 1)$ respectively. If there is a path between a and b in $\mathcal{G}_e^*(G)$, then $\langle a \rangle \subset \langle b \rangle$. In particular, if both a and b have order p , then, $\langle a \rangle = \langle b \rangle$.*

Proof Let $a = a_1 \sim a_2 \sim \dots \sim a_m = b$ be a path between a and b . Now $a = a_1 \sim a_2$ implies that there exists $x \in G$ such that $a_1, a_2 \in \langle x \rangle$. As a result, $a_1 \in \langle a_2 \rangle$ (since a cyclic group has a unique subgroup corresponding to each divisor of the order of the cyclic group). Now, $a_2 \sim a_3$ and G is a p -group, then either $a_2 \in \langle a_3 \rangle$ or $a_3 \in \langle a_2 \rangle$. Clearly for both of the cases $a_1 \in \langle a_3 \rangle$. Continuing this process we can conclude that $a \in \langle b \rangle$. □

Lemma 2.7 *Let G be any non-cyclic group. For any dominating vertex $v (\neq e)$ of G there exists a prime p dividing $o(v)$ such that G has a unique subgroup of order p .*

Proof Let $v \neq e$ be a dominating vertex and $o(v) = m$. Let p be a prime divisor of m and $m = rp$. We claim that $H = \langle v^r \rangle$ is the unique subgroup of order p in G . Consider $x \in G$ such that $o(x) = p$. Since v is dominating vertex, we have $x \sim v$. Thus, there exists a cyclic subgroup A such that $x, v \in A$. Then $o(x) = p$ implies that

$x \in \langle v \rangle$. If $x = v^q$, by division algorithm it can be shown that q has to be a multiple of r and thus $x \in H$. This completes the proof. \square

We next move on to the most important result of this section.

Theorem 2.8 *For any group G , the graph $\mathcal{P}^*(G)$ is connected if and only if the graph $\mathcal{G}_e^*(G)$ is connected.*

Proof The forward implication is easy. That is, if $\mathcal{P}^*(G)$ is connected then $\mathcal{G}_e^*(G)$ is of course connected. We prove the other direction. Let, $\mathcal{G}_e^*(G)$ be connected and $a, b \in \mathcal{P}^*(G)$. As $\mathcal{G}_e^*(G)$ is connected, there exists a path $a = a_1 \sim a_2 \sim \dots \sim a_m = b$. Now, $a_i \sim a_{i+1}$ in $\mathcal{G}_e^*(G) \Rightarrow$ there exist $b_i \in G$ such that both a_i and $a_{i+1} \in \langle b_i \rangle$. In that case, $a_i \sim b_i \sim a_{i+1}$. Therefore, we have $a = a_1 \sim b_1 \sim a_2 \sim b_2 \sim a_3 \dots \sim a_m = b$ in $\mathcal{P}^*(G)$. This completes the proof. \square

Therefore, for any graph G , the information about the connectivity of one of the two graphs $\mathcal{P}^*(G)$ and $\mathcal{G}_e^*(G)$ gives information about the connectivity of the other one.

3 Proofs of Main Results About Vertex Connectivity of $\mathcal{G}_e(G)$ When G is Abelian

Proof of Theorem 1.1 First suppose that G is non-cyclic abelian p -group. Clearly G has at least two distinct cyclic subgroups $H_1 = \langle a \rangle$ and $H_2 = \langle b \rangle$ of order p . Now by Lemma 2.6, there is no path joining a and b in $\mathcal{G}_e^*(G)$, otherwise $H_1 = H_2$. The proof is complete. \square

Proof of Theorem 1.2 G is non-cyclic abelian p -group. Therefore, by Theorem 1.1, $\kappa(\mathcal{G}_e(G)) = 1$.

For the converse part, let G be a finite abelian group which is not a p -group. Let, p_1, p_2, \dots, p_k be the prime factors of $|G|$. Let, $a, b \in G$ and $o(a) = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ and $o(b) = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$. We consider the following two cases:

Case 1: There exists distinct i and j with $r_i \neq 0$ and $s_j \neq 0$. Then the elements $p_1^{r_1} p_2^{r_2} \dots p_{i-1}^{r_{i-1}} p_{i+1}^{r_{i+1}} \dots p_k^{r_k} a$ and $p_1^{s_1} p_2^{s_2} \dots p_{j-1}^{s_{j-1}} p_{j+1}^{s_{j+1}} \dots p_k^{s_k} b$ are of order $p_i^{r_i}$ and $p_j^{s_j}$ respectively. Thus, by Lemma 2.5, $p_1^{r_1} p_2^{r_2} \dots p_{i-1}^{r_{i-1}} p_{i+1}^{r_{i+1}} \dots p_k^{r_k} a$ and $p_1^{s_1} p_2^{s_2} \dots p_{j-1}^{s_{j-1}} p_{j+1}^{s_{j+1}} \dots p_k^{s_k} b$ are adjacent. Therefore we have

$$a \sim p_1^{r_1} p_2^{r_2} \dots p_{i-1}^{r_{i-1}} p_{i+1}^{r_{i+1}} \dots p_k^{r_k} a \sim p_1^{s_1} p_2^{s_2} \dots p_{j-1}^{s_{j-1}} p_{j+1}^{s_{j+1}} \dots p_k^{s_k} b \sim b.$$

That is, there exists a path of length ≤ 3 between a and b . We observe that this case takes care of everything except when both $o(a)$ and $o(b)$ are power of the same prime p_ℓ for some $1 \leq \ell \leq k$ which we consider next.

Case 2: $o(a) = p_\ell^{r_\ell}$ and $o(b) = p_\ell^{s_\ell}$. Let, c be an element of order p_i in G with $i \neq \ell$. Then by Lemma 2.5, we have $a \sim c \sim b$. Thus, $\mathcal{G}_e^*(G)$ is connected. This completes the proof. \square

By Theorem 2.8, we immediately get the following corollary on the connectivity of the power graphs.

Corollary 1 *Let G be a finite non-cyclic abelian group. Then $\kappa(\mathcal{P}(G))$ is equal to 1 if and only if G is a p -group.*

Proof of Theorem 1.4 We show that (e, x) is a dominating vertex, where e is the identity element of the group G and x is a any element of the group \mathbb{Z}_n . Consider an arbitrary vertex (g, y) of the graph $\mathcal{G}_e(G \times \mathbb{Z}_n)$.

Case 1: Let $g = e$. Let a be a generator of the cyclic group \mathbb{Z}_n . Now $y \in \mathbb{Z}_n$ implies that $(e, y), (e, x) \in \langle (e, a) \rangle$ and so $(e, x) \sim (e, y)$.

Case 2: Let $g \neq e$ and $y = 0$. [Here 0 actually means the additive identity of the group \mathbb{Z}_n]. We show that $(g, 0), (e, x) \in \langle (g, a) \rangle$. First we show that $(e, a) \in \langle (g, a) \rangle$. Let $o(g) = m$. Now $\gcd(|G|, n) = 1$ implies that $\gcd(m, n) = 1$. Then by Euler’s theorem $m^{\phi(n)} = n\ell + 1, \ell \in \mathbb{N}$. Therefore, $(g, a)^{m^{\phi(n)}} = (g^{m^{\phi(n)}}, a^{m^{\phi(n)}}) = (e, a^{n\ell+1}) = (e, a)$. Hence, $(e, a) \in \langle (g, a) \rangle$. Now we show that $(g, 0) \in \langle (g, a) \rangle$. It is given that $\gcd(m, n) = 1$. So, by the Euler’s theorem, $n^{\phi(m)} = mk + 1, k \in \mathbb{N}$. Hence $(g, a)^{n^{\phi(m)}} = (g^{n^{\phi(m)}}, 0) = (g^{mk+1}, 0) = (g, 0)$. Consequently, $(g, 0) \in \langle (g, a) \rangle$.

Case 3: Let $g \neq e$ and $y \neq 0$. We show that $(g, y), (e, x) \in \langle (g, a) \rangle$. Already we have proved that $(g, 0), (e, x) \in \langle (g, a) \rangle$. Since a is a generator of \mathbb{Z}_n , $(e, y) \in \langle (e, a) \rangle \subset \langle (g, a) \rangle$. Hence $(g, y) = (g, 0)(e, y) \in \langle (g, a) \rangle$.

To finish the proof we have to show that if (g, z) is a dominating vertex, then g must be the identity of G . Let

$$G = \mathbb{Z}_{p_1^{t_{11}}} \times \mathbb{Z}_{p_1^{t_{12}}} \times \mathbb{Z}_{p_1^{t_{13}}} \times \cdots \times \mathbb{Z}_{p_1^{t_{1k_1}}} \times \mathbb{Z}_{p_2^{t_{21}}} \times \mathbb{Z}_{p_2^{t_{22}}} \times \cdots \\ \times \mathbb{Z}_{p_2^{t_{2k_2}}} \times \cdots \times \mathbb{Z}_{p_r^{t_{r1}}} \times \mathbb{Z}_{p_r^{t_{r2}}} \times \cdots \times \mathbb{Z}_{p_r^{t_{rk_r}}},$$

where $k_i \geq 2$ and $t_{i1} \leq t_{i2} \leq \cdots \leq t_{ik_i}$, for all $i \in [r]$. Let

$$v = (x_{p_1^{t_{11}}}, x_{p_1^{t_{12}}}, \cdots, x_{p_1^{t_{1k_1}}}, x_{p_2^{t_{21}}}, \dots, x_{p_2^{t_{2k_2}}}, \dots, x_{p_r^{t_{r1}}}, x_{p_r^{t_{r2}}}, \dots, x_{p_r^{t_{rk_r}}}, z)$$

be a dominating vertex. We will prove that, for each $i \in [r]$ and $j \in [k_i], x_{p_i^{t_{ij}}} = x_{p_i^{t_{i2}}} = \cdots = x_{p_i^{t_{ij}}} = 0$. [Here 0 actually means the additive identity of the group $\mathbb{Z}_{p_i^{t_{ij}}}$]. Consider the element

$$v' = (0, 0, \dots, 0, g_{p_1^{t_{1k_1}}}, 0, 0, \dots, g_{p_2^{t_{2k_2}}}, 0, \dots, 0, g_{p_r^{t_{rk_r}}}, z'),$$

where $g_{p_i^{t_{ik_i}}}$ is a generator of the cyclic group $\mathbb{Z}_{p_i^{t_{ik_i}}}$ for each $i \in [r]$ and z' is a generator of \mathbb{Z}_n . As v is a dominating vertex of the graph $\mathcal{G}_e(G)$, we have $v \sim v'$ in $\mathcal{G}_e(G)$. Clearly, v' is an element of maximum ordered. So, we have $v \in \langle v' \rangle$. As a result, for each $i \in [r]$ and $j \in [k_i - 1], x_{p_i^{t_{ij1}}} = x_{p_i^{t_{ij2}}} = \cdots = x_{p_i^{t_{ij}}} = 0$, i.e.,

$$v = (0, 0, \dots, 0, x_{p_1^{t_{1k_1}}}, 0, 0, \dots, 0, x_{p_2^{t_{2k_2}}}, 0, \dots, 0, x_{p_r^{t_{rk_r}}}, z).$$

Now we show that $x_{p_i^{t_{ik_i}}} = 0$, for all $i \in [r]$. Suppose at least one of the $x_{p_i^{t_{ik_i}}}$ is non-zero. Without any loss of generality we assume that $x_{p_1^{t_{1k_1}}} \neq 0$. Consider $v_1 =$

$(x, 0, 0, \dots, 0, 0, 0)$, (last zero is the identity of cyclic group Z_n) where $x \in \mathbb{Z}_{p_1^{t_{1k_1}}}$ with $o(x) = p_1$. Then p_1 divides $o(v)$. If $v \sim v_1$, then there exists a cyclic subgroup C of $G \times \mathbb{Z}_n$ such that $v, v_1 \in C$. Then $o(v_1) = p_1$ and p_1 divides $o(v)$. This implies that $v_1 \in \langle v \rangle$, which contradicts that $x \neq 0$. This completes the proof. \square

We next consider the connectivity of the proper enhanced power graph $\mathcal{G}_e^{**}(G)$, when G is abelian.

Proof of Theorem 1.5 First we show that if G_1 is a p -group, then $\mathcal{G}_e^{**}(G)$ is disconnected. By Theorem 1.4, order of each element of $\mathcal{G}_e^{**}(G)$ is divisible by p . So applying the proof of the lemma 2.6, we get that for two elements $a = (x, 0)$ and $b = (x', y')$ of $\mathcal{G}_e^{**}(G)$, with $o((x, 0)) = p$ and $x' \neq e$, if there exists any path joining a and b , then $\langle (x, 0) \rangle$ is contained in or equal to $\langle (x', y') \rangle$. In particular, if both a and b have order p , then the existence of a path joining a and b implies that $\langle a \rangle = \langle b \rangle$. Since, G_1 is noncyclic abelian p -group, there exist two elements a and b of order p such that $\langle a \rangle \neq \langle b \rangle$. So by our previous observation, a is not path connected to b .

Conversely, suppose that G_1 is non- p -group. Then we show that $\mathcal{G}_e^{**}(G)$ is connected. Here we have two cases.

Case 1: Let $(x_1, 0)$ and $(x_2, 0)$ be two elements of $G_1 \times \mathbb{Z}_n$ such that $x_1 \neq e$ and $x_2 \neq e$. Then by same argument as in proof of converse part of Theorem 1.2, $(x_1, 0)$ and $(x_2, 0)$ are path connected in $\mathcal{G}_e^{**}(G)$.

Case 2: Let $(x_1, y_1), (x_2, y_2) \in V(\mathcal{G}_e^{**}(G))$. Clearly, $(x_1, y_1) \sim (x_1, 0)$ and $(x_2, 0) \sim (x_2, y_2)$. Again there is a path between $(x_1, 0)$ and $(x_2, 0)$ in $\mathcal{G}_e^{**}(G)$ by Case 1. Therefore, (x_1, y_1) and (x_2, y_2) are path connected in $\mathcal{G}_e^{**}(G)$. Hence the graph $\mathcal{G}_e^{**}(G)$ is connected. This completes the proof. \square

Proof of Theorem 1.6 Let $H = \langle a \rangle$ be a cyclic subgroup of G , where $a = (a_{11}, 0, \dots, 0, a_{21}, 0, \dots, a_{r1}, 0, \dots, 0)$, and $a_{i1} \in \mathbb{Z}_{p_i^{t_{i1}}}$ such that $o(a_{i1}) = p_i^{t_{i1}}$, for $i = 1, 2, \dots, r$. H is maximal cyclic subgroup of G .

Now we show that for any $b \in G \setminus H$, there is no edge between b and any element in $Gen(a)$. If possible there exists $x \in Gen(a)$ such that $b \sim x$ in $\mathcal{G}_e(G)$. Then there exists a cyclic subgroup K of G such that $b, x \in K$. Again H is a maximal cyclic subgroup of G which is also generated by x . Therefore, $K = \langle a \rangle = H$. Hence a contradiction as $b \in G \setminus H$. Clearly, if we remove the identity and non-identity non-generators elements from the cyclic subgroup H , then the graph will be disconnected and the number of deleted vertices is $p_1^{t_{11}} p_2^{t_{21}} \dots p_r^{t_{r1}} - \phi(p_1^{t_{11}} p_2^{t_{21}} \dots p_r^{t_{r1}})$. Hence the result. \square

From Theorem 1.6, we immediately have the following corollary on the vertex connectivity of power graphs of any non-cyclic abelian group.

Corollary 2 *Let G be a non-cyclic abelian group such that*

$$G \cong \mathbb{Z}_{p_1^{t_{11}}} \times \mathbb{Z}_{p_1^{t_{12}}} \times \mathbb{Z}_{p_1^{t_{13}}} \times \dots \times \mathbb{Z}_{p_1^{t_{1k_1}}} \times \mathbb{Z}_{p_2^{t_{21}}} \times \mathbb{Z}_{p_2^{t_{22}}} \\ \times \dots \times \mathbb{Z}_{p_2^{t_{2k_2}}} \times \dots \times \mathbb{Z}_{p_r^{t_{r1}}} \times \mathbb{Z}_{p_r^{t_{r2}}} \times \dots \times \mathbb{Z}_{p_r^{t_{rk_r}}},$$

where $k_i \geq 1$ and $1 \leq t_{i1} \leq t_{i2} \leq \dots \leq t_{ik_i}$, for all $i \in [r]$. Then

$$\kappa(\mathcal{P}(G)) \leq p_1^{t_{11}} p_2^{t_{21}} \cdots p_r^{t_{r1}} - \phi(p_1^{t_{11}} p_2^{t_{21}} \cdots p_r^{t_{r1}}).$$

Proof of Theorem 1.7 Proof of this theorem follows from Theorems 1.4 and 1.5. \square

3.1 The Number of Components of $\mathcal{G}_e^{**}(G)$ When it is Disconnected

So far, we have characterized the abelian groups for which the proper enhanced power graph is disconnected. In this context, the natural question that comes to our mind is the number of connected components of the subgraph $\mathcal{G}_e^{**}(G)$. By Theorem 1.5, the proper enhanced power graph $\mathcal{G}_e^{**}(G)$ for a finite abelian group G is disconnected when G is either a non-cyclic p -group or $G \cong G_1 \times \mathbb{Z}_n$ where G_1 is an p -group and $\gcd(p, n) = 1$. Here, we explicitly count the number of components for those G .

Theorem 3.1 *Let G be a finite abelian p -group. Suppose that*

$$G = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}},$$

where $r \geq 2$ and $t_1 \leq t_2 \leq \cdots \leq t_r$. Then, the number of components of $\mathcal{G}_e^{**}(G)$ is $\frac{p^r - 1}{p - 1}$.

Proof It is easy to show that there are $p^r - 1$ elements of order p . For any element a of order p , the $p - 1$ non-zero scalar multiples of a must be in the same component. Moreover, by Lemma 2.6, if any two elements of order p are connected by a path, then one of them must be the multiple of another. Henceforth, there are exactly $p - 1$ members of order p in any component. Thus, the number of connected components of $\mathcal{G}_e^{**}(G)$ is $\frac{p^r - 1}{p - 1}$. \square

It is quite interesting to note that the number of components of the proper enhanced power graph of a finite abelian non-cyclic p -group is independent of the exponent t_i 's. In the next result, we prove that this phenomenon is observed also in the case when $G \cong G_1 \times \mathbb{Z}_n$ where G_1 is an p -group and $\gcd(p, n) = 1$. Let $C(\mathcal{G}_e^{**}(G))$ be the set of connected components of the proper enhanced power graph $\mathcal{G}_e^{**}(G)$.

Theorem 3.2 *Let G be an abelian group such that $G \cong \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}} \times \mathbb{Z}_n$, where $r \geq 2$ and $\gcd(p, n) = 1$. Then, the number of components of $\mathcal{G}_e^{**}(G)$ is $\frac{p^r - 1}{p - 1}$.*

Proof Let, $G_1 = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}}$. By Theorem 3.1, the number of connected components of $\mathcal{G}_e^{**}(G_1)$ is $\frac{p^r - 1}{p - 1}$. Let, $C_1, C_2, \dots, C_{\frac{p^r - 1}{p - 1}}$ be the components of $C(\mathcal{G}_e^{**}(G_1))$. Define $f : C(\mathcal{G}_e^{**}(G_1)) \rightarrow C(\mathcal{G}_e^{**}(G_1 \times \mathbb{Z}_n))$ by

$$f(C_i) = C_i \times \mathbb{Z}_n.$$

At first, we show that there is no path in between $C_i \times \mathbb{Z}_n$ and $C_j \times \mathbb{Z}_n$ for $1 \leq i < j \leq \frac{p'-1}{p-1}$. Let, there exists an path between (a_1, b_1) and (a_2, b_2) where $a_1 \in C_i, a_2 \in C_j$ and $b_1, b_2 \in \mathbb{Z}_n$. If possible, let $(a_1, b_1) \sim (c_1, d_1) \sim (c_2, d_2) \sim \dots \sim (c_{m-1}, d_{m-1}) \sim (a_2, b_2)$ in $\mathcal{G}_e^{**}(G)$ where $c_1, c_2, \dots, c_{m-1} \in G_1$ and $d_1, d_2, \dots, d_{m-1} \in \mathbb{Z}_n$. Then c_1, c_2, \dots, c_{m-1} must be non-zero elements of G_1 . This proves that a_1 and a_2 are connected by a path in $\mathcal{G}_e^*(G_1)$ which contradicts the fact that C_i and C_j are distinct connected components of $\mathcal{G}_e^{**}(G_1)$. Therefore, the number of components of $\mathcal{G}_e^{**}(G)$ is at least $\frac{p'-1}{p-1}$.

Moreover, it is clear that the number of elements of order p in G is $\frac{p'-1}{p-1}$. Any element of order $> p$ is adjacent to an element of order p . Therefore, the number of components of $\mathcal{G}_e^{**}(G)$ should be exactly equal to $\frac{p'-1}{p-1}$. The proof is complete. \square

4 Vertex Connectivity of Some Non-abelian Groups

In this section, we discuss the vertex connectivity of some interesting classes of non-abelian groups. We start with the dihedral groups. We need the structures of these groups to determine the vertex connectivity. For $n \geq 2$, the dihedral group of order $2n$ is defined by the following presentation:

$$D_{2n} = \langle r, s : r^n = s^2 = e, rs = sr^{-1} \rangle.$$

We also consider the generalized quaternion groups Q_{2^n} . Let $x = \overline{(1, 0)}$ and $y = \overline{(0, 1)}$. Then $Q_{2^n} = \langle x, y \rangle$, where

- (1) x has order 2^{n-1} and y has order 4,
- (2) every element of Q_{2^n} can be written in the form x^a or $x^a y$ for some $a \in \mathbb{Z}$,
- (3) $x^{2^{n-2}} = y^2$,
- (4) for each $g \in Q_{2^n}$ such that $g \in \langle x \rangle$, such that $g x g^{-1} = x^{-1}$.

For more information about D_{2n} , and Q_{2^n} see [13, 16, 27].

Theorem 4.1 *Let G be the dihedral group of order $2n$. Then $\kappa(\mathcal{G}_e(G))$ is equal to 1. Moreover, the number of components of $\mathcal{G}_e^{**}(G)$ is $n + 1$.*

Proof Consider the following $n + 1$ sets:

$$S_1 = \{rs\}, S_2 = \{r^2s\}, \dots, S_{n-1} = \{r^{n-1}s\}, S_n = \{s\}, S_{n+1} = \{r, r^2, \dots, r^{n-1}\}.$$

We observe that $G \setminus \{e\} = \cup_{i=1}^{n+1} S_i$ and for $1 \leq i < j \leq n$. Moreover, the power of any element of S_i must be in S_i itself in $G \setminus \{e\}$. Therefore, there can be no edge between S_i and S_j for distinct i, j . This completes the proof. \square

Theorem 4.2 *For $n \geq 3$, let Q_{2^n} be the generalized quaternion group. Then the vertex connectivity of $\mathcal{G}_e(Q_{2^n})$ is 2. Moreover the number of components of $\mathcal{G}_e^{**}(Q_{2^n})$ is $2^{n-2} + 1$.*

Proof Q_{2^n} is generalized quaternion group, so Q_{2^n} is a 2-group and it has a unique minimal subgroup of order 2. Let $g \in Q_{2^n}$ such that $o(g) = 2$. Then by Lemma 2.2, g and e are adjacent to all other vertices in $\mathcal{G}_e(Q_{2^n})$. For this reason to disconnect the graph we have to delete the vertices e, g . Consequently, $\kappa(\mathcal{G}_e(Q_{2^n})) \geq 2$. Now we show that after removing the vertices e and g from $V(\mathcal{G}_e(Q_{2^n}))$, the graph $\mathcal{G}_e^{**}(Q_{2^n})$ will be disconnected. Let $x \in Q_{2^n}$ such that $o(x) = 2^{n-1}$ and $H = \langle x \rangle$. We will prove that there is no edge between the vertices in H and $Q_{2^n} \setminus H$. If for any $y \in Q_{2^n} \setminus H$ is adjacent to a vertex of H , then y should belong to H , (since H is the only subgroup of order 2^{n-1} and there is no other subgroup of order $> 2^{n-1}$) a contradiction. So the graph is disconnected. Moreover it is clear that there is no edge between the vertices in $\text{Gen}(y)$ and $\text{Gen}(z)$, where $y, z \in Q_{2^n} \setminus H$ such that $o(y) = 4 = o(z)$ and $\langle y \rangle \neq \langle z \rangle$. Hence the number of components in $\mathcal{G}_e^{**}(Q_{2^n})$ is $2^{n-2} + 1$ (as the number of 4-ordered element in $Q_{2^n} \setminus H$ is 2^{n-1} and $\phi(4) = 2$). \square

Corollary 3 *Let Q_{2^n} be the generalized quaternion group. Then the enhanced power graph $\mathcal{G}_e^*(Q_{2^n})$ is connected but the proper enhanced power graph $\mathcal{G}_e^{**}(Q_{2^n})$ is disconnected.*

We next consider the family of symmetric groups \mathfrak{S}_n . [16] is a good reference for this. Recall from Sect. 1 that $\mathcal{P}(G)$ denotes the power graph of G . Let, $\mathcal{P}^*(G)$ denote the power graph of G after deleting the identity. Doostabadi et al. [15] proved the following theorem on the vertex connectivity of power graphs [15].

Theorem 4.3 *Let $G = \mathfrak{S}_n$ be a symmetric group with $n \geq 3$. Then*

- (1) *If $n \geq 3$ and neither n nor $n - 1$ is a prime, then $\mathcal{P}^*(G)$ is connected.*
- (2) *If n is such that either n or $n - 1$ is a prime, then $\mathcal{P}^*(G)$ is disconnected.*

We prove an analogous result corresponding to the enhanced power graph of \mathfrak{S}_n . For this, we first prove that $\mathcal{G}_e(\mathfrak{S}_n)$ has no dominating vertex.

Lemma 4.4 *For $n \geq 3$, the enhanced power graph $\mathcal{G}_e(\mathfrak{S}_n)$ has no dominating vertex other than identity. Therefore, for $n \geq 3$, the graphs $\mathcal{G}_e^*(\mathfrak{S}_n)$ and $\mathcal{G}_e^{**}(\mathfrak{S}_n)$ coincide.*

Proof We first prove it when n is composite. Let $a \neq e$ be a dominating vertex. By Lemma 2.7, there exists a prime p dividing $o(a)$ such that G has a unique subgroup of order p . But we can take $C_1 = \langle (1, 2, \dots, p) \rangle$ and $C_2 = \langle (2, 3, \dots, p + 1) \rangle$ and arrive at a contradiction. We next consider the case when n is prime, say $n = p$. Let $a \neq e$ be a dominating vertex. Then, $a \sim \langle (1, 2, \dots, p) \rangle$. So, they are contained in a cyclic subgroup, say A of G . Now, there cannot be a subgroup of G which properly contains $\langle (1, 2, \dots, p) \rangle$. Hence, $a \in \langle (1, 2, \dots, p) \rangle$ and consequently $\langle a \rangle = \langle (1, 2, \dots, p) \rangle$. Again, since a is a dominating vertex, $a \sim (1, 2)$. Now, applying the similar argument as above, we see that $(1, 2) \in \langle a \rangle$, which is not possible. The proof is complete. \square

Theorem 4.5 *For positive integers $n \geq 3$, $\kappa(\mathcal{G}_e(\mathfrak{S}_n)) =$ if and only if either n or $n - 1$ is prime.*

Proof By Lemma 4.4, we have $\mathcal{G}_e^*(\mathfrak{S}_n) = \mathcal{G}_e^{**}(\mathfrak{S}_n)$. So, by Theorem 2.8, $\mathcal{G}_e^{**}(\mathfrak{S}_n)$ is connected if and only if $\mathcal{P}^*(\mathfrak{S}_n)$ is connected. Now, the proof is complete using Theorem 4.3. \square

Let, $\mathcal{A}_n \subseteq \mathfrak{S}_n$ be the alternating group. The family of alternating groups is an interesting subgroup of the set of even permutations in \mathfrak{S}_n . Doostabadi et al. proved the following theorem on the vertex connectivity of power graphs [15].

Theorem 4.6 *Let $G = \mathcal{A}_n$ be the alternating group and $n \geq 4$. Then*

- (1) If $n, n-1, n-2, n/2, (n-1)/2, (n-2)/2$ are not primes, then $\mathcal{P}^*(G)$ is connected.
- (2) If n is such that any one of $n, n-1, n-2, n/2, (n-1)/2, (n-2)/2$ is prime, then $\mathcal{P}^*(G)$ is not connected.

We start with showing that $\mathcal{G}_e(\mathcal{A}_n)$ has no dominating vertex.

Lemma 4.7 *For $n \geq 4$, the enhanced power graph $\mathcal{G}_e(\mathcal{A}_n)$ has no dominating vertex other than identity. Therefore, for $n \geq 4$, the graphs $\mathcal{G}_e^*(\mathcal{A}_n)$ and $\mathcal{G}_e^{**}(\mathcal{A}_n)$ coincide.*

Proof Let, $a \neq e$ be a dominating vertex. By Lemma 2.7, there exists a prime p such that G has a unique subgroup of order p . If $p \geq 4$, take $C_1 = \langle (1, 2, 3, \dots, p) \rangle$ and $C_2 = \langle (1, 3, 2, \dots, p) \rangle$. If $p = 3$, take $C_1 = \langle (1, 2, 3) \rangle$ and $C_2 = \langle (2, 3, 4) \rangle$. When $p = 2$, let $C_1 = \langle (1, 2)(3, 4) \rangle$ and $C_2 = \langle (1, 3)(2, 4) \rangle$. Thus, in each case, we can verify that both C_1 and C_2 are even permutations and this contradicts Lemma 2.7. This completes the proof. \square

From Theorem 4.6 and Lemma 4.7, we prove the following theorem.

Theorem 4.8 *For positive integers $n \geq 7$, $\kappa(\mathcal{G}_e(\mathcal{A}_n)) = 1$ if and only if one of $n, n-1, n-2, n/2, (n-1)/2, (n-2)/2$ is prime.*

Proof By Lemma 4.7, we have $\mathcal{G}_e^*(\mathcal{A}_n) = \mathcal{G}_e^{**}(\mathcal{A}_n)$. Now, we are done using Theorems 2.8 and 4.6. \square

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