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ORIGINAL PAPER

Sub-Ramsey Numbers for Matchings

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Abstract
Given a graph G and a positive integer k, the sub-Ramsey number $sr(G, k)$ is defined to be the minimum number m such that every K_m whose edges are colored using every color at most k times contains a subgraph isomorphic to G all of whose edges have distinct colors. In this paper, we will concentrate on $sr(nK_2, k)$ with nK_2 denoting a matching of size n. We first give upper and lower bounds for $sr(nK_2, k)$ and exact values of $sr(nK_2, k)$ for some n and k. Afterwards, we show that $sr(nK_2, k) = 2n$ when n is sufficiently large and $k < \frac{n}{8}$ by applying the Local Lemma.

Keywords Sub-Ramsey number · Rainbow matchings · Local lemma

1 Introduction

All graphs considered in this paper are simple and finite. For terminology and notation not defined here, we refer the reader to [5].

Let G be a graph and k a positive integer. An edge-coloring of G is a mapping $C : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We call C *k-bounded* if each color appears at most k times. If $k = 1$, i.e., all edges of G have distinct colors, then we say that G is rainbow. The *sub-Ramsey number* $sr(G, k)$ is defined to be the minimum number m such that every k-bounded edge-coloring of K_m contains a rainbow subgraph isomorphic to G.

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The study of sub-Ramsey theory dates back to 1975 when Fred Galvin posed the Advanced Problem 6034 in the American Mathematical Monthly [12]. The problem description is as follows:

Suppose that the edges of the complete graph on n vertices are colored so that no color appears more than k times. (1) If $n \geq k+2$, show that there is a rainbow triangle. (2) Show that this is not necessarily so if $n = k+1$.

Galvin [12] gave a solution to this problem. After that, some related generalizations began to emerge. In Ref. [3] the authors gave the definition of the sub-Ramsey number and proved that $sr(G,k) \leq r(G,k)$ (the definition of $r(G, k)$ will be given later), and hence each $sr(G, k)$ is guaranteed to be finite. So far the sub-Ramsey number has been considered for several special graph classes-complete graphs, paths, cycles and stars. Denote by K_n, P_n and C_n the complete graph, path and cycle of n vertices respectively, and $K_{1,n}$ the star with $n+1$ vertices. For complete graphs, Alspach et al. [3] proved that

$$k(n-1)+1 \leq sr(K_n,k) \leq \frac{n(n-1)(n-2)(k-1)+3}{4},$$

where $n \geq 4$ and $k \geq 2$. After that Hell et al. [16] showed that

$$cn^{3/2} \leq sr(K_n,k) \leq (2n-3)(n-2)(k-1)+3$$

for some constant c, which improved the results of Alspach et al. For paths and cycles, Hahn and Thomassen proved that $sr(P_n,k) = sr(C_n,k) = n$ when $n \geq ck^3$ in Ref. [15]. Albert et al. [1] later showed that if n is sufficiently large and $k \leq cn$ for $c < \frac{1}{32}$, then $sr(C_n,k) = n$. The sub-Ramsey numbers of stars have not been completely solved so far. The authors of Refs. [9, 13, 14] gave upper and lower bounds for $sr(K_{1,n},k)$ and some exact values of $sr(K_{1,n},k)$ when n or k is fixed. In addition to this, sub-Ramsey numbers for arithmetic progressions are also studied [2, 4]. In this paper, we will concentrate on the sub-Ramsey number of nK_2 which is a matching of n independent edges.

First, according to the definition of $sr(nK_2,k)$, we can get the following simple results:

1. $sr(nK_2,k) \geq 2n$.
   Note that K_m contains no nK_2 for $m < 2n$.
2. $sr(nK_2,k) \geq \left\lfloor \frac{3+\sqrt{1+8k(n-1)}}{2} \right\rfloor$.

   Since K_m can be colored with at most $n-1$ colors when $\binom{m}{2} \leq k(n-1)$, we have

$$sr(nK_2,k) \geq \left\lfloor \frac{1+\sqrt{1+8k(n-1)}}{2} \right\rfloor + 1 \geq \left\lfloor \frac{3+\sqrt{1+8k(n-1)}}{2} \right\rfloor.$$

3. $sr(nK_2,1) = 2n$.

When $m = 2n$ and $k = 1$, K_m must contain a rainbow nK_2, and hence $sr(nK_2, 1) \leq 2n$. Since $sr(nK_2, 1) \geq 2n$, we can get $sr(nK_2, 1) = 2n$. Therefore, we assume $k \geq 2$ in the rest of the paper.

Let $r(G, k)$ denote the Ramsey number for graph G, that is, $r(G, k)$ is the minimum number m such that if the edges of K_m are partitioned into at most k classes then some class contains a subgraph isomorphic to G. The following proposition shows the relationship between $sr(G, k)$ and $r(G, k)$.

Proposition 1 *(Alspach et al. [3]) Let G be a given graph and k a positive integer. Then* $sr(G, k) \leq r(G, k)$.

So knowing what is the value of $r(nK_2, k)$ is necessary to solve for $sr(nK_2, k)$. In 1975, the Ramsey numbers for matchings are given by Cockayne and Lorimer [7].

Theorem 1 (*Cockayne and Lorimer* [7]) *If* $n_1, n_2, \ldots, n_c$ *are positive integers and* $n_1 = max(n_1, n_2, \ldots, n_c)$, *then* $r(n_1K_2, n_2K_2, \ldots, n_cK_2) = n_1 + 1 + \sum_{i=1}^{c}(n_i - 1)$.

From this theorem we can know that if $c = k$ and $n_1 = n_2 = \cdots = n_k = n$ then $r(nK_2, k) = n(k+1) + 1 - k$, so $sr(nK_2, k) \leq r(nK_2, k) = n(k+1) + 1 - k$. Our following result improves the upper bound of $sr(nK_2, k)$ from $O(nk)$ to $O(nk^{\frac{1}{2}})$.

Theorem 2 *Let* $n \geq 3$ *and* $k \geq 5$ *be two integers. Then*

$$sr(nK_2, k) \leq \left\lfloor \frac{5 + \sqrt{1 + 4n(n-1)(k-1)}}{2} \right\rfloor.$$

It is easy to verify that the upper bound of Theorem 2 is also true for $sr(3K_2, 3)$ and $sr(nK_2, 4)$ with $n \in \{3, 4, 5, 6\}$. Our next theorems give some exact values of $sr(nK_2, k)$ when n or k is fixed. It is clear that $sr(K_2, k) = 2$ so that $n \geq 2$ is assumed.

Theorem 3 $sr(2K_2, k) = \max\left\{5, \left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor \right\}$.

Theorem 4

(1) $sr(nK_2, 2) = 2n$ *for all* $n \geq 3$;
(2) $sr(nK_2, 3) = 2n$ *for all* $n \geq 3$;
(3) $sr(nK_2, 4) = 2n$ *for all* $n \geq 3$;
(4) $2n \leq sr(nK_2, 5) \leq 2n + 1$.

Theorem 4 shows that $sr(nK_2, 2) = sr(nK_2, 3) = sr(nK_2, 4) = 2n$ attains the lower bound $2n$ and the upper bound of Theorem 2 for some n. A case is now considered which shows that $sr(3K_2, 5) = 7$. It is probably the case that the upper bound of Theorem 2 is closer to the truth than the lower bound $2n$ for some k.

Proposition 2 $sr(3K_2, 5) = 7$.

It became clear that $sr(nK_2, k) \leq sr(nK_2, k+1)$. Judging from these results, the gap between the lower bound $2n$ and the upper bound of Theorem 2 is not large. A question mentioned in Erdős et al. [8] is that of how fast can we allow k to grow and guarantee that there exists a rainbow Hamilton cycle in a k-bounded edge-colored K_n. After that, this question was considered by many researchers. Albert et al. [1] proved the following theorem, which showed that the growth rate of k can be linear.

Theorem 5 (*Albert et al.* [1]) *If n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < \frac{1}{32}$, then any k-bounded coloring of K_n contains a rainbow Hamilton cycle.*

For other results related to the above question, we recommend papers [10, 11, 15]. From Theorem 5 we can easily conclude that if n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < \frac{1}{16}$, then any k-bounded coloring of K_{2n} contains a rainbow nK_2. Here, we show that $c < \frac{1}{8}$ is enough by giving the following theorem.

Theorem 6 *If n is sufficiently large and $k < \frac{n}{8}$, then $sr(nK_2, k) = 2n$.*

We give the proofs of our results in the rest of this paper. Before that, we need to introduce a definition. An (H, k)-colored graph G is a k-bounded edge-colored G so that no $H \subseteq G$ is rainbow. We use $C(G)$ to denote the set of colors appearing on the edges of G.

2 Proof of Theorem 2

Let K_m be (nK_2, k)-colored and let the colors used be $c_1, c_2, \ldots, c_p$. Denote by m_i the number of edges colored c_i, where $i = 1, 2, \ldots, p$, then $m_i \le k$ and

$$\sum_{i=1}^{p} m_i = \binom{m}{2} = \frac{m(m-1)}{2}.$$

Let W denote the number of unordered pairs of disjoint edges of K_m colored by the same color. Clearly,

$$W \le \sum_{i=1}^{p} \binom{m_i}{2} = \sum_{i=1}^{p} m_i \frac{m_i - 1}{2} \le \sum_{i=1}^{p} m_i \frac{k-1}{2} = \binom{m}{2} \frac{k-1}{2}. \tag{1}$$

Let $\mathscr{G}$ denote the set of all subgraphs of K_m isomorphic to nK_2. Each unordered pairs of disjoint edges colored by the same color appears in

$$\frac{\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-(2n-2)}{2}}{(n-2)!}$$

graphs in $\mathscr{G}$. But K_m contains

$$\frac{\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\cdots\binom{m-(2n-2)}{2}}{n!}$$

nK_2's each of which must contain a pair of edges of the same color by K_m be (nK_2, k)-colored. Thus,

$$W\frac{\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-(2n-2)}{2}}{(n-2)!} \geq \frac{\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\cdots\binom{m-(2n-2)}{2}}{n!}.$$

By inequality (1), we have

$$\binom{m}{2}\frac{k-1}{2}\frac{\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-(2n-2)}{2}}{(n-2)!} \geq \frac{\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\cdots\binom{m-(2n-2)}{2}}{n(n-1)(n-2)!}.$$

Finally,

$$\frac{k-1}{2} \geq \frac{\binom{m-2}{2}}{n(n-1)}.$$

This yields

$$m \leq \left\lfloor \frac{5+\sqrt{1+4n(n-1)(k-1)}}{2} \right\rfloor.$$

Since the upper bound must be greater than or equal to the lower bound, we assume that $k \geq 5$. Then we have

$$\left\lfloor \frac{5+\sqrt{1+4n(n-1)(k-1)}}{2} \right\rfloor \geq \left\lfloor \frac{5+\sqrt{1+16n(n-1)}}{2} \right\rfloor$$
$$\geq \left\lfloor \frac{5+\sqrt{16(n-1)(n-1)}}{2} \right\rfloor$$
$$= \left\lfloor 2n+\frac{1}{2} \right\rfloor$$
$$= 2n.$$

Hence the theorem holds.

3 Proof of Theorem 3

Consider the complete graph K_4 on the vertex set $\{v_1, v_2, v_3, v_4\}$. Clearly, K_4 contains

$$\frac{\binom{4}{2}\binom{2}{2}}{2} = 3$$

matchings of size 2. Color each of the three matchings with one color (see Fig. 1).

If K_4 is edge-colored with 3 colors such that $C(v_1v_2) = C(v_3v_4) = 1$, $C(v_1v_3) = C(v_2v_4) = 2$ and $C(v_1v_4) = C(v_2v_3) = 3$, then it contains no rainbow $2K_2$. So, $sr(2K_2, k) \geq 5$. Note that we already showed that $sr(2K_2, k) \geq \left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor$. Now, we proceed by proving the following lemma which also appears in Ref. [6].

Lemma 1 *Any edge-coloring of K_m $(m \geq 5)$ with at least 2 colors contains a rainbow subgraph isomorphic to $2K_2$.*

Proof For $m \geq 5$, let the edges of K_m be colored with at least 2 colors. Suppose that K_m contains no rainbow $2K_2$. Let $e_1 = v_1v_2$ be an arbitrary edge of K_m. Assume that $C(e_1) = 1$ and $U = V(K_m) - \{v_1, v_2\}$. Then $C(e) = 1$ for all edges $e \in E(K_m[U])$. Moreover, $C(e) = 1$ for all edges $e \in [\{v_1, v_2\}, U]$, since$|U| \geq 3$, where $[\{v_1, v_2\}, U]$ is the set of edges between $\{v_1, v_2\}$ and U. Then K_m is monochromatic, a contradiction. □

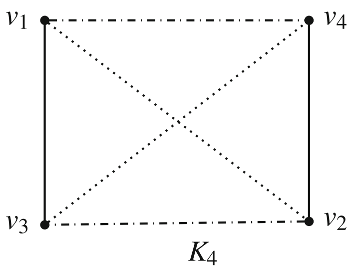


Fig. 1 2-bounded edge-coloring of K_4 contains no rainbow $2K_2$

If $\binom{m}{2} > k$, which yields $m \geq \left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor$, then, clearly, K_m is colored such that at least two colors are used. By Lemma 1, K_m contains a rainbow $2K_2$. Therefore, $sr(2K_2, k) \leq \left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor$. In conclusion, we have this theorem.

4 Proof of Theorem 4

The lower bounds are obvious. For the upper bounds of (1)–(3) of Theorem 4, we prove them by contradiction. Before giving the proofs, we first make some assumptions. Suppose that $m = 2n$ and K_m is (nK_2, k)-colored where $k \in \{2, 3, 4\}$. Let $C(E(K_m)) = \{c_1, c_2, \ldots, c_p\}$ and denote by e_i the number of edges colored c_i. Then $e_i \leq k$ and

$$\sum_{i=1}^{p} e_i = \binom{2n}{2} = \frac{2n(2n-1)}{2} = 2n^2 - n \leq kp.$$

Let $\mathscr{G}$ denote the set of all subgraphs of K_m isomorphic to nK_2, s denote the number of unordered pairs of disjoint edges of K_m colored by the same color. Clearly,

$$s \leq \sum_{i=1}^{p} \binom{e_i}{2} = \sum_{i=1}^{p} e_i \frac{e_i - 1}{2} \leq \sum_{i=1}^{p} e_i \frac{k-1}{2} = \frac{(2n^2 - n)(k-1)}{2}, \tag{2}$$

and each pair of edges of the same color appears in

$$\frac{\binom{2n-4}{2}\binom{2n-6}{2}\cdots\binom{2}{2}}{(n-2)!}$$

graphs in $\mathscr{G}$ if the edges are vertex-disjoint. Note that K_m contains

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

subgraphs isomorphic to nK_2 each of which must contain a pair of edges of the same color. Thus,

$$s \frac{\binom{2n-4}{2}\binom{2n-6}{2}\cdots\binom{2}{2}}{(n-2)!} \geq \frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}. \tag{3}$$

Then we have

(1) When $k = 2$, it follows that $s \leq \frac{2n^2-n}{2}$ from inequality (2), and hence

$$\frac{2n^2 - n}{2} \geq \frac{\binom{2n}{2}\binom{2n-2}{2}}{n(n-1)}$$

by inequality (3). This yields $\frac{1}{2} \leq n \leq 2$. But $n \geq 3$, a contradiction.

(2) When $k = 3$, $s \leq 2n^2 - n$ can be obtained from inequality (2). Hence, we have

$$2n^2 - n \geq \frac{\binom{2n}{2}\binom{2n-2}{2}}{n(n-1)}$$

by inequality (3). This yields $\frac{1}{2} \leq n \leq 3$. Recall that $n \geq 3$. We have $n = 3$. Then, by Theorem 2,

$$sr(3K_2, 3) \leq \left\lfloor \frac{5 + \sqrt{1 + 4 \times 3 \times 2 \times 2}}{2} \right\rfloor = 6.$$

(3) Similarly, we know that $s \leq \frac{3(2n^2-n)}{2}$ when $k = 4$ by inequality (2). Then, we have

$$\frac{3(2n^2 - n)}{2} \geq \frac{\binom{2n}{2}\binom{2n-2}{2}}{n(n-1)}$$

by inequality (3). This yields $\frac{1}{2} \leq n \leq 6$. Recall that $n \geq 3$. We have $n \in \{3, 4, 5, 6\}$. Then, by Theorem 2,

$$sr(nK_2, 4) \leq \left\lfloor \frac{5 + \sqrt{1 + 4 \times n \times (n-1) \times 3}}{2} \right\rfloor \leq 2n.$$

(4) By Theorem 2,

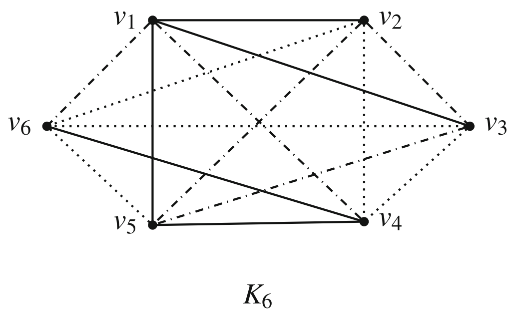
$$sr(nK_2, 5) < \frac{5 + \sqrt{1 + 4n(n-1)(5-1)}}{2} < 2n + 2.$$

So $2n \leq sr(nK_2, 5) \leq 2n + 1$.

5 Proof of Proposition 2

By Theorem 4 (4), we have $sr(3K_2, 5) \leq 7$. Now, consider K_6 on the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Color the edges $\{v_2v_4, v_2v_6, v_3v_4, v_3v_6, v_5v_6\}$ with red color and the edges $\{v_1v_2, v_1v_3, v_1v_5, v_4v_5, v_4v_6\}$ with blue color and the edges $\{v_1v_4, v_1v_6, v_2v_3, v_2v_5, v_3v_5\}$ with green color. It is routine to verify that any completion of this coloring to a 5-bounded edge-coloring will result in a $(3K_2, 5)$-colored graph. Thus, $sr(3K_2, 5) \geq 7$ (see Fig. 2). This proposition holds.

Fig. 2 5-bounded edge-coloring of K_6 contains no rainbow $3K_2$



6 Proof of Theorem 6

It is clear that $sr(nK_2, k) \geq 2n$. To prove that $sr(nK_2, k) \leq 2n$, we just have to show that if n is sufficiently large and $k < \frac{n}{8}$, then any k-bounded edge-coloring of K_{2n} contains a rainbow perfect matching. This proof technique follows the lines of modification of the Local Lemma listed by Albert et al. in Ref. [1].

Lemma 2 (*Albert et al.* [1]) *Let $A_1, A_2, \ldots, A_N$ denote events in some probability space. Suppose that for each i there is a partition of $[N] \backslash \{i\}$ into X_i and Y_i. Let $m = \max\{|Y_i| : i \in [N]\}$ and $\beta = \max\{Pr(A_i | \bigcap_{j \in X} \overline{A_j}) : i \in [N], X \subseteq X_i\}$. If there exists $0 \leq \alpha < 1$ such that $\alpha(1 - m\alpha) \geq \beta$ then $Pr(\bigcap_{i=1}^{N} \overline{A_i}) > 0$.*

Let $K = K_{2n}$ be a k-bounded edge-colored complete graph satisfying n is sufficiently large and $k < \frac{n}{8}$. Now, construct a graph G whose vertex set is the edge set of K and two edges e, f of K correspond to the two end-vertices of an edge of G if and only if $C(e) = C(f)$. Thus a set of vertices of G is independent if and only if it corresponds to a rainbow set of edges of K. Then we only need to prove that K contains a perfect matching whose edge set is an independent vertex set in G.

Let H be a perfect matching chosen randomly and independently from the set of $\frac{(2n)!}{2^n n!}$ perfect matchings of K. Let $\{e_i f_i : 1 \leq i \leq N\}$ be the edge set of G and

$$A_i = \{H : e_i, f_i \in E(H)\}.$$

We will show that $Pr(\bigcap_{i=1}^{N} \overline{A_i}) > 0$ to prove the conclusion we want. For $1 \leq i \leq N$, let

$$Y_i = \{j \neq i : \text{one of } e_j, f_j \text{ shares a vertex with one of } e_i, f_i\}$$

and $X_i = [N] \backslash (Y_i \cup \{i\})$. Obviously, $Y_i \leq 8nk$, so $m \leq 8nk$. Let $X \subseteq X_i$, then no edge in X shares an end-vertex with either e_i or f_i. Let $e_i = u_1 u_2$, $f_i = v_1 v_2$ and M be a perfect matching containing both e_i and f_i but no edges of X. Consider two edges $a = a_1 a_2$ and $b = b_1 b_2$ of M. There are at least $(n-2)(n-3) \cdot 4$ choices for a, b. For each such M, we construct $M_{a,b}$ by removing the edges e_i, f_i, a, b from M and adding 4 edges $u_1 a_1$, $u_2 a_2$, $v_1 b_1$, $v_2 b_2$ which does not contain an edge of X. Then, taking $F(M) = \{M_{a,b} : a, b \text{ as above}\}$, we have $|F(M)| \geq 4(n-2)(n-3)$ and

$F(M) \cap F(M') = \emptyset$ for $M \neq M'$. Let $\mathscr{M}$ denote the set of perfect matchings containing e_i and f_i. Then, we have

$$Pr\left(A_i | \bigcap_{j \in X} \overline{A_j}\right) = \sum_{M \in \mathscr{M}} Pr\left(H = M | \bigcap_{j \in X} \overline{A_j}\right)$$
$$\leq \frac{1}{4n^2 - 20n + 25} \sum_{M \in \mathscr{M}} Pr\left(H \in \{M\} \cup F(M) | \bigcap_{j \in X} \overline{A_j}\right)$$
$$\leq \frac{1}{4n^2 - 20n + 25}.$$

Thus,

$$\beta \leq \frac{1}{4n^2 - 20n + 25}.$$

We choose

$$\alpha = \frac{1 - \sqrt{1 - (8 + \epsilon)k/n}}{16nk},$$

then

$$\alpha(1 - m\alpha) \geq \frac{8 + \epsilon}{32n^2} = \frac{1 + \epsilon/8}{4n^2}.$$

Thus $\alpha(1 - m\alpha) \geq \beta$ when $\epsilon > 0$ is sufficiently small and n is sufficiently large which means $Pr(\bigcap_{i=1}^{N} \overline{A_i}) > 0$ by Lemma 2. Therefore, we can find a rainbow perfect matching in K. This completes the proof of Theorem 6.

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