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ORIGINAL PAPER

Stirling Pairs of Permutations

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Abstract
Stirling permutations are permutations π of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ in which those integers between the two occurrences of an integer are greater than it. We identify a permutation π of $\{1, 1, 2, 2, \ldots, n, n\}$ as a pair of permutations (π_1, π_2) which we call a Stirling pair. We characterize Stirling pairs using the weak Bruhat order and the notion of a 312-avoiding permutation. We give two algorithms to determine if a pair of permutations is a Stirling pair.

Keywords Permutation · Inversion · Weak Bruhat order · 312-Avoiding permutation

Mathematics Subject Classification 05A05

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$, and let $\mathcal{T}_n$ denote the *set of* 2*-permutations of* $\{1, 2, \ldots, n\}$, that is, the set of permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. A *Stirling permutation of order n and length $2n$* is a 2-permutation in $\mathcal{T}_n$ such that the following *Stirling property* holds:

($*$) For each $k = 1, 2, \ldots, n$, the integers between the two occurrences of k are greater than k.

In particular, in a Stirling permutation of order n, the two n's must be adjacent. An example of a Stirling permutation of length 12 is

$$\pi = (1, 2, 2, 4, 5, 6, 6, 5, 4, 3, 3, 1). \tag{1}$$

The set of Stirling permutations in $\mathcal{T}_n$, that is, the *set of Stirling permutations of order n and length $2n$*, is denoted by $\mathcal{Q}_n$. Deleting the two n's in a Stirling permutation of order n, we are left with a Stirling permutation of order $n - 1$. Given a Stirling permutation of order $n-1$, there are $2n-1$ places to insert the n's, and hence it follows

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by induction that the number of Stirling permutations of length $2n$ is $(2n-1)!!$ where *double factorial* $(2n-1)!!$ equals $(2n-1)(2n-3)(2n-5)\cdots 1$ [2]. The number of Stirling permutations of length 2 equals 1, the number of Stirling permutations of length 4 equals $3!! = 3$, and the number of Stirling permutations of length 6 equals $5!! = 5 \cdot 3 \cdot 1 = 15$. Stirling permutations were introduced by Gessel and Stanley [4] in the context of Stirling polynomials. A number of recent papers [3,5–10] have investigated enumerative and other properties of Stirling permutations.

Our purpose in this paper is to investigate Stirling permutations considered as a pair of permutations of $\{1, 2, \ldots, n\}$ and to discuss some of the consequences of this viewpoint. Each 2-permutation $\pi \in \mathcal{T}_n$ determines the unique pair of permutations (π_1, π_2) of $\{1, 2, \ldots, n\}$ obtained, respectively, from the first, π_1, and second, π_2, occurrences (from left to right) of the integers $\{1, 2, \ldots, n\}$. We say that π *induces* π_1 and π_2, and we write $\pi \to (\pi_1, \pi_2)$. Thus we have a mapping

$$\mathcal{T}_n \to \mathcal{S}_n \times \mathcal{S}_n$$

from the set of 2-permutations of $\{1, 2, \ldots, n\}$ to pairs of permutations of $\{1, 2, \ldots, n\}$. This mapping is surjective but not injective. For example, we have both

$$(1, 1, 2, 2, 3, 3) \to ((1, 2, 3), (1, 2, 3)) \quad \text{and} \quad (1, 2, 3, 1, 2, 3) \to ((1, 2, 3), (1, 2, 3)).$$

If π is a Stirling permutation of order n and $\pi \to (\pi_1, \pi_2)$, then we call (π_1, π_2) a *Stirling permutation pair of order n* corresponding to the Stirling permutation π. By restricting to $\mathcal{Q}_n$, we obtain a mapping

$$\mathcal{Q}_n \to \mathcal{S}_n \times \mathcal{S}_n. \tag{2}$$

This mapping (2) is not surjective for $n \geq 3$. For instance, let $\pi_1 = (1, 2, 3)$ and $\pi_2 = (3, 1, 2)$ be permutations in $\mathcal{S}_n$. Since in a 2-permutation of order 3 with $\pi \to (\pi_1, \pi_2)$, 3 is repeated first followed by the repeat of 1, then that of 2. Thus the only permutation π of $\{1, 1, 2, 2, 3, 3\}$ such that $\pi \to (\pi_1, \pi_2)$ is $(1, 2, 3, 3, 1, 2)$, and this is not a Stirling permutation since there is a 1 between the two 2's.

We denote the *set of Stirling permutation pairs of order n* by $\mathcal{S}_n^{\times 2}$ and we now write

$$\phi_n : \mathcal{Q}_n \to \mathcal{S}_n^{\times 2} \tag{3}$$

to denote the surjective *Stirling mapping* ϕ_n from the set $\mathcal{Q}_n$ of Stirling permutations of order n to the set $\mathcal{S}_n^{\times 2}$ of Stirling permutation pairs of order n. Later we shall observe that ϕ_n is actually a bijection. From the example π in (1) we get the Stirling permutation pair (π_1, π_2) as follows:

$$\phi_6(\pi) = (\pi_1, \pi_2) \text{ where } \pi_1 = (1, 2, 4, 5, 6, 3) \quad \text{and} \quad \pi_2 = (2, 6, 5, 4, 3, 1). \tag{4}$$

By way of explanation, we note that e.g. the 4 in position 3 of π_1 implies that the first occurrence of 4 occurs after the first occurrences of 1 and 2 and before the first occurrences of 5, 6, and 3; the 5 in position 3 in π_2 implies that that the second occurrence of 5 occurs after the second occurrences of 2 and 6 and before the second occurrences of 4, 3, and 1.

Given any permutation $\pi_1 = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, then (π_1, π_1) is a Stirling permutation pair with corresponding Stirling permutation $(i_1, i_1, i_2, i_2, \ldots, i_n, i_n)$. Thus every permutation of $\{1, 2, \ldots, n\}$ occurs as the first permutation of a Stirling permutation pair and, likewise, as the second permutation. The Stirling permutation $(1, 1, 2, 2, \ldots, n, n)$ is called the *identity Stirling permutation* of order n, and we have $\phi_n : (1, 1, 2, 2, \ldots, n, n) \to (\iota_n, \iota_n)$ where ι_n is the *identity permutation* $(1, 2, \ldots, n)$ of order n. For the Stirling permutation $(1, 2, \ldots, n, n, \ldots, 2, 1)$ we have $\phi_n : (1, 2, \ldots, n, n, \ldots, 2, 1) \to (\iota_n, \iota_n^*)$ where ι_n^* is the *anti-identity permutation* $(n, n-1, \ldots, 1)$.

A permutation pair (π_1, π_2) of order n, obtained as the first and second occurrences of any 2-permutation $\pi \in \mathcal{T}_n$, determines an $n \times n$ $(0, 1, 2)$-matrix $E_\pi = E_{(\pi_1, \pi_2)} := P_{\pi_1} + P_{\pi_2}$ where P_{π_k} equals the $n \times n$ permutation matrix corresponding to the permutation π_k $(k = 1, 2)$. All row and column sums of E_π equal 2. We call a $(0, 1, 2)$-matrix with all row and column sums equal to 2 a 2-*permutation matrix*, and we use *Stirling permutation matrix* if it arises in this way from a Stirling permutation.[1] Thus from example (4) we get the following Stirling permutation matrix E_π without any 2's:

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & 1 & & & \end{bmatrix} + \begin{bmatrix} & 1 & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ 1 & & & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & & & & 1 \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & 1 & & & 1 \\ 1 & & 1 & & & \end{bmatrix}. \tag{5}$$

Here and throughout, empty cells in a matrix are assumed to contain 0's. For the identity Stirling permutation of order n we have $P_{\pi_1} = P_{\pi_2} = I_n$ and hence $E_\pi = 2I_n$. The Stirling permutation matrix E_π in (5) can also be written as a sum of two other permutation matrices, namely,

$$E_\pi = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \\ & & 1 & & & \end{bmatrix} + \begin{bmatrix} & 1 & & & & \\ & & & & & 1 \\ & & & 1 & & \\ & & & & 1 & \\ & & 1 & & & \\ 1 & & & & & \end{bmatrix}$$

corresponding to permutations $\pi_1' = (1, 2, 5, 4, 6, 3)$ and $\pi_2' = (2, 6, 4, 5, 3, 1)$ where, however, (π_1', π_2') is not a Stirling permutation pair. The reason is that in a 2-permutation π' such that $\phi_6 : \pi' \to (\pi_1', \pi_2')$, 4 will occur between the two 5's.

[1] We note that our terminology is consistent in that just as a Stirling permutation is a 2-permutation, a Stirling permutation matrix is a 2-permutation matrix.

As we shall see later, given a Stirling permutation matrix E_π, there may be many ways to write E as a sum of permutation matrices but there is a unique Stirling pair (π_1, π_2) so that $E_\pi = P_{\pi_1} + P_{\pi_2}$.

Before summarizing the contents of this paper we consider the following example.

Example 1.1 Consider the Stirling permutation $\pi = (1, 2, 2, 4, 5, 6, 6, 5, 4, 3, 3, 1)$ of order 6 with corresponding Stirling permutation pair

$$\pi_1 = (1, 2, 4, 5, 6, 3) \quad \text{and} \quad \pi_2 = (2, 6, 5, 4, 3, 1).$$

The following four 2-permutations of $\{1, 2, 3, 4, 5, 6\}$ have π_1 and π_2 as the permutations corresponding to the first and second occurrences of the integers $1, 2, 3, 4, 5, 6$:

$$(1, 2, 2, 4, 5, 6, 6, 5, 4, 3, 3, 1), (1, 2, 2, 4, 5, 6, 6, 5, 3, 4, 3, 1),$$

$$(1, 2, 2, 4, 5, 6, 6, 3, 5, 4, 3, 1), (1, 2, 2, 4, 5, 6, 6, 5, 3, 4, 3, 1).$$

As is easily verified, only the first, namely the given π, is a Stirling permutation. □

We now summarize the contents of this paper. In the next section we give some elementary properties of Stirling permutations and Stirling permutation pairs, and we give two algorithms for constructing a Stirling permutation from a Stirling permutation pair. We show, in particular, that the mapping $\phi_n : \mathcal{Q}_n \to \mathcal{S}_n^{\times 2}$ from Stirling permutations to Stirling permutation pairs in (3) is a bijection. As a consequence it follows that a Stirling permutation matrix can be uniquely decomposed into a sum of permutation matrices corresponding to a Stirling permutation pair. In Sect. 3 we review some elementary properties of permutations including the weak Bruhat order and discuss their connection to Stirling permutation pairs. In the final section we make some additional comments and discuss some questions currently being considered.

2 Stirling Permutation Pairs

To familiarize the reader with Stirling permutations and Stirling permutation pairs, in the next lemma we collect some elementary properties of Stirling permutations and Stirling permutation pairs.

We recall that a permutation $\sigma = (j_1, j_2, \ldots, j_n)$ of $\{1, 2, \ldots, n\}$ is a 312-*avoiding permutation*[2] provided that there does not exist $1 \le k < l < p \le n$ such that $j_k > j_l$, $j_k > j_p$ and $j_p > j_l$. It is known (see [2, pages 150–151]) that the number of 312-avoiding permutations of $\{1, 2, \ldots, n\}$ equals the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. As a permutation matrix, a 312-avoiding permutation is one that does not have any 3×3 submatrix equal to

[2] The symbolic "312" can also be thought of as the out-of-order "LSM" as in L(arge), S(mall), and M(edium).

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Lemma 2.1 *Let n be a positive integer, and let π be a Stirling permutation of order n with corresponding Stirling permutation pair $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$. Then the following properties hold:*

(a) *For any $1 \leq q < p \leq n$, the subsequence of π determined by p and q is not equal to any of*

$$p, q, p, q; \quad p, q, q, p; \quad q, p, q, p$$

and so equals one of

$$p, p, q, q; \quad q, q, p, p; \quad q, p, p, q.$$

(b) *Let $\pi_1 = (1, 2, \ldots, n)$. Then π_2 is a 312-avoiding permutation.*
(c) *Let $\pi_1 = (1, 2, \ldots, n)$ and suppose that π_2 begins with n. Then $\pi_2 = (n, n-1, \ldots, 1)$.*
(d) *Let $i_k = n$ and $j_l = n$. Then $k \geq l$ and $\{j_1, j_2, \ldots, j_l\} \subseteq \{i_1, i_2, \ldots, i_k\}$. More generally, let $i_k = p$ and $j_l = p$. If p immediately follows p in π, then $\{j_1, j_2, \ldots, j_l\} \subseteq \{i_1, i_2, \ldots, i_p\}$.*
(e) *For any sequence σ let $\overleftarrow{\sigma}$ be the sequence obtained by reversing the order of the terms of σ. Let π be a Stirling permutation of order n with corresponding Stirling pair (π_1, π_2).Then $\overleftarrow{\sigma}$ is a Stirling permutation with corresponding Stirling pair $(\overleftarrow{\pi_2}, \overleftarrow{\pi_1})$.*
(f) *If $\pi_1 = (n, n-1, \ldots, 2, 1)$ and (π_1, π_2) is a Stirling pair, then $\pi_2 = \pi_1$.*
(g) *If $\pi_2 = (1, 2, \ldots, n)$ and (π_1, π_2) is a Stirling pair, then $\pi_1 = \pi_2$.*

Proof The simple proofs of the individual assertions are below.

(a) This is an immediate consequence of the Stirling property.
(b) Consider $1 \leq k < l < t \leq n$. If $j_k > j_t > j_l$ then, since $\pi_1 = (1, 2, \ldots, n)$, $\pi_1 = (\ldots, j_l, \ldots, j_t, \ldots, j_k, \ldots)$. Since $\pi_2 = (j_1, j_2, \ldots, j_n)$, then in π, j_k is repeated before j_l and j_l is repeated before j_t. Thus

$$\pi = (\ldots, j_l, \ldots, j_t, \ldots, j_k, \ldots, j_k, \ldots, j_l, \ldots, j_t, \ldots).$$

Since $j_l < j_t$, we contradict that π is a Stirling permutation.
(c) We have $\pi = (1, 2, \ldots, n, j_1 = n, j_2, \ldots, j_n)$. If π_2 were not in decreasing order, then we contradict the definition of a Stirling permutation.
(d) Since $i_k = n = j_l$, n is the lth integer repeated in π and must immediately follow the n in π_1. The integers $j_1, j_2, \ldots, j_{l-1}$ are repeated before n is and so $\{j_1, j_2, \ldots, j_{l-1}\} \subseteq \{i_1, i_2, \ldots, i_{k-1}\}$ and hence $k \geq l$. The second assertion is the obvious generalization with n replaced with an arbitrary $p \in \{1, 2, \ldots, n\}$.
(e) This follows in a straightforward way from the definitions.

(f) Suppose to the contrary that $\pi_2 = (\ldots, a, \ldots, b, \ldots)$ where $a < b$. Then in a Stirling permutation π with Stirling pair (π_1, π_2), a is repeated before b is repeated. Then $\pi = (\ldots, b, \ldots, a, \ldots, a, \ldots, b, \ldots)$, a contradiction since $a < b$.

(g) Suppose to the contrary that $\pi_1 = (\ldots, a, \ldots, b, \ldots)$ where $a > b$. Since $\pi_2 = (1, 2, \ldots, n)$, in a Stirling permutation π with Stirling pair (π_1, π_2), b is repeated before a is repeated. Then $\pi = (\ldots, a, \ldots, b, \ldots, b, \ldots, a, \ldots)$, a contradiction since $b < a$.

□

Before giving two algorithms to construct a Stirling permutation given a Stirling permutation pair (π_1, π_2), we prove the following somewhat more substantial property of Stirling permutation pairs.

Lemma 2.2 *Let $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$ be a Stirling pair of permutations of $\{1, 2, \ldots, n\}$ with a corresponding Stirling permutation π. Let $i_1 = p$ and let k be such that $j_k = p$. Then*

(a) $\{i_2, \ldots, i_k\} = \{j_1, \ldots, j_{k-1}\}$ *and thus* $\{i_{k+1}, \ldots, i_n\} = \{j_{k+1}, \ldots, j_n\}$*;*
(b) $\pi = (i_1 = p, a_1, a_2, \ldots, a_{2(k-1)}, j_k = p, b_1, b_2, \ldots, b_{2(n-k)})$ *where,* $(a_1, a_2, \ldots, a_{2(k-1)})$ *is a 2-permutation of* $\{i_2, \ldots, i_k\}$ *and* $(b_1, b_2, \ldots, b_{2(n-k)})$ *is a 2-permutation of* $\{i_{k+1}, \ldots, i_n\}$*, both with the Stirling property.*
(c) $\{1, 2, \ldots, p-1\} \subseteq \{i_{k+1}, i_{k+2}, \ldots, i_n\} \cap \{j_{k+1}, j_{k+2}, \ldots, j_n\}$.

Proof Since π is a Stirling permutation, $j_1, j_2, \ldots, j_{k-1}$ must all be repeated in π before j_k is repeated. Thus between the two p's in π, each of $j_1, \ldots, j_{k-1}$ appears twice. Suppose that in π there is a q occuring between the two p's where $q \notin \{j_1, j_2, \ldots, j_{k-1}\}$. Thus q is repeated in π after j_k is repeated. Then in π we have the subsequence $i_1 = p, i_r = q, j_k = p, j_s = q$ for some r and s with $1 < r < k$ and $k < s \leq n$. If $p < q$, then q, p, q contradicts that π is a Stirling permutation while if $p > q$, then p, q, p contradicts that π is a Stirling permutation. Thus (a) holds. Since π is a Stirling permutation, the 2-permutation $(a_1, a_2, \ldots, a_{2(k-1)})$ (regarded as a 2-permutation of $\{1, 2, \ldots, k-1\}$), and the 2-permutation $(b_1, b_2, \ldots, b_{2(n-k)})$ (regarded as a 2-permutation of $\{1, 2, \ldots, n-k-1\}$), satisfy the Stirling property and thus are Stirling permutations of $\{1, 2, \ldots, k-1\}$ and $\{1, 2, \ldots, n-k\}$, respectively. Assertion (c) follows from (a) and (b). □

Lemma 2.2 inductively gives the following corollary.

Corollary 2.3 *Let (π_1, π_2) be a Stirling pair of permutations. Then there is a unique corresponding Stirling permutation; equivalently, the mapping (3) given by $\phi_n : \mathcal{Q}_n \to \mathcal{S}_n^{\times 2}$ is a bijection.*

Lemma 2.2 is the basis for an algorithm for construction of a Stirling permutation from a Stirling permutation pair.

Algorithm (I) for Construction of a Stirling permutation with $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$

(i) Begin with π_1.
(ii) Let $i_1 = p$ and let k be such that $j_k = p$. Put a $j_k = p$ immediately after i_k in π_1.
(iii) Repeat recursively with $\pi_1' = (i_2, \ldots, i_k)$ and with $\pi_2' = (j_1, j_2, \ldots, j_{k-1})$ (regarded as permutations of $\{1, 2, \ldots, k-1\}$) and then recursively with $\pi_1'' = (i_{k+1}, \ldots, i_n)$ and $\pi_2'' = (j_{k+1}, \ldots, j_n)$ (regarded as permutations of $\{k+1, k+2, \ldots, n\}$).
(iv) Output the resulting π.

Theorem 2.4 *Let* $\pi_1 = (i_1, i_2, \ldots, i_n)$ *and* $\pi_2 = (j_1, j_2, \ldots, j_n)$ *be a Stirling permutation pair. Then Algorithm* (I) *produces the unique Stirling permutation with Stirling permutation pair* (π_1, π_2).

Proof Let π be a Stirling permutation with Stirling pair (π_1, π_2). By Lemma 2.2, j_k immediately follows i_k in π. Moreover, it follows from this lemma, that applying the algorithm recursively to the pairs (π_1', π_2') and (π_1'', π_2'') results in the unique Stirling permutation corresponding to (π_1, π_2). □

We now give a second algorithm to construct a Stirling permutation of order n from a Stirling permutation pair (π_1, π_2). Both algorithms (I) and (II) show that a Stirling permutation pair is determined by a unique Stirling permutation.

Algorithm (II) for Construction of a Stirling permutation with $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$.

(i) Set τ_0 equal to π_1.
(ii) For $k = 1, 2, \ldots, n$, let τ_k be obtained from τ_{k-1} by inserting a second j_k into τ_{k-1} immediately following j_k in τ_{k-1} and the second occurrence of j_{k-1} in τ_{k-1} whichever comes later.
(iii) Output the permutation τ_n of $\{1, 1, 2, 2, \ldots, n, n\}$.

Theorem 2.5 *Let* $\pi_1 = (i_1, i_2, \ldots, i_n)$*, and let* $\pi_1 = (j_1, j_2, \ldots, j_n)$ *be a Stirling permutation pair with corresponding Stirling permutation* π*. Then Algorithm* (II) *produces the unique Stirling permutation with Stirling permutation pair* (π_1, π_2).

Proof It follows from the algorithm that the second occurrences of the integers $1, 2, \ldots, n$ do occur in the required order $j_1, j_2, \ldots, j_n$. The algorithm clearly produces a 2-permutation of $\{1, 2, \ldots, n\}$. We claim that τ_n is the Stirling permutation π whose corresponding Stirling permutation pair is (π_1, π_2). We show by induction on k that τ_k satisfies the Stirling property, that is, all integers between the two j_k's in τ_k are greater than j_k, and hence that $\tau_n = \pi$.

First assume that $k = 1$ and $j_1 = p$. If there exists in π an integer q with q between the two occurrences of p then, since q is repeated in π after p and π contains as a subsequence p, q, p, q violating the Stirling property. Thus in π, the two p's are consecutive and this agrees with τ_1.

We now proceed by induction and assume that $k > 1$. If j_k is placed in τ_k immediately following the j_k in τ_{k-1}, then τ_k satisfies the Stirling property since τ_{k-1} does. Assume that in π the second j_{k-1} follows the first j_k in τ_{k-1} and that there exists an i_q between the second j_{k-1} and the second j_k. Then i_q is repeated after the second

j_k and hence π contains the subsequence j_k, i_q, j_k, i_q contradicting again the Stirling property. Thus in this case, j_k is repeated immediately following the second j_{k-1} in π. Thus τ_k satisfies the Stirling property. Hence by induction $\tau_n = \pi$. □

We now give two examples of these algorithms with $n = 7$ one of which uses the the identity permutation as π_1.

Example 2.6 Let $\pi_1 = (1, 2, 3, 4, 5, 6, 7)$ and $\pi_2 = (4, 3, 2, 1, 6, 7, 5)$, a 312-avoiding permutation. Applying Algorithms (I) and (II) we get:

Algorithm (I)	Algorithm (II)
1234567	1234567
12341567	12344567
123421567	123443567
1234321567	1234432567
12344321567	12344321567
123443215667	123443215667
1234432156677	1234432156677
12344321566775	12344321566775

.

Now let $\pi_1 = (3, 4, 6, 5, 7, 1, 2)$ and $\pi_2 = (6, 4, 3, 7, 5, 2, 1)$. Applying Algorithms (I) and (II) we get:

Algorithm (I)	Algorithm (II)
3465712	3465712
34635712	34665712
346435712	346645712
3466435712	3466435712
34664357512	34664357712
346643577512	346643577512
3466435775122	3466435775122
34664357751221	34664357751221

.

□

Given a permutation π_1 of $\{1, 2, \ldots, n\}$, we define its *Stirling right-multiplicity* $|\pi_1|_r$ to be the number of permutations π_2 of $\{1, 2, \ldots, n\}$ such that (π_1, π_2) is a Stirling permutation pair. We have $|\pi_1|_r \geq 1$ since (π_1, π_1) is always a Stirling permutation pair. Let

$$M_r = \max\{|\pi_1|_r : \pi_1 \text{ a permutation of } \{1, 2, \ldots, n\}\},$$

the *maximum Stirling right-multiplicity* of a permutation in $\mathcal{S}_n$. Similarly, we define the *Stirling left-multiplicity* of a permutation π_2 of $\{1, 2, \ldots, n\}$ to be the number $|\pi_2|_l$ of permutations π_1 of $\{1, 2, \ldots, n\}$ such that (π_1, π_2) is a Stirling permutation pair. In investigating these numbers it follows from (e) of Lemma 2.1 that it is enough to

Table 1 Stirling permutations of order 3

π	π_1	π_2
(1,1,2,2,3,3)	(1,2,3)	(1,2,3)
(2,2,1,1,3,3)	(2,1,3)	(2,1,3)
(1,2,2,1,3,3)	(1,2,3)	(2,1,3)
(1,1,2,3,3,2)	(1,2,3)	(1,3,2)
(2,2,1,3,3,1)	(2,1,3)	(2,3,1)
(1,2,2,3,3,1)	(1,2,3)	(2,3,1)
(1,1,3,3,2,2)	(1,3,2)	(1,3,2)
(2,2,3,3,1,1)	(2,3,1)	(2,3,1)
(1,2,3,3,2,1)	(1,2,3)	(3,2,1)
(2,3,3,2,1,1)	(2,3,1)	(3,2,1)
(1,3,3,2,2,1)	(1,3,2)	(3,2,1)
(1,3,3,1,2,2)	(1,3,2)	(3,1,2)
(3,3,1,1,2,2)	(3,1,2)	(3,1,2)
(3,3,2,2,1,1)	(3,2,1)	(3,2,1)
(3,3,1,2,2,1)	(3,1,2)	(3,2,1)

consider Stirling right-multiplicities since $|\pi_2|_l = |\overleftarrow{\pi_1}|_r$. We begin with the following lemma concerning the minimum Stirling right multiplicity.

Lemma 2.7 *Let $\pi_1 = (i_1, i_2, \ldots, i_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Then $|\pi_1|_r \geq 1$ with equality if and only if π_1 is the anti-identity permutation $\iota_n^* = (n, n-1, \ldots, 1)$.*

Proof As already remarked, (π_1, π_1) is always a Stirling permutation pair so that we have $|\pi_1|_r \geq 1$. Now assume that $\pi_1 = (n, n-1, \ldots, 1)$ and let π_2 be a permutation such that (π_1, π_2) is a Stirling permutation pair with corresponding Stirling permutation π. Since n always immediately follows n in a Stirling permutation of order n, π_2 must begin with n and $\pi = (n, n, \ldots)$. It now follows by induction that $\pi_2 = (n, n-1, \ldots, 1)$ and $\pi = (n, n, n-1, n-1, \ldots, 1, 1)$. Thus $|\pi_1|_r = 1$.

Now let $\pi_1 = (i_1, i_2, \ldots, i_n) \neq (n, n-1, \ldots, 1)$. Then there exists a k such that $i_k < i_{k+1}$. With $\pi_2 = (i_1, i_2, \ldots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \ldots, i_n)$, we have $\pi_2 \neq \pi_1$ where $(i_1, i_1, \ldots, i_{k-1}, i_{k-1}, i_k, i_{k+1}, i_{k+1}, i_k, i_{k+2}, i_{k+2}, \ldots, i_n, i_n)$ is a Stirling permutation with corresponding Stirling permutation pair (π_1, π_2). Hence $|\pi_1|_r \geq 2$. □

Example 2.8 The 15 Stirling permutations π of order 3 along with their corresponding Stirling permutation pairs (π_1, π_2) are given in Table 1.

Note that each pair (π_1, π_2) in the Table 1 corresponds to exactly one Stirling permutation as verified by Corollary 2.3. Note also that for $\pi_1 = (1, 2, 3)$ we are missing only $\pi_2 = (3, 1, 2)$; for $\pi_2 = (3, 2, 1)$ we are missing only $(2, 1, 3)$. The numbers $|\pi_1|_r$'s and the numbers $|\pi_2|_l$'s are given in the Table 2.

There are twenty-one 3×3 2-permutation matrices. The fifteen 3×3 Stirling permutation matrices corresponding to each of the Stirling permutation pairs (π_1, π_2) in Table 1 are given in Table 3.

Table 2 Left- and right-multiplicities

π_1	$\lvert\pi_1\rvert_r$	π_2	$\lvert\pi_2\rvert_l$
(1,2,3)	5	(1,2,3)	1
(1,3,2)	3	(1,3,2)	2
(2,1,3)	2	(2,1,3)	2
(2,3,1)	2	(2,3,1)	3
(3,1,2)	2	(3,1,2)	2
(3,2,1)	1	(3,2,1)	5

Table 3 3×3 Stirling permutation matrices

$$\left[\begin{array}{c|c|c} 2 & & \\ \hline & 2 & \\ \hline & & 2 \end{array}\right], \left[\begin{array}{c|c|c} & 2 & \\ \hline 2 & & \\ \hline & & 2 \end{array}\right], \left[\begin{array}{c|c|c} 1 & 1 & \\ \hline 1 & 1 & \\ \hline & & 2 \end{array}\right], \left[\begin{array}{c|c|c} 2 & & \\ \hline & 1 & 1 \\ \hline & 1 & 1 \end{array}\right], \left[\begin{array}{c|c|c} & 2 & \\ \hline 1 & & 1 \\ \hline 1 & & 1 \end{array}\right],$$
$$\left[\begin{array}{c|c|c} 1 & 1 & \\ \hline & 1 & 1 \\ \hline 1 & & 1 \end{array}\right], \left[\begin{array}{c|c|c} 2 & & \\ \hline & & 2 \\ \hline & 2 & \end{array}\right], \left[\begin{array}{c|c|c} & 2 & \\ \hline & & 2 \\ \hline 2 & & \end{array}\right], \left[\begin{array}{c|c|c} 1 & & 1 \\ \hline & 2 & \\ \hline 1 & & 1 \end{array}\right], \left[\begin{array}{c|c|c} & 1 & 1 \\ \hline & 1 & 1 \\ \hline 2 & & \end{array}\right],$$
$$\left[\begin{array}{c|c|c} 1 & & 1 \\ \hline & 1 & 1 \\ \hline 1 & 1 & \end{array}\right], \left[\begin{array}{c|c|c} 1 & & 1 \\ \hline 1 & & 1 \\ \hline & 2 & \end{array}\right], \left[\begin{array}{c|c|c} & & 2 \\ \hline 2 & & \\ \hline & 2 & \end{array}\right], \left[\begin{array}{c|c|c} & & 2 \\ \hline & 2 & \\ \hline 2 & & \end{array}\right], \left[\begin{array}{c|c|c} & & 2 \\ \hline 1 & 1 & \\ \hline 1 & 1 & \end{array}\right].$$

Table 4 3×3 Non-Stirling permutation matrices

$$\left[\begin{array}{c|c|c} 1 & & 1 \\ \hline 1 & 1 & \\ \hline & 1 & 1 \end{array}\right], \left[\begin{array}{c|c|c} & 1 & 1 \\ \hline 1 & & 1 \\ \hline 1 & 1 & \end{array}\right], \left[\begin{array}{c|c|c} & 1 & 1 \\ \hline 1 & 1 & \\ \hline 1 & & 1 \end{array}\right], \left[\begin{array}{c|c|c} 1 & 1 & \\ \hline 1 & & 1 \\ \hline & 1 & 1 \end{array}\right], \left[\begin{array}{c|c|c} 1 & 1 & \\ \hline & & 2 \\ \hline 1 & 1 & \end{array}\right], \left[\begin{array}{c|c|c} & 1 & 1 \\ \hline 2 & & \\ \hline & 1 & 1 \end{array}\right].$$

The six 3×3 2-permutation matrices that are not Stirling permutation matrices are given in Table 4.

□

3 Stirling Permutation Pairs and the Weak Bruhat Order

We begin this section with an exposition of some fairly standard facts concerning permutations in order to make it easier for the reader to understand their implications for Stirling permutation pairs.

Let $\sigma = (k_1, k_2, \ldots, k_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Recall that an *inversion* of σ is a pair (k_p, k_q) such that $p < q$ but $k_p > k_q$. Note that an inversion is defined to be the ordered pair (k_p, k_q), not the ordered pair (p, q) of positions; if (k_p, k_q) is an inversion of π, then (p, q) is an inversion of its inverse π^{-1}. Thus an inversion of σ is a pair of integers in σ that are out of their natural order. The *set of inversions* of the permutation σ is denoted by $\mathcal{I}(\sigma)$. For the identity permutation ι_n we have $\mathcal{I}(\iota_n) = \emptyset$. The anti-identity permutation ι_n^* has the largest number of inversions, indeed has all possible inversions: $\mathcal{I}(\iota_n^*) = \{(a, b) : n \geq a > b \geq 1\}$. The permutation σ can be brought to the identity ι_n by a sequence of *transpositions* $(k_p, k_q) \to (k_q, k_p)$, each chosen in such a way as to reduce the number of inversions (not necessarily the set of inversions in general) by exactly 1. For example, we have $(\mathbf{3}, 4, 1, \mathbf{2}) \to (\mathbf{2}, 4, 1, \mathbf{3})$ reduces the number of inversions from 4 to 3, with the set of inversions changing from $\{(3, 1), (3, 2), (4, 1), (4, 2)\}$ to $\{(2, 1), (4, 1), (4, 3)\}$.

An inversion of $\sigma = (k_1, k_2, \ldots, k_n)$ of the form (k_p, k_{p+1}) is called an *adjacent inversion*. If (k_p, k_{p+1}) is an adjacent inversion of the permutation σ, then the transposition $(k_p, k_{p+1}) \to (k_{p+1}, k_p)$ is called an *adjacent transposition*. An adjacent transposition applied to a permutation σ has the sole effect of removing exactly one inversion (an adjacent inversion) from $\mathcal{I}(\pi)$ and thus, in particular, reduces the number of inversions by exactly one.

For completeness, we note the following basic property of permutations.

Lemma 3.1 *A permutation of* $\{1, 2, \ldots, n\}$ *is uniquely determined by its set of inversions. A permutation is not, in general, uniquely determined by its set of adjacent inversions.*

Proof Briefly, let $\pi = (i_1, i_2, \ldots, i_n)$ be a permutation of $\{1, 2 \ldots, n\}$ with inversion set $\mathcal{I}(\pi)$. If $i_1 = 1$, then 1 does not contribute to any inversions of π and the first assertion follows by induction. Let $i_p = 1$ for some $p > 1$, and let $X_1 = \{i_j : 1 \leq j < p\}$ and $X_2 = \{i_j : p < j \leq n\}$. If $a \in X_1$ and $b \in X_2$ satisfy $a > b$, then $(a, b) \in \mathcal{I}(\pi)$. The first assertion now follows by applying induction to $(i_1, \ldots, i_{p-1})$ and $(i_{p+1}, \ldots, i_n)$.

For the second assertion, the permutations $(4, 1, 2, 3)$ and $(2, 4, 1, 3)$ of $\{1, 2, 3, 4\}$ have exactly one adjacent inversion, namely $(4, 1)$ in both instances, and their sets of inversions are $\{(4, 1), (4, 2), (4, 3)\}$ and $\{(2, 1), (4, 1), (4, 3)\}$, respectively. □

Inversions play a role in Stirling permutations.

Lemma 3.2 *Let* π_1 *and* π_2 *be permutations of* $\{1, 2, \ldots, n\}$ *such that* (π_1, π_2) *is a Stirling permutation pair. Then* $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$.

Proof Suppose $(a, b) = (i_p, i_q) \in \mathcal{I}(\pi_1)$. Thus a precedes b in π_1 and $a > b$. Suppose that b precedes a in π_2. Then in the Stirling permutation corresponding to (π_1, π_2), b is repeated before a is repeated. Thus the relative positions of a and b in π are a, b, b, a, a contradiction since $b < a$. Hence (a, b) is also an inversion in π_2. □

Note that since $\mathcal{I}(\iota_n) = \emptyset$, Lemma 3.2 holds vacuously if $\pi_1 = \iota_n$.

Corollary 3.3 *Let* π_1 *and* π_2 *be permutations of* $\{1, 2, \ldots, n\}$ *such that both* (π_1, π_2) *and* (π_2, π_1) *are Stirling permutation pairs. Then* $\pi_1 = \pi_2$.

Proof Lemma 3.2 implies that π_1 and π_2 have the same set of inversions, and hence by Lemma 3.1, $\pi_1 = \pi_2$. □

If π_1 and π_2 are permutations of $\{1, 2, \ldots, n\}$, then π_2 as a permutation relative to π_1 is the permutation $\pi_2\pi_1^{-1}$ (π_1^{-1} followed by π_2). The cycles γ of this permutation are the *cycles of* π_2 *relative to* π_1. For example, let $n = 8$ and let

$$\pi_1 = (3, 5, 1, 7, 2, 8, 4, 6) \text{ and } \pi_2 = (6, 8, 5, 3, 1, 2, 4, 7).$$

Then

$$5 \overset{\pi_1^{-1}}{\to} 2 \overset{\pi_2}{\to} 8 \overset{\pi_1^{-1}}{\to} 6 \overset{\pi_2}{\to} 2 \overset{\pi_1^{-1}}{\to} 5 \overset{\pi_2}{\to} 1 \overset{\pi_1^{-1}}{\to} 3 \overset{\pi_2}{\to} 5.$$

determines the 4-cycle

$$5 \to 8 \to 2 \to 1 \to 5.$$

of π_2 relative to π_1. Replacing in π_2 this cycle with its reverse cycle

$$5 \to 1 \to 2 \to 8 \to 5$$

gives a new permutation $\pi_2' = (6, 5, 1, 3, 2, 8, 4, 7)$. This operation is called *cycle-substitution*. In a similar way we obtain a new permutation π_1' and we refer to a *symmetric cycle-substitution* of π_1 and π_2. In terms of the corresponding permutation matrices we have the following:

$$P_{\pi_1} = \begin{bmatrix} & & 1 & & & & & \\ & & & & 1 & & & \\ 1 & & & & & & & \\ & & & & & & 1 & \\ & 1 & & & & & & \\ & & & & & & & 1 \\ & & & 1 & & & & \\ & & & & & 1 & & \end{bmatrix} \text{ and } P_{\pi_2} = \begin{bmatrix} & & & & & 1 & & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & 1 & & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & 1 & & & & \\ & & & & & & 1 & \end{bmatrix}$$

gives

$$P_{\pi_1'} = \begin{bmatrix} & & 1 & & & & & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & & 1 & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & 1 & & & & \\ & & & & & 1 & & \end{bmatrix} \text{ and } P_{\pi_2'} = \begin{bmatrix} & & & & & 1 & & \\ & & & & 1 & & & \\ 1 & & & & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & & & & & 1 \\ & & & 1 & & & & \\ & & & & & & 1 & \end{bmatrix}.$$

If π_2 has several cycles relative to π_1, then one may simultaneously do more than one cycle substitution. Of course if one substitutes all of these cycles, then π_2 becomes π_1. Otherwise, we refer to a *partial cycle substitution*.

Lemma 3.4 *Let $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$ be two permutations of $\{1, 2, \ldots, n\}$ with $i_k \neq j_k$ for $k = 1, 2, \ldots, n$ such that $\mathcal{I}(\pi_1) \subset \mathcal{I}(\pi_2)$. Let π_1' and π_2' be obtained from π_1 and π_2 by one or more, but not all, symmetric partial cycle substitutions. Then $\mathcal{I}(\pi_1')$ and $\mathcal{I}(\pi_2')$ are unrelated, that is, $\mathcal{I}(\pi_1') \nsubseteq \mathcal{I}(\pi_2')$ and $\mathcal{I}(\pi_2') \nsubseteq \mathcal{I}(\pi_1')$.*

Proof For ease of language we argue under the assumption of one symmetric cycle substitution γ. The assumptions imply that π_2 contains an inversion which is not an inversion of π_1 involving indices not belonging to the cycle γ. The cycle substitution interchanges certain, but not all, of the inversions of π_1 and π_2. It thus follows that $\mathcal{I}(\pi_2') \nsubseteq \mathcal{I}(\pi_1')$ and $\mathcal{I}(\pi_1') \nsubseteq \mathcal{I}(\pi_2')$. □

Corollary 3.5 *Let Q be an $n \times n$ Stirling permutation matrix. Then there is a unique Stirling permutation pair (π_1, π_2) such that $Q = P_{\pi_1} + P_{\pi_2}$.*

Proof Since Q is a Stirling permutation, there is a Stirling permutation pair (π_1, π_2) such that $Q = P_{\pi_1} + P_{\pi_2}$. Since any permutation matrix R such that $R \leq Q$ (entrywise) is obtained by a cycle substitution involving P_{π_1} and P_{π_2}, the corollary is a direct consequence of Lemmas 3.2 and 3.4. □

Let π_1 and π_2 be two permutations of $\{1, 2, \ldots, n\}$. The *weak Bruhat order* (see e.g [1]) is a lattice order $\preceq_{\mathrm{b}}$ on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$ defined by:

$$\pi_1 \preceq_{\mathrm{b}} \pi_2 \text{ provided that } \mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2).$$

It is known [1,11] that this is equivalent to: π_1 can be obtained from π_2 by a sequence of adjacent transpositions.[3] If π_1 can be obtained from π_2 by a sequence of adjacent transpositions, then clearly $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$. For completeness we briefly verify the converse [11].

Lemma 3.6 *Let π_1 and π_2 be two permutation of $\{1, 2, \ldots, n\}$ such that $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$. Then by a sequence of adjacent transpositions we may reduce π_2 to π_1.*

Proof Let $\pi_1 = (r_1, r_2, \ldots, r_n)$ and $\pi_2 = (s_1, s_2, \ldots, s_n)$. Let j be the minimum value of i such that $s_j \neq r_j$. Thus $r_1 = s_1, \ldots, r_{j-1} = s_{j-1}, r_j \neq s_j$. Define an integer $k > j$ by $r_j = s_k$. Thus

$$\pi_1 = (r_1 = s_1, \ldots, r_{j-1} = s_{j-1}, r_j = s_k, \ldots, r_l = s_j, \ldots) \text{ (for some } l\text{) and}$$
$$\pi_2 = (s_1 = r_1, \ldots, s_{j-1} = r_{j-1}, s_j \neq r_j, \ldots, s_{k-1}, s_k = r_j, \ldots).$$

Since $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$, we have that $s_j < r_j$; otherwise, $r_j > s_j$ and $(r_j, s_j) \in \mathcal{I}(\pi_2)$ but $(r_j, s_j) \notin \mathcal{I}(\pi_2)$, a contradiction. We also have $s_{k-1} > s_k$; otherwise,

[3] As the name suggests, the weak Bruhat order is a weakening of the notion of the *Bruhat order* $\preceq_B$ on $\mathcal{S}_n$ whereby $\pi_1 \preceq_B \pi_2$ provided π_1 can be obtained from π_2 by a sequence of transpositions each of which reduces the number of inversions by one but, in general, does not simply remove one inversion from $\mathcal{I}(\pi_2)$. For example, $\pi_1 = (4, 2, 1, 3)$ can be gotten from $\pi_2 = (4, 3, 1, 2)$ by the one transposition that interchanges 3 and 2, but π_1 cannot be gotten from π_2 by adjacent transpositions. The Bruhat order and weak Bruhat order on $\mathcal{S}_n$ are special instances of the corresponding orders on Coxeter groups. See [1].

$s_k > s_{k-1}$ and $(s_k, s_{k-1}) \in \mathcal{I}(\pi_1)$, but $(s_{k-1}, s_k) \notin \mathcal{I}(\pi_1)$. Applying the adjacent interchange (s_{k-1}, s_k) to π_2 gives a permutation π_2' such that $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2')$ and $\mathcal{I}(\pi_2')\backslash\mathcal{I}(\pi_1)$ has one fewer adjacent inversion than $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$. The lemma now follows by induction. □

We now extend the definition of a 312-avoiding permutation. Being a 312-avoiding permutation places restrictions on the inversions of the permutation. In fact, the permutation $\sigma = (j_1, j_2, \ldots, j_n)$ is 312-avoiding is equivalent to:

(312) If $1 \le k < l < p \le n$ and (j_k, j_l) and (j_k, j_p) are inversions, then (j_l, j_p) is also an inversion, that is, j_k, j_l, j_p is a decreasing subsequence of σ.
Now let $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$ be two permutations of $\{1, 2, \ldots, n\}$. Then we define π_2 to be a 312-*avoiding permutation relative to* π_1 (or, (π_1, π_2) is a 312-*avoiding permutation pair*) provided the following hold:

(312-i) $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$, and
(312-ii) If $1 \le k < l < p \le n$ and (j_k, j_l) and (j_k, j_p) are inversions in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$, then (j_l, j_p) is also an inversion in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$.

If (312-i) and (312-ii) hold, then

(312-iii) j_k, j_l, j_p is a decreasing subsequence of π_2 and j_p, j_l, j_k is an increasing subsequence of π_1.

Lemma 3.7 *Let* π_1 *and* π_2 *be two permutations of* $\{1, 2, \ldots, n\}$ *such that* (π_1, π_2) *is a Stirling pair of permutations. Then* π_2 *is* 312-*avoiding permutation relative to* π_1.

Proof Let $\pi_1 = (i_1, i_2, \ldots, i_n)$ and $\pi_2 = (j_1, j_2, \ldots, j_n)$ with corresponding Stirling permutation π. First suppose (312-i) does not hold. Then there exists $k < l$ such that $i_k > i_l$ but i_l comes before i_k in π_2. Thus i_l is repeated before i_k in π, and hence π contains i_k, i_l, i_l, i_k as a subsequence. Since $i_l < i_k$, this contradicts that π is a Stirling permutation. Hence (312-i) holds.

Now suppose that (312-ii) does not hold. Then there exists $1 \le k < l < p \le n$ and inversions (j_k, j_l) and (j_k, j_p) in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$ such that either

(a) (j_l, j_p) is not an inversion in $\mathcal{I}(\pi_2)$, or
(b) (j_l, j_p) is an inversion in both $\mathcal{I}(\pi_2)$ and $\mathcal{I}(\pi_1)$.

First suppose that (a) holds. Then j_k, j_l, j_p is a subsequence of π_2 with $j_l < j_p$, and either j_p, j_l, j_k or j_l, j_k, j_p is a subsequence of π_1. Hence $j_p, j_l, j_k, j_k, j_l, j_p$ or $j_l, j_p, j_k, j_k, j_l, j_p$ is a subsequence of π which, because $j_l < j_p$, contradicts that π is a Stirling permutation.

Now suppose that (b) holds. Since (j_l, j_p) is an inversion in π_2, $j_l > j_p$. Since (j_l, j_p) is also an inversion in π_1, j_l comes before j_p in π_1. Since (j_k, j_l) and (j_k, j_p) are not inversions in π_1, j_l precedes j_k in π_1 and j_p precedes j_k in π_1. Thus in π_1 we have the subsequence j_l, j_p, j_k. Thus $j_l, j_p, j_k, j_k, j_l, j_p$ is a subsequence of π with $j_p < j_l$ contradicting that π is a Stirling permutation.

Thus (312-i) and (312-ii) hold. □

We now characterize Stirling permutations in general by characterizing Stirling permutation pairs.

Theorem 3.8 *Let* $\pi_1 = (i_1, i_2, \ldots, i_n)$ *and* $\pi_2 = (j_1, j_2, \ldots, j_n)$ *be permutations of* $\{1, 2, \ldots, n\}$. *Then* (π_1, π_2) *is a Stirling permutation pair if and only if* (π_1, π_2) *is a* 312*-avoiding pair of permutations. Moreover, if* (π_1, π_2) *is a Stirling permutation pair, then both Algorithm* (I) *and Algorithm* (II) *produce the corresponding Stirling permutation.*

Proof By Lemma 3.7, if (π_1, π_2) is a Stirling permutation pair, then (π_1, π_2) is a 312-avoiding permutation pair. Now assume that (π_1, π_2) is a 312-avoiding permutation pair.

We show by induction on k that under this assumption, the permutations τ_k produced by Algorithm (II) satisfy the Stirling property and thus $\pi = \tau_n$ is a Stirling permutation with corresponding Stirling permutation pair (π_1, π_2). One should keep in mind the obvious that $\{i_1, i_2, \ldots, i_n\} = \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

It is instructive to first argue the case where $j_1 = i_n$. Since $\pi_2 = (j_1, j_2, \ldots, j_n)$ and $j_1 = i_n$, Algorithm (II) gives the 2-permutation

$$\pi = (i_1, i_2, \ldots, i_n, i_n = j_1, j_2, \ldots, j_n), \tag{6}$$

and this is the only possibility for a Stirling permutation corresponding to (π_1, π_2). We need to show that (6) is a Stirling permutation. We must have $i_n = n$, for otherwise (n, i_n) is an inversion of π_1 but not of π_2.

Assume to the contrary that (6) is not a Stirling permutation. Then there exists $a = i_r$ and $a = j_s$ and an integer b either from π_1 or π_2 such that $b < a$ with b between the two a's. So $b = i_p$ for some p with $r < p < n$, or $b = j_q$ for some integer q with $1 < q < s$.

Suppose first that $b = i_p$ with $r < p < n$. Then (a, b) is an inversion of π_1 and hence of π_2. So $b = j_k$ for some $k > s$. We thus have $n = j_1 > a = j_s > b = j_k$ but $i_n = n > i_r = a > i_p = b$. Then (n, a) and (n, b) are inversions in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$ and so (a, b) must be an inversion in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$. This contradicts (312-ii) since (a, b) is an inversion in $\mathcal{I}(\pi_1)$.

Now suppose that $b = j_q$ with $1 \le q < s$. Then (a, b) is not an inversion of π_2 and so is not an inversion of π_1. Thus $b = i_k$ for some $k < p$. Thus (n, a), (n, b) are inversions in $\mathcal{I}(\pi_2)\backslash\mathcal{I}(\pi_1)$; moreover, (a, b) is not an inversion in $\mathcal{I}(\pi_1)$ but also not an inversion in $\mathcal{I}(\pi_2)$, contradicting (312-ii).

We now consider the general case.

In Algorithm II, τ_1 is obtained from π_1 by inserting j_1 immediately after the occurrence of j_1 in π_1 and this implies that τ_1 satisfies the Stirling property. Now let $k > 1$. By induction τ_{k-1} satisfies the Stirling property. We now insert j_k into τ_{k-1} according to rule (ii) of Algorithm (II). There are two cases to consider.

Case 1 j_{k-1} was inserted in τ_{k-1} somewhere *before* the occurrence of j_k in π_1. Hence j_k is inserted into τ_{k-1} immediately after the occurrence of j_k in τ_{k-1}. Since the j_k's occur consecutively in τ_k and τ_{k-1} satisfies the Stirling property, τ_k satisfies the Stirling property.

Case 2 j_{k-1} was inserted in τ_{k-1} somewhere *after* the occurrence of j_k in π_1. Hence, in constructing τ_k, j_k is inserted in τ_{k-1} immediately after the second occurrence of j_{k-1} in τ_{k-1}. So we have $\pi_1 = (\ldots, j_k, \ldots, j_{k-1}, \ldots)$ and

$\tau_k = (\ldots, j_k, \ldots, j_{k-1}, \ldots, j_{k-1}, j_k, \ldots)$. If $j_k > j_{k-1}$, then $(j_k, j_{k-1}) \in \mathcal{I}(\pi_1)$ but, since j_{k-1} precedes j_k in π_2, $(j_k, j_{k-1}) \notin \mathcal{I}(\pi - 1)$, a contradiction. Thus $j_{k-1} > j_k$.

Consider an integer a in τ_k between the two j_{k-1}'s:

$$\tau_k = (\ldots, j_k, \ldots, j_{k-1}, \ldots, a, \ldots, j_{k-1}, j_k, \ldots).$$

Since τ_{k-1} satisfies the Stirling property, $a > j_{k-1}$ and hence $a > j_k$.

Now consider an integer b in τ_k between the first j_k and first j_{k-1} in τ_k

$$\tau_k = (\ldots, j_k, \ldots, b, \ldots, j_{k-1}, \ldots, j_{k-1}, j_k, \ldots).$$

(So this j_k and j_{k-1} belong to π_1.) Suppose to the contrary that $b < j_k$. Then $j_{k-1} > j_k > b$. First consider the possibility that b comes from π_1. Then $(j_k, b) \in \mathcal{I}(\pi_1)$ and so we must have $(j_k, b) \in \mathcal{I}(\pi_2)$, that is, $b = j_l$ for some $l > k$. In π_1 we have the subsequence $j_k, j_l = b, j_{k-1}$, while in π_2 we have the subsequence $j_{k-1}, j_k, j_l = b$ where $j_{k-1} > j_k > b$. This contradicts property (312-iii) of a 312-avoiding pair of permutations.

Hence b must come from π_2 and, in addition, b must now occur in π_1 before j_k giving:

$$\tau_k = (\ldots, b, \ldots, j_k, \ldots, b, \ldots, j_{k-1}, \ldots, j_{k-1}, j_k, \ldots).$$

The fact that $b = j_l$, where $l > k$, was inserted after the j_k in π_1 and that j_k is not yet repeated contradicts Algorithm (II). We conclude that there are no integers in τ_k between the two j_k's which are smaller than j_k. Hence τ_k satisfies the Stirling property, and Algorithm (II) produces a Stirling permutation with Stirling permutation pair (π_1, π_2). Since Algorithm (I) produces a Stirling permutation when (π_1, π_2) is a Stirling permutation pair, Algorithm (I) also produces a Stirling permutation when (π_1, π_2) is a Stirling permutation pair. □

4 Coda

We conclude with a number of examples and questions.

Let n be a positive integer and let π_1 and π_2 be in $\mathcal{S}_n$. In order that (π_1, π_2) be a Stirling pair of permutations we must have $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$ and so $\pi_1 \preceq_b \pi_2$. Suppose that $\mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2)$ and $|\mathcal{I}(\pi_2) \backslash \mathcal{I}(\pi_1)| = 1$ so that π_2 covers π_1 in the weak Bruhat order. Then it follows from Theorem 3.8 that, by default, (π_1, π_2) is a 312-avoiding pair of permutations. Suppose we define a new relation $\preceq_*$ on $\mathcal{S}_n$ by $\pi_1 \preceq_* \pi_2$ provided (π_1, π_2) is a Stirling pair. Then all the cover relations $\pi_1 \preceq_b \pi_2$ in the weak Bruhat order on $\mathcal{S}_n$ automatically satisfy $\pi_1 \preceq_* \pi_2$. But the relation $\preceq_*$ is not a partial order on $\mathcal{S}_n$ because, as shown in the next example, it is not transitive.

Example 4.1 Let $n = 4$ and consider the following maximal chain in the weak Bruhat order on $\mathcal{S}_4$:

$$(1,2,3,4) \preceq_b (1,2,4,3) \preceq_b (1,4,2,3) \preceq_b (4,1,2,3) \preceq_b (4,2,1,3) \preceq_b (4,2,3,1) \preceq_b (4,3,2,1).$$

Then we also have

$$(1,2,3,4) \preceq_* (1,2,4,3) \preceq_* (1,4,2,3) \preceq_* (4,1,2,3) \preceq_* (4,2,1,3) \preceq_* (4,2,3,1) \preceq_* (4,3,2,1).$$

But transitivity fails. For instance, as is easily checked, $(1,2,4,3) \not\preceq_* (4,1,2,3)$ but we do have $(1,2,4,3) \preceq_* (4,2,1,3)$. □

Example 4.2 We know that $|(1,2,3,4)|_r = 14$ but with one transposition increasing the number of inversions we get $(1,3,2,4)$ where $|(1,3,2,4)|_r = 9$. The π_2's that work for $(1,2,3,4)$ are:

$$(4,3,2,1), (3,4,2,1), (3,2,1,4), (3,2,4,1), (2,1,3,4), (2,1,4,3), (2,3,1,4), (2,3,4,1), (2,4,3,1), (1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,3,2).$$

Those that work for $(1,3,2,4)$ are

$$(1,3,2,4), (1,3,4,2), (3,1,2,4), (3,1,4,2), (3,2,1,4), (3,2,4,1), (3,4,1,2), (3,4,2,1), (4,2,3,1).$$

Now let $\pi_1 = (1,3,2,4)$ also with one inversion. Then there are seven π_2's that give Stirling permutations, namely,

$$1324, 1342, 3124, 3142, 3214, 3241, 3421.$$

□

Question 4.3 Is there a type of Bruhat order on Stirling permutations of the multiset $\{1,1,2,2,\ldots,n,n\}$ in terms of their corresponding Stirling pairs and associated $(0,1,2)$-matrix with row and column sums equal to 2. (A permutation pair π_1 and π_2 gives a $(0,1,2)$-matrix A with all row and column sums equal to 2.) □

Question 4.4 Is there an algorithm that allows one to generate all Stirling permutations of order n from the identity Stirling permutation $(1,1,2,2,\ldots,n,n)$ of order n. What kinds of transformations take a Stirling permutation matrix into another Stirling permutation matrix? □

Finally, we mention that Stirling permutations have been extended to more general r-permutations of $\{1,2,\ldots,n\}$ [5,9].

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References

1. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups, Graduate Texts in Mathematics, vol. 231. Springer, New York (2005)
2. Bóna, M.: Combinatorics of Permutations, 2nd edn. CRC Press, Boca Raton (2012)
3. Duh, G.-H., Lin, Y.-C., Ma, S.-M., Yeh, Y.-N.: Some statistics on Stirling permutations and Stirling derangements. Discrete Math. **341**, 2478–2484 (2018)
4. Gessel, I., Stanley, R.: Stirling polynomials. J. Combin. Theory Ser. A **72**, 95–117 (2005)
5. Haglund, J., Visontai, M.: Stable multivariate Eulerian permutations and generalized Stirling permutations. Eur. J. Combin. **33**, 477–487 (2012)
6. Ma, S.-M., Yeh, Y.-N.: Stirling permutations, cycle structure of permutations and perfect matchings. Electr. J. Combin. **22**(4), 4.42 (2015)
7. Ma, S.-M., Ma, J., Yeh, Y.-N.: The ascent-plateau statistics on Stirling permutations. Electr. J. Combin. **26**(2), P2.5 (2019)
8. Ma, S.-M., Yeh, Y.-N.: Eulerian permutations, Stirling permutations of the second kind and perfect matchings. Electr. J. Combin. **24**(4), 4.27 (2017)
9. Park, S.K.: The r-multipermutations. J. Combin. Theory Ser. A **67**, 44–71 (1994)
10. Remmel, J.B., Wilson, A.T.: Block patterns in Stirling permutations (2014). arXiv:1402.3358
11. Yanagimoto, T., Okamoto, M.: Partial orderings of permutations and monotonicity of a rank correlation statistic. Ann. Inst. Stat. Math. **21**, 489–506 (1969)