



On Restricted Colorings of (d,s) -Edge Colorable Graphs

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Abstract

A cycle is *2-colored* if its edges are properly colored by two distinct colors. A (d, s) -*edge colorable graph* G is a d -regular graph that admits a proper d -edge coloring in which every edge of G is in at least $s - 1$ 2-colored 4-cycles. Given a (d, s) -edge colorable graph G and a list assignment L of forbidden colors for the edges of G satisfying certain sparsity conditions, we prove that there is a proper d -edge coloring of G that avoids L , that is, a proper edge coloring φ of G such that $\varphi(e) \notin L(e)$ for every edge e of G . Additionally, this paper also contains a discussion of graphs belonging to the family of (d, s) -edge colorable graphs.

Keywords (d, s) -Edge colorable graph

1 Introduction

A graph G is *k -edge list colorable* or *k -edge choosable* if for every assignment of lists of at least k colors to the edges of G , there is a proper edge coloring of G using only colors from the lists. The *list chromatic index* or *edge choosability* $\chi'_l(G)$ of a graph G is the minimum number k such that G is k -edge list colorable. The most famous conjecture about list coloring states that $\chi'_l(G) = \chi'(G)$, where $\chi'(G)$ is the chromatic index of G , referring to the smallest number of colors needed to color the edges of G to obtain a proper edge coloring. In 1994, Galvin [1] proved this conjecture for bipartite multigraphs, his result also answers a question of Dinitz (1979) [2] about a generalization of Latin squares which can be formulated as a result of list edge coloring of the complete bipartite graph $K_{d,d}$, that $\chi'_l(K_{d,d}) = d$.

Meanwhile, Häggkvist [3] worked with sparser lists. His conjecture of avoiding arrays can be rewritten in the language of graph theory to state that there exists a fixed $0 < \beta \leq \frac{1}{3}$ such that if each edge e of $K_{d,d}$ is assigned a list $L(e)$ of at most βd

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colors from $\{1, \dots, d\}$ and at every vertex v each color is forbidden on at most βd edges adjacent to v , then there is a proper d -edge coloring φ of $K_{d,d}$ that *avoids* the lists, i.e., $\varphi(e) \notin L(e)$ for every edge e of $K_{d,d}$; if such a coloring exists, then L is *avoidable*. For the case when d is a power of two, Andr en proved that such a β exists in [4]; the full conjecture was later settled in the affirmative in [5].

Casselgren et al. [6] demonstrated that a similar result holds for the family of hypercube graphs. A benefit of working with the complete bipartite graph $K_{d,d}$ ($d = 2^t$, $t \in \mathbb{N}$) and the d -dimensional hypercube graph Q_d ($d \in \mathbb{N}$) is that they are both regular graphs that have proper edge colorings in which every edge is in $(d - 1)$ 2-colored 4-cycles. The purpose of this paper is to study this type of problem for regular graphs where the number of 2-colored 4-cycles each edge is contained in can be smaller.

To be more specific, we consider the family of (d, s) -edge colorable graphs:

Definition 1 A d -regular graph G is called (d, s) -edge colorable if it admits a proper d -edge coloring in which every edge of G is contained in at least $(s - 1)$ 2-colored 4-cycles.

Remark For a (d, s) -edge colorable graph G , let \mathcal{H} be the set of all proper d -edge colorings that satisfy the condition in Definition 1. Since $|\mathcal{H}| \geq 1$, we can pick one proper d -edge coloring $h \in \mathcal{H}$ (h can be chosen arbitrarily) and let it be our *standard coloring*. A *standard matching* M of G is a maximum set of edges of G all of which have the same color in the standard coloring h .

The *distance* between two edges e and e' is the number of edges in a shortest path between an endpoint of e and an endpoint of e' . The t -neighborhood of an edge e is the graph induced by all edges of distance at most t from e .

Note that since the number of 2-colored 4-cycles containing an edge e of G is at most $(d - 1)$, s can not exceed d . Throughout, we shall assume that the standard coloring h for the edges of G uses the set of colors $\{1, \dots, d\}$. Next, similarly to [6], using the colors $\{1, \dots, d\}$ we define a β -sparse list assignment for the edges of a (d, s) -edge colorable graph.

Definition 2 A list assignment L for a (d, s) -edge colorable graph G is β -sparse if the list of each edge is a (possibly empty) subset of $\{1, \dots, d\}$, and

- (i) $|L(e)| \leq \beta s$ for each edge $e \in G$;
- (ii) for every vertex $v \in V(G)$, each color in $\{1, \dots, d\}$ occurs in at most βs lists of edges incident to v ;
- (iii) for every 6-neighborhood W of G , and every standard matching M of G , any color appears at most βs times in lists of edges of M contained in W .

The 6-neighborhood used in Definition 2 is solely due to proof technical reasons. The neighborhood size might be decreased, but this seems to be out of reach with current method. We can now formulate our main result.

Theorem 1 For any positive integers n, d, s such that $s \geq 11$ and any positive β such that $\beta \leq 2^{-11}sd^{-1}(2n)^{-2^9ds^{-2}}$; if G is a (d, s) -edge colorable graph of order n then any β -sparse list assignment L for G is avoidable.

The rest of the paper is organized as follows. In Sect. 2, after introducing some terminology and notation, we prove Theorem 1. Section 3 contains several corollaries that are deduced directly from the proof of Theorem 1. In Sect. 4, we give some examples of classes of graphs that belong to the family of (d, s) -edge colorable graphs.

2 Proof the Main Theorem

Given a (d, s) -edge colorable graph G of order n , for a vertex $u \in G$, we denote by E_u the set of edges with one endpoint being u , and for a (partial) edge coloring f of G , let $f(u)$ denote the set of colors on edges in E_u under f . If two edges uv and zt of G are in a 2-colored 4-cycle in G then the edges uv and zt are *parallel*.

Given a proper coloring h' of the edges of G , for an edge $e \in G$, any edge $e' \in G$ ($e' \neq e$) belongs to at most one 2-colored 4-cycle containing e . This property is obvious if e' and e are not adjacent; in the case when they have the same endpoint u , assume $e = uv$, $e' = uv'$ and $uvv_1v'u$ and $uvv_2v'u$ are two 2-colored 4-cycles containing e and e' ; then $h'(vv_1) = h'(vv_2) = h'(uv')$, a contradiction since vv_1 and vv_2 are adjacent.

Consider a β -sparse list assignment L for G and a proper edge coloring ϕ of G . An edge e of G is called a *conflict edge* (of ϕ with respect to L) if $\phi(e) \in L(e)$. An *allowed cycle* (under ϕ with respect to L) of G is a 4-cycle $C = uvztu$ in G that is 2-colored under ϕ , and such that interchanging colors on C yields a proper d -edge coloring ϕ' of G where none of uv, vz, zt, tu is a conflict edge. We call such an interchange a *swap* in ϕ .

In the following, G is a (d, s) -edge colorable graph of order n , L is a β -sparse list assignment for G , and h is the standard coloring of G . For simplicity of notation, we shall omit floor and ceiling signs whenever these are not crucial. Below we outline the proof of Theorem 1.

Step I. We prove that there exists a permutation ρ of the elements of the set $\{1, \dots, d\}$ such that in the proper d -edge coloring h' obtained by applying ρ to the colors used in h , locally, each standard matching in G contains “sufficiently few” conflict edges with L , and each vertex u of G satisfies that E_u contains “sufficiently few” conflict edges, and that each edge of G belongs to “many” allowed cycles. These conditions shall be more precisely articulated below.

Step II. We find a set of allowed cycles in h' such that each conflict edge belongs to one of them, with no two of the cycles intersecting, and swap on those cycles to obtain a proper d -edge coloring h'' of G which avoids L .

In the proof we shall verify that it is possible to perform Steps I–II described above to obtain a proper d -edge coloring of G that avoids L . This is done by proving a lemma in each step.

Step I We use Lemma 1 to prove that there exists a permutation ρ such that applying ρ to the colors used in h , we obtain a required proper d -edge coloring h' .

Lemma 1 *Let $0 < \gamma, \tau < 1$ be parameters such that $\beta \leq \gamma$ and*

$$n \frac{(\beta s)^{\gamma s}}{(\gamma s)!} + \frac{nd (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau - 2\beta)s)!} < 1 \tag{1}$$

Then there is a permutation ρ of $\{1, \dots, d\}$, such that applying ρ to the set of colors $\{1, \dots, d\}$ used in h , we obtain a proper d -edge coloring h' of G satisfying the following:

- (a) *For every 6-neighborhood W of G , and every standard matching M of G , at most γs edges of $M \cap E(W)$ are conflict edges.*
- (b) *For each vertex u in G , E_u contains at most γs conflict edges.*
- (c) *Each edge in G belongs to at least $(1 - \tau)s$ allowed cycles.*

Proof Let A, B, C be the number of permutations which do not fulfill the conditions (a), (b), (c), respectively. Let X be the number of permutations satisfying the three conditions (a), (b), (c). There are $d!$ ways to permute the colors, so we have

$$X \geq d! - A - B - C$$

We will now prove that X is greater than 0.

Let's start by counting A . Since all edges that are in the same standard matching have the same color under h and for every 6-neighborhood W of G , and every standard matching M of G , any color appears at most βs times in lists of edges of M contained in W , we have that the maximum number of conflict edges in a subset of a given standard matching contained in a 6-neighborhood is βs .

Since $\gamma \geq \beta$, this means that all permutations satisfy condition (a); so $A = 0$.

To estimate B , let u be a fixed vertex of G , and let P be a set of size γs ($|P| = \gamma s$) of edges from E_u . For a vertex v adjacent to u , if uv is a conflict edge, then the colors used in h should be permuted in such a way that in the resulting coloring h' , the color of uv is in $L(uv)$. Since $|L(uv)| \leq \beta s$, there are at most $(\beta s)^{\gamma s}$ ways to choose which colors from $\{1, 2, \dots, d\}$ to assign to the edges in P so that all edges in P are conflict. The rest of the colors can be arranged in any of the $(d - \gamma s)!$ possible ways. In total this gives at most

$$\binom{d}{\gamma s} (\beta s)^{\gamma s} (d - \gamma s)! = \frac{d! (\beta s)^{\gamma s}}{(\gamma s)!}$$

permutations that do not satisfy condition (b) on vertex u . There are n vertices in G , so we have

$$B \leq n \frac{d!(\beta s)^{\gamma s}}{(\gamma s)!}$$

To estimate C , let uv be a fixed edge of G . Each 2-colored 4-cycle $\mathcal{C} = uvztu$ containing uv is uniquely defined by an edge zt which is parallel with uv . Moreover, a permutation ζ contributes to C if and only if there are at least τs choices for zt so that C is not allowed. We shall count the number of ways ζ could be constructed for this to happen. First, note that for each choice of a color c_1 from $\{1, \dots, d\}$, for the standard matching which contains uv , there are up to $2\beta s$ cycles that are not allowed because of this choice. This follows from the fact that there are at most βs choices for t (or z) such that $L(ut)$ (or $L(vz)$) contains c_1 . So for a permutation ζ to contribute to C , ζ must satisfy that at least $(\tau - 2\beta)s$ cycles containing uv are forbidden because of the color assigned to the standard matching containing ut and vz .

Let C_{uv} be the set of edges that are parallel with uv . It is obvious that $s - 1 \leq |C_{uv}| \leq d - 1$. Let $S \subseteq C_{uv}$, $|S| = (\tau - 2\beta)s$, such that for every edge $zt \in S$, the 2-colored 4-cycle $\mathcal{C} = uvztu$ is not allowed because of the color assigned to ut and vz . There are at most $\binom{d-1}{(\tau-2\beta)s}$ ways to choose S . Furthermore, $L(uv)$ and $L(zt)$ contain at most βs colors each, so there are at most $2\beta s$ choices for a color for the standard matching containing ut and vz that would make \mathcal{C} disallowed because of the color assigned to this standard matching. The remaining colors can be permuted in $(d - 1 - (\tau - 2\beta)s)!$ ways. Thus, the total number of permutations σ with not enough allowed cycles for a given edge is bounded from above by

$$d \binom{d - 1}{(\tau - 2\beta)s} (2\beta s)^{(\tau-2\beta)s} (d - 1 - (\tau - 2\beta)s)!$$

Notice that any d -regular graph G of order n has $\frac{nd}{2}$ edges, so the total number of permutations σ that have too few allowed cycles for at least one edge is bounded from above by

$$\begin{aligned} C &\leq \frac{nd}{2} d \binom{d - 1}{(\tau - 2\beta)s} (2\beta s)^{(\tau-2\beta)s} (d - 1 - (\tau - 2\beta)s)! \\ &= \frac{nd d! (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau - 2\beta)s)!} \end{aligned}$$

Hence,

$$\begin{aligned} X &\geq d! - n \frac{d!(\beta s)^{\gamma s}}{(\gamma s)!} - \frac{nd d! (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau - 2\beta)s)!} \\ &= d! \left(1 - n \frac{(\beta s)^{\gamma s}}{(\gamma s)!} - \frac{nd (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau - 2\beta)s)!} \right) \end{aligned}$$

By assumption, we now deduce that $X > 0$. □

Step II We use Lemma 2 to prove that by performing a sequence of swaps on disjoint allowed 2-colored 4-cycles in h' , we obtain a proper d -edge coloring h'' of G which avoids L .

Lemma 2 *Let h' be a proper d -edge coloring satisfying conditions (a), (b), (c) of Lemma 1 and γ, τ be parameters satisfying condition of Lemma 1. If there exists ϵ such that $0 < \epsilon < 1$ and*

$$s - \tau s - 9\gamma s - 3\epsilon s - \frac{20\gamma}{\epsilon}d - 3 > 0 \quad (2)$$

Then, by performing a sequence of swaps on disjoint allowed 2-colored 4-cycles in h' , we obtain a proper d -edge coloring h'' of G which avoids L .

Proof For constructing h'' from h' , we will perform a number of swaps on G , and we shall refer to this procedure as P -swap. We are going to construct a set P of disjoint allowed 2-colored 4-cycles such that each conflict of h' with L belongs to one of them. An edge that belongs to a 2-colored 4-cycle in P is called *used* in P -swap. Suppose we have included a 2-colored 4-cycle C in P . Since for every 6-neighborhood W of G , and every standard matching M of G , the number of conflict edges in $M \cap E(W)$ is not greater than γs , for every 5-neighborhood W of G , the total number of edges in W that are used in P -swap is at most $4\gamma ds$. A vertex u in G is P -overloaded if E_u contains at least ϵs edges that are used in P -swap; note that each used edge is incident to two vertices, thus no more than $\frac{2 \times 4\gamma ds}{\epsilon s} = \frac{8\gamma}{\epsilon}d$ vertices of each 4-neighborhood are P -overloaded. A standard matching M in G is P -overloaded in a t -neighborhood W if $M \cap E(W)$ contains at least ϵs edges that are used in P -swap; note that for each 5-neighborhood W , no more than $\frac{4\gamma}{\epsilon}d$ standard matchings of G are P -overloaded in W .

Using these facts, let us now construct our set P by steps; at each step we consider a conflict edge e and include an allowed 2-colored 4-cycle containing e in P . Initially, the set P is empty. Next, for each conflict edge $e = uv$ in G , there are at least $s - \tau s$ allowed cycles containing e . We choose an allowed cycle $uvztu$ which contains e and satisfies the following:

- (1) z and t and the standard matching that contains vz and ut are not P -overloaded in the 4-neighborhood W_e of e ; this eliminates at most $\frac{2 \times 8\gamma}{\epsilon}d + \frac{4\gamma}{\epsilon}d = \frac{20\gamma}{\epsilon}d$ choices. Note that with this strategy for including 4-cycles in P , after completing the construction of P , every vertex is incident with at most $2\gamma s + (\epsilon s - 1) + 2 = 2\gamma s + \epsilon s + 1$ edges that are used in P -swap. Furthermore, after we have constructed the set P , no standard matching contains more than $2\gamma s + \epsilon s + 1$ edges that are used in S -swap in a 1-neighborhood of G ; this follows from the fact that every 1-neighborhood W' in G that ut, vz or zt belongs to is contained in W_e .

- (2) None of the edges vz, zt, ut are conflict, or used before in P -swap. All possible choices for these edges are in the 1-neighborhood W_e of e in G . Since no vertex in W_e or subset of a standard matching that is in W_e contains more than γs conflict edges and P -swap uses at most $2\gamma s + \epsilon s + 1$ edges at each vertex and in each subset of a standard matching contained in W_e , these restrictions eliminate at most $3\gamma s + 3(2\gamma s + \epsilon s + 1)$ or $9\gamma s + 3\epsilon s + 3$ choices.

It follows that we have at least

$$s - \tau s - 9\gamma s - 3\epsilon s - \frac{20\gamma}{\epsilon}d - 3$$

choices for an allowed cycle $uvztu$ which contains uv . By assumption, this expression is greater than zero, so we conclude that there is a cycle satisfying these conditions, and thus we may construct the set P by iteratively adding disjoint allowed 2-colored 4-cycles such that each cycle contains a conflict edge. After this process terminates we have a set P of disjoint allowed cycles; we swap on all the cycles in P to obtain a coloring h'' which avoids L . \square

We can now prove Theorem 1.

Proof of Theorem 1 Firstly, using Stirling’s approximation $x! \geq x^x e^{-x}$, we have

$$\begin{aligned} & n \frac{(\beta s)^{\gamma s}}{(\gamma s)!} + \frac{nd (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau-2\beta)s)!} \\ & \leq n \frac{e^{\gamma s} (\beta s)^{\gamma s}}{(\gamma s)^{\gamma s}} + \frac{nd e^{(\tau-2\beta)s} (2\beta s)^{(\tau-2\beta)s}}{2 ((\tau-2\beta)s)^{(\tau-2\beta)s}} \\ & \leq n \left(\frac{e\beta}{\gamma}\right)^{\gamma s} + \frac{nd}{2} \left(\frac{2e\beta}{\tau-2\beta}\right)^{(\tau-2\beta)s}. \end{aligned}$$

By assumption, we have $\beta \leq 2^{-11}sd^{-1}(2n)^{-2^9ds^{-2}}$. Since $s \leq d$ and $(2n)^{-2^9ds^{-2}} \leq 1$, it follows that $\beta \leq 2^{-11}sd^{-1} \leq 2^{-11}$. Let $\gamma = 2^{-9}sd^{-1}$, then $\beta \leq \gamma$ and

$$n \left(\frac{e\beta}{\gamma}\right)^{\gamma s} < n \left(\frac{2^2 2^{-11}sd^{-1}(2n)^{-2^9ds^{-2}}}{2^{-9}sd^{-1}}\right)^{2^{-9}sd^{-1}s} = n(2n)^{-1} = \frac{1}{2}. \tag{3}$$

Let $\tau = 2^{-7}$, then $\tau - 2\beta \geq 2^{-7} - 2 \cdot 2^{-11} > 2^{-8}$ and

$$\frac{\tau - 2\beta}{2e\beta} > \frac{2^{-8}}{2e2^{-11}sd^{-1}(2n)^{-2^9ds^{-2}}} > \frac{1}{(2n)^{-2^9ds^{-2}}} > (2n)^{2^9ds^{-2}} > 1$$

This implies

$$\left(\frac{\tau - 2\beta}{2e\beta}\right)^{(\tau-2\beta)s} > \left(\frac{\tau - 2\beta}{2e\beta}\right)^{2^{-8}s} > \left((2n)^{2^9ds^{-2}}\right)^{2^{-8}s} = (2n)^{2ds^{-1}} > (2n)^2$$

Using the fact that $d < n$, we have

$$\frac{nd}{2} \left(\frac{2e\beta}{\tau - 2\beta} \right)^{(\tau-2\beta)s} < \frac{nd}{2} \frac{1}{(2n)^2} < \frac{1}{8}. \tag{4}$$

Combining (3) and (4), we obtain $n \left(\frac{e\beta}{\gamma} \right)^{\gamma s} + \frac{nd}{2} \left(\frac{2e\beta}{\tau - 2\beta} \right)^{(\tau-2\beta)s} < 1$. This implies $n \frac{(\beta s)^{\gamma s}}{(\gamma s)!} + \frac{nd}{2} \frac{(2\beta s)^{(\tau-2\beta)s}}{((\tau - 2\beta)s)!} < 1$. Since the values of γ, τ satisfy the conditions in Lemma 1, there is a permutation ρ of the colors in the standard d -edge coloring h of G from which we obtain a proper d -edge coloring h' of G satisfying the conditions (a), (b), (c) in Lemma 1. Furthermore, since $d \geq s \geq 11$, if we let $\epsilon = 2^{-3}$, then we have

$$s - \tau s - 9\gamma s - 3\epsilon s - \frac{20\gamma}{\epsilon} d - 3 > 0.$$

It follows from Lemma 2 that there exists a proper d -edge coloring h'' of G which avoids L .

Remark

- (1) If $2^{-11}sd^{-1}(2n)^{-2^9 ds^{-2}} < 1/s$, then $\beta s < 1$, Theorem 1 becomes obvious. So our result is only meaningful if $2^{-11}sd^{-1}(2n)^{-2^9 ds^{-2}} \geq 1/s$.
- (2) Our proof relies heavily on the fact that every edge is contained in a large number of 2-colored 4-cycles. It would be interesting to investigate if a similar result holds for graphs containing a certain amount of 2-colored $2c$ -cycles ($c \in \mathbb{N}, c > 2$).

□

3 Corollaries

The following corollaries are deduced directly from the proof of Theorem 1.

Corollary 1 *For any positive integers n, d and any constant κ such that $11 \leq \kappa d \leq d$, if G is a $(d, \kappa d)$ -edge colorable graph G of order n then there exists positive constants c_1, c_2 such that for any $\beta \leq c_1(2n)^{-c_2 d^{-1}}$, any β -sparse list assignment L for G is avoidable.*

Corollary 2 *For any positive integers n, d and any constants c, κ such that $11 \leq \kappa d \leq d$ and $d \geq c \log n$, if G is a $(d, \kappa d)$ -edge colorable graph G of order n then there exists a positive constant β such that any β -sparse list assignment L for G is avoidable.*

Note that the complete bipartite graph $K_{d,d}$ ($d = 2^t, t \in \mathbb{N}$) and the d -dimensional hypercube graph Q_d ($d \in \mathbb{N}$) are both (d, d) -edge colorable graphs satisfying the

condition in Corollary 2. Thus this corollary generalizes the results in [4–6]. The next corollary examines the condition on d and s so that for every (d, s) -edge colorable graph G of order n and any β -sparse list assignment L for G satisfying that the length of every list in L is constant, there is a proper d -edge coloring of G which avoids L .

Corollary 3 *For any positive integers n, d, s and any constant c such that $s \geq 11$ and $\frac{1}{2}(2^{-11}c^{-1}s^2d^{-1})^{2^{-9}d^{-1}s^2} \geq n$, if G is a (d, s) -edge colorable graph G of order n then any $\frac{c}{s}$ -sparse list assignment L for G is avoidable.*

If the length of every list in L is bounded by a power of s , we have a slightly different condition as follows.

Corollary 4 *For any positive integers n, d, s and any constant c such that $s \geq 11$ and $\frac{1}{2}(2^{-11}s^{2-c}d^{-1})^{2^{-9}d^{-1}s^2} \geq n$, if G is a (d, s) -edge colorable graph G of order n then any s^{c-1} -sparse list assignment L for G is avoidable.*

A distance- t matching is a matching where any two edges are at distance at least t from each other. Consider an arbitrary list assignment L' (not necessarily β -sparse) for a (d, s) -edge colorable graph. In general, it is difficult to determine if L' is avoidable or not. However, if the edges with forbidden lists are placed on a distance-3 matching, our method in fact immediately yields the following.

Corollary 5 *Let L' be a list assignment for the edges of a (d, s) -edge colorable graph G such that for each edge e of G , $L'(e) \leq s - 1$. If every edge e satisfying $L'(e) \neq \emptyset$ belongs to a distance-3 matching in G , then L' is avoidable.*

4 Families of (d, s) -Edge Colorable Graphs

The results in [3] and [4] allow us to conclude that the complete bipartite graph $K_{d,d}$ ($d = 2^t, t \in \mathbb{N}$) and the hypercube graph Q_d ($d \in \mathbb{N}$) are (d, d) -edge colorable graphs. Compared to the hypercube graph Q_d ($d \in \mathbb{N}$), the complete bipartite graph $K_{d,d}$ ($d = 2^t, t \in \mathbb{N}$) is much denser, so it is interesting to see how this complete bipartite graph behaves if some edges are removed. Lemma 3 considers the case when we take a set of k standard matchings out of $K_{d,d}$ ($d = 2^t, t \in \mathbb{N}$).

Lemma 3 *The graph G obtained by removing k standard matchings from the complete bipartite graph $K_{d,d}$ ($d = 2^t, t \in \mathbb{N}$) is a $(d - k, d - k)$ -edge colorable graph.*

Proof It is straightforward that G is a $(d - k)$ -regular graph. Let h be the standard coloring of $K_{d,d}$ such that all edges in a standard matching of $K_{d,d}$ receive the same color in h . We define the proper $(d - k)$ -edge coloring h' of G from h by retaining the color of every non-deleted edge in $E(G)$. For any edge e of G , removing one standard matching from $K_{d,d}$ eliminates one 2-colored 4-cycle that contains e . Hence, the number of 2-colored 4-cycles containing e is $d - 1 - k$, which implies that G is a $(d - k, d - k)$ -edge colorable graph. \square

Recall that the Cartesian product $G = G_1 \square G_2$ of the graphs G_1 and G_2 is a graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and where two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in G whenever $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2 , or $u_2 = v_2$ and u_1 is adjacent with v_1 in G_1 . The following lemma concerns the (d, s) -edge colorability of Cartesian products of graphs.

Lemma 4 *Let G_1 be a (d_1, s_1) -edge colorable graph and G_2 be a (d_2, s_2) -edge colorable graph. Then $G = G_1 \square G_2$ of graphs G_1 and G_2 is a (d, s) -edge colorable graph with $d = d_1 + d_2$ and $s = \min\{d_1 + s_2, d_2 + s_1\}$.*

Proof By the definition of the Cartesian product of graphs, it is straightforward that G is d -regular graph with $d = d_1 + d_2$. Let h_1 be the standard coloring of G_1 and h_2 be the standard coloring of G_2 such that the set of colors in h_1 and the set of colors in h_2 are disjoint. We define an edge coloring h of G : for two adjacent vertices $u = (u_1, u_2)$ and $v = (u_1, v_2)$ in G , the edge uv is given the color $h(uv) = h_2(u_2v_2)$ and for two adjacent vertices $u = (u_1, u_2)$ and $v = (v_1, u_2)$ in G , the edge uv is given the color $h(uv) = h_1(u_1v_1)$. Thus h is a proper d -edge coloring of G .

Note that an edge uv of G with $u = (u_1, u_2)$ and $v = (u_1, v_2)$ is contained in a 2-colored 4-cycle $uvztu$ with $z = (u_i, v_2)$ and $t = (u_i, u_2)$ (u_i is neighbour of u_1 in G_1). Furthermore, if $u_2v_2z_2t_2u_2$ is a 2-colored 4-cycle in G_2 , then $uvz't'u$ with $z' = (u_1, z_2)$ and $t' = (u_1, t_2)$ is a 2-colored 4-cycle in G . Since the degree of u_1 is d_1 and every edge of G_2 is in at least $s_2 - 1$ 2-colored 4-cycles, an edge uv of G with $u = (u_1, u_2)$ and $v = (u_1, v_2)$ belongs to at least $d_1 + s_2 - 1$ 2-colored 4-cycles. Similarly, an edge uv of G with $u = (u_1, u_2)$ and $v = (v_1, u_2)$ belongs to at least $d_2 + s_1 - 1$ 2-colored 4-cycles. Therefore, we can conclude that G is a (d, s) -edge colorable graph with $d = d_1 + d_2$ and $s = \min\{d_1 + s_2, d_2 + s_1\}$. \square

In the remaining part of this section, we examine some other graphs that belong to the family of (d, s) -edge colorable graphs.

Let G be a finite group and let S be a generating set of G such that S does not contain the identity element e , $|S| = d$ and $S = S^{-1}$ (which means if $a \in S$ then $a^{-1} \in S$). The undirected Cayley graph $\text{Cay}(G, S)$ over the set S is defined as the graph whose vertex set is G and where two vertices $a, b \in G$ are adjacent whenever $\{ab^{-1}, ba^{-1}\} \subseteq S$. It is straightforward that $\text{Cay}(G, S)$ is a d -regular graph, Lemmas 5 and 6 show that if S satisfies some further conditions then $\text{Cay}(G, S)$ is a (d, s) -edge colorable graph.

Lemma 5 *Let $\text{Cay}(G, S)$ be an undirected Cayley graph on a group G over the generating set $S \subseteq G \setminus \{e\}$. If $a = a^{-1}$ for every $a \in S$, $|S| = d$ and there exists a subset $S_c \subseteq S$, $|S_c| = s$, satisfying that every element of S_c is commutative with all elements in S , then $\text{Cay}(G, S)$ is a (d, s) -edge colorable graph.*

Proof Let h be the proper d -edge coloring of $\text{Cay}(G, S)$ such that every edge uv in $\text{Cay}(G, S)$ is colored a if $uv^{-1} = vu^{-1} = a \in S$. For an edge uv colored a , consider an arbitrary element $b \in S_c$ and let $z = vb$, $t = ub$, then the edges vz and ut are colored b in h . Furthermore, since b is commutative with a , i.e. $ab = ba$, we have $z = vb = uab = uba = ta$. This implies that there is an edge between z and t , and

this edge is colored a in h . Hence, $uvztu$ is a 2-colored 4-cycle. Because $|S_c| = s$, each edge of $\text{Cay}(G, S)$ is in at least $s - 1$ 2-colored 4-cycles. It follows that $\text{Cay}(G, S)$ is a (d, s) -edge colorable graph.

Lemma 6 *Let $\text{Cay}(G, S)$ be an undirected Cayley graph on an Abelian group G over the generating set $S \subseteq G \setminus \{e\}$, $S = S^{-1}$ and $|S| = d$. Let $S_k = \{s_1, s_2, \dots, s_k\}$ be a subset of S such that $S_k \cup S_k^{-1} = S$ and S_k does not contain two different elements $s_i \neq s_j$ satisfying that $s_i = s_j^{-1}$. If S has the following properties:*

- (i) every element s_i of S_k has even order d_i ($s_i^{d_i} = s_i^0 = e$);
- (ii) for every element $g \in G$, there is exactly one sequence (x_1, x_2, \dots, x_k) ($x_i \in [0, d_i - 1]$ for $i \in [1, k]$) such that $g = s_1^{x_1} s_2^{x_2} \dots s_k^{x_k}$;

then $\text{Cay}(G, S)$ is a (d, d) -edge colorable graph.

Proof We write $S = \{s_1, \dots, s_k, s_1^{-1}, \dots, s_k^{-1}\}$; note that the size of S may not be $2k$, since S may contain some element s_x with $s_x = s_x^{-1}$. Consider an edge uv of $\text{Cay}(G, S)$; without loss of generality assume that $v = us_i$ ($u = vs_i^{-1}$) for some $i \in [1, k]$. The condition (ii) implies that there exists exactly one sequence (x_1, x_2, \dots, x_k) such that $u = s_1^{x_1} \dots s_i^{x_i} \dots s_k^{x_k}$ and $v = s_1^{x_1} \dots s_i^{(x_i+1) \bmod d_i} \dots s_k^{x_k}$. We color the edge uv by color s_i if x_i is even, and by color s_i^{-1} if x_i is odd. By repeating this for all edges of $\text{Cay}(G, S)$, we obtain the proper d -edge coloring h .

Given an edge e of $\text{Cay}(G, S)$, let u and v be the two endpoints of e , where $v = us_i$ (for some $i \in [1, k]$). Let $u = s_1^{x_1} \dots s_i^{x_i} \dots s_k^{x_k}$; x_i is called the *power of s_i in u* ($i \in [1, k]$) and denoted by $p_u(s_i)$. Consider an arbitrary element $s \in S$, if $s = s_j \in S_k$, let $z = vs_j$, $t = us_j$, then $h(vz) = h(ut) = s_j$ or $h(vz) = h(ut) = s_j^{-1}$ since $p_u(s_j) = p_v(s_j)$ and $p_t(s_j) = p_z(s_j) = p_u(s_j) + 1$. Furthermore, since G is an Abelian group, we have $z = vs_j = us_i s_j = us_j s_i = ts_i$. Thus there is an edge between z and t , and $h(uv) = h(tz)$ since $p_u(s_i) = p_t(s_i)$ and $p_v(s_i) = p_z(s_i) = p_u(s_i) + 1$; hence $uvztu$ is a 2-colored 4-cycle. If $s \in S_k^{-1}$, we proceed similarly. Because $|S| = d$, each edge of $\text{Cay}(G, S)$ is in at least $d - 1$ 2-colored 4-cycles. It follows that $\text{Cay}(G, S)$ is a (d, d) -edge colorable graph. □

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