



N -Flips in Triangulations with Two Odd Degree Vertices

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Abstract

In this paper, we prove that any two triangulations G and G' on the sphere with exactly two odd degree vertices can be transformed into each other by two local transformations, called an N -flip and a P_2 -flip, preserving the parity of degree of each vertex, if $|V(G)| = |V(G')|$. This is an analogy of the same result for triangulations with each vertex even degree [7], but we prove such a fact does not hold for triangulations with at least four odd degree vertices.

Keywords Even triangulations · Surface · Transformation

1 Introduction

A *triangulation* G on a closed surface F^2 is a fixed embedding of a simple graph on F^2 such that each face of G is bounded by a 3-cycle, where a k -cycle means a cycle of length k . A k -vertex is a vertex of degree k and an *even* (resp., *odd*) vertex is a vertex of even (resp., odd) degree. For a vertex v of G , the *link* of v is the boundary cycle of the 2-cell region formed by all faces incident to v . A k -region is a 2-cell region bounded by a closed walk of length k . A face (or a region) R of a graph G is *even* (resp., *odd*) if R is bounded by a closed walk of even (resp., odd) length.

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Let $v_1v_2v_3$ and $v_1v_3v_4$ be two triangular faces of a triangulation G sharing an edge v_1v_3 . A *diagonal flip* of v_1v_3 is to replace the diagonal v_1v_3 with v_2v_4 in the quadrilateral $v_1v_2v_3v_4$, as shown in Fig. 1. When the transformation breaks the simpleness of G , we do not apply it.

Wagner proved that any two triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of diagonal flips [13]. Moreover, Negami has extended this result to all closed surfaces and proved that any two triangulations on the same closed surface can be transformed into each other by diagonal flips if they have the same and sufficiently large number of vertices [9]. For related results, see the survey [10].

Observe that a diagonal flip changes the parity of degree of vertices. Hence we consider the following transformation, called an N -flip, preserving the parity of degree of each vertex in a triangulation. This was first defined by Nakamoto et al. in [7].

Let G be a triangulation. Suppose that G has a hexagonal region bounded by a cycle $v_1v_2v_3v_4v_5v_6$ with edges v_1v_5, v_2v_4, v_2v_5 and there is no other vertices and edges in the region. The N -flip of the path $v_1v_5v_2v_4$ is to remove v_1v_5, v_2v_4, v_2v_5 and add v_1v_3, v_3v_6, v_4v_6 , as shown in Fig. 2. When the transformation breaks the simpleness of G , we do not apply it. Two triangulations G and G' are N -equivalent, denoted by $G \sim_N G'$, if G and G' can be transformed into each other by a sequence of N -flips.

A triangulation G is *even* if each vertex of G has even degree. Since an even triangulation G on the sphere has a unique 3-coloring [12], $V(G)$ can uniquely be decomposed into three partite sets $V_R(G), V_B(G)$ and $V_Y(G)$ colored by red, blue and yellow, respectively. We call $(|V_R(G)|, |V_B(G)|, |V_Y(G)|)$ the *tripartition size* of G . An N -flip always preserves the tripartition of a triangulation, as shown in Fig. 2. Hence if G and G' are N -equivalent, then they have the same tripartition size. Nakamoto et al. [7] proved that the converse also holds.

Theorem 1 (Nakamoto et al. [7]) *Two even triangulations on the sphere can be transformed into each other by a sequence of N -flips if and only if they have the same tripartition size.*

If two even triangulations G and G' have the same number of vertices but distinct tripartition size, then G and G' cannot be transformed into each other only by N -flips. So consider another transformation changing the tripartition size. Let v be a vertex of G with the link $v_1 \dots v_k$ ($k \geq 3$). Put two vertices x, y on the edge vv_1 and join them to v_2 and v_k . The P_2 -flip of x, y is to move x, y to the edge vv_2 and join them to v_1 and v_3 , as shown in Fig. 3. Similarly to N -flips, if the transformation breaks the simpleness of G , we do not apply it. A P_2 -flip preserves the parity of

Fig. 1 The diagonal flip

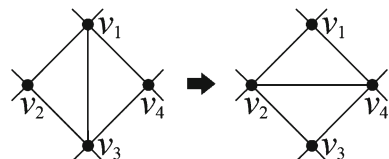


Fig. 2 The N -flip

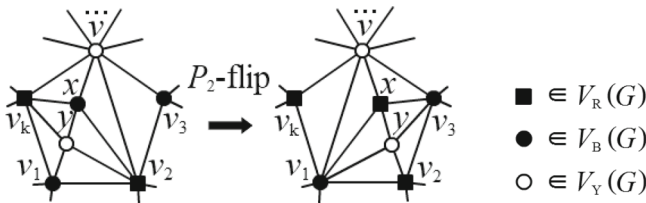
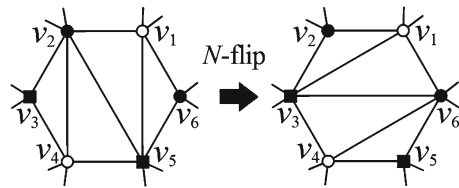


Fig. 3 The P_2 -flip

degree of each vertex, but it changes the tripartition size, as we can see that in Fig. 3. Then by using P_2 -flips in addition to N -flips, they proved the following theorem:

Theorem 2 (Nakamoto et al. [7]) *Any two even triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of N - and P_2 -flips.*

The N - and P_2 -flips in even triangulations have also been considered for non-spherical surfaces [5, 8]. Since every non-spherical surface admits non-3-colorable even triangulations, we can find two even triangulations with the same number of vertices which cannot be transformed by the two operations, that is, a 3-colorable one and a non-3-colorable one. We note that the N - and P_2 -flips preserve the 3-colorability of graphs. Moreover, the paper [5] has introduced an algebraic invariant, called the *monodromy*, such that if two even triangulations on a surface can be transformed by the two operations, then their invariants coincide [4, 5]. For other related researches, see [6] for example.

In this paper, we would like to consider whether two triangulations G and G' on the sphere can be transformed by N - and P_2 -flips if G and G' have the same number of vertices and the same number of odd degree vertices.

A k -odd triangulation is one with exactly k odd vertices. We note that k must be even by the Handshaking Lemma. In this paper, we prove the following result for 2-odd triangulations:

Theorem 3 *Any two 2-odd triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of N - and P_2 -flips.*

For even triangulations, the P_2 -flip is necessary to change the tripartition size. (Also the N -flip is necessary to transform an even triangulation on the sphere without adjacent 4-vertices into another.) However, since a 2-odd triangulation on

the sphere is not 3-colorable, what can we say about the necessity of P_2 -flips in 2-odd triangulations?

We prove the following theorem for 2-odd triangulations without P_2 -flips, introducing the *color factor* $U(G)$ of a 2-odd triangulation G , where the color factor of G is a unique vertex subset of G whose definition will be given in Sect. 2.

Theorem 4 *Let G and G' be 2-odd triangulations on the sphere with $|V(G)| = |V(G')|$ and $|U(G)| = |U(G')|$. Then G and G' can be transformed into each other only by N -flips.*

Theorems 3 and 4 are an analogy of Theorems 2 and 1 for 2-odd triangulations, respectively. However, we surprisingly give the following negative result for k -odd triangulations with any even integer $k \geq 4$.

Theorem 5 *For any even integer $k \geq 4$ and any integer $h \geq 1$, there exists a pair of k -odd triangulations G and G' on the sphere with $|V(G)| = |V(G')| \geq h$ which cannot be transformed into each other by N - and P_2 -flips.*

In Sect. 2, we define the notion “face subdivision” and “color factor” which plays an essential role, and using them, we prove Theorem 5. In Sect. 3, we establish a *generating theorem* for 2-odd triangulations on the sphere. In Sect. 4, we show several lemmas and give a proof of Theorem 3. Finally, in Sect. 5, we prove Theorem 4.

2 Face Subdivision and Color Factor

Let D be a simple graph 2-cell embedded on a surface F^2 . The *face subdivision* of D , denoted by $\text{FS}(D)$, is the triangulation on F^2 obtained from D by putting a single vertex into each face of D and joining it to all vertices on the corresponding boundary walk. The set of added vertices in $\text{FS}(D)$ is the *color factor* of $\text{FS}(D)$. It should be observed that each vertex of D has even degree in $\text{FS}(D)$, and hence no two odd vertices are adjacent in $\text{FS}(D)$. A triangulation G on F^2 is a *face subdivision* if there exists a graph D on F^2 such that $G = \text{FS}(D)$. In this case, D is the *frame* of G , and an edge $e \in E(G)$ is a *frame edge* (resp., a *spoke*) if $e \in E(D)$ (resp., $e \notin E(D)$).

The following proposition holds for 2-odd triangulations on the sphere.

Proposition 6 Every 2-odd triangulation on the sphere is a face subdivision with a unique frame.

The proof of Proposition 6 is postponed to the next section. Before it, we describe several facts obtained by Proposition 6, assuming that the proposition is true.

Let G be a 2-odd triangulation. Then, by Proposition 6, the color factor $U(G)$ can be taken in G uniquely. Moreover, we see that an N -flip always preserves $U(G)$ but a P_2 -flip does not, as shown in Fig. 4. Hence we have the following.

Proposition 7 *Let G be a triangulation on a surface F^2 , and let G' and G'' be triangulations obtained from G by a single N -flip and a single P_2 -flip, respectively.*

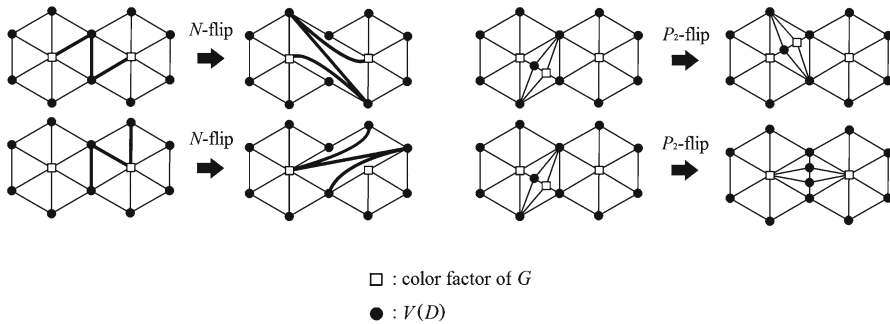


Fig. 4 N -flips and P_2 -flips in a face subdivision G

If G is a face subdivision, then so are both G' and G'' , and furthermore, $|U(G)| = |U(G')|$.

Proof Let v_1v_5, v_5v_2 and v_2v_4 be diagonals of a 6-region in G which are flipped by an N -flip, and let v_1v_3, v_3v_6 and v_6v_4 be the resulting diagonals (as in Fig. 2). Then, by the definition of the face subdivision, one of the following always holds:

- (i) Both v_1 and v_4 are color factors of G , or
- (ii) either v_2 or v_5 is a color factor of G .

Otherwise, G has a face consisting only of vertices which are not in the color factor, or G has adjacent two vertices in the color factor, a contradiction. Therefore, as shown in the left of Fig. 4, the proposition holds since any N -flip preserves $U(G)$. Similarly, we can prove that G'' is a face subdivision (see the right of Fig. 4). \square

By Proposition 7, we can prove Theorem 5.

Proof of Theorem 5 Let $k \geq 4$ be an even integer and let $h \geq 1$ be an integer. Let Q be a 3-connected quadrangulation (i.e., one with each face bounded by a 4-cycle) on the sphere with m vertices. Take $\frac{k}{2}$ distinct faces $f_1, \dots, f_{\frac{k}{2}}$ in Q such that $f_1 = xyzw$ shares the edge xy with f_2 , where we have $\frac{k}{2} \leq m - 2$ since Q has $m - 2$ faces by Euler’s formula. Put a single diagonal to f_i for $i \in \{1, \dots, \frac{k}{2}\}$ so that f_1 receives the diagonal xz . Then resulting graph Q' has exactly k triangular faces and all others quadrilateral. We note that Q' is simple. (For, if two added diagonals join the same pair of vertices, say p, q , then $Q - \{p, q\}$ is disconnected, contrary to the 3-connectedness of Q .) Let $G = FS(Q')$, which is a k -odd triangulation with $m + (m - 2 - \frac{k}{2}) + k = 2m + \frac{k}{2} - 2$ vertices. Taking m large, we may suppose that $|V(G)| \geq h$.

Let $a, b \in V(G)$ be two vertices of the color factor of G which are added to the triangular faces xyz and xzw , respectively. Let G' be the triangulation obtained from G by a single diagonal flip of the edge xz . Then G' is a k -odd triangulation, in which x and z are odd, a and b are even and the degree of other vertices are unchanged. Observe that x is adjacent to a 3-vertex in G' , say c , since two 3-vertices are added to two triangular faces of f_2 . Hence x and c are two neighboring odd vertices in G' , and

hence G' is never a face subdivision, by the observation in the first paragraph of this section. By Proposition 7, G and G' are k -odd triangulations with $|V(G)| = |V(G')| \geq h$ but cannot be transformed into each other by N - and P_2 -flips. \square

3 A Generating Theorem for 2-Odd Triangulations

In the rest of this paper, a 2-odd triangulation means one on the sphere. Let G be a 2-odd triangulation and let v be a 4-vertex in G with link $v_1v_2v_3v_4$. The 4-contraction of v at $\{v_1, v_3\}$ is to remove v , identify v_1 and v_3 , and replace two pairs of multiple edges $\{v_1v_2, v_3v_2\}$ and $\{v_1v_4, v_3v_4\}$ with two single edges v_1v_2 and v_1v_4 , respectively, as shown in Fig. 5. We call the new vertex arisen by the identification $v_1 = v_3$ the *image* of the 4-contraction and denoted by $[v]_{v_1=v_3}$. If this operation breaks the simpleness or changes the number of odd vertices in G , then we do not apply it. The inverse operation is called a 4-splitting. By the definition, a 4-contraction transforms G into a 2-odd triangulation with $|V(G)| - 2$ vertices.

Let $u_1u_2u_3$ be a face in G and let $v_1v_2v_3$ be a 3-cycle in G such that 3-cycles $v_iu_ju_k$ and $v_i v_j u_k$ bound faces for $\{i, j, k\} = \{1, 2, 3\}$, where these six vertices form the *octahedron* in G . The *octahedron removal* is to remove u_1, u_2 and u_3 (see Fig. 6), and the inverse operation is an *octahedron addition*. Note that these transformations always preserve the simpleness of graphs and the number of odd vertices in G (Fig. 7).

By using the above operations, we establish a generating theorem for 2-odd triangulations. A 2-odd triangulation G is *minimal* if neither a 4-contraction nor an octahedron removal can be applied to G .

Lemma 8 *Let x and y be odd vertices of a 2-odd triangulation. Then x and y are not adjacent.*

Proof It is shown in [3] that if a 2-odd triangulation on any surface has two adjacent odd vertices, then it is not 4-colorable. Thus, by Four Color Theorem [1], two odd vertices are not adjacent in G . \square

The following lemma is useful in a proof of Theorem 10. In fact, the following is already proved for even triangulations on any surface in [11]. We can mostly apply their proof to the proof of the following lemma since each inner vertex in the region has even degree. However, to make the paper be self-contained, we write the proof of the following.

Lemma 9 *Let G be a 2-odd triangulation on a surface which is minimal with respect to 4-contractions and octahedron removals.*

Fig. 5 The 4-contraction

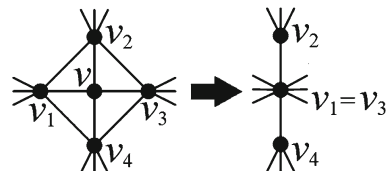


Fig. 6 The octahedron removal

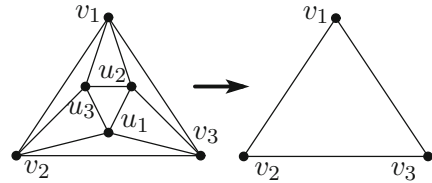
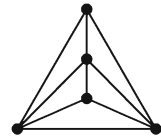


Fig. 7 The graph $C_3 + \bar{K}_2$



- (i) Let Δ be a 3-region of G which contains no odd vertex in its interior. Then Δ bounds a face of G .
- (ii) Let Γ be a 4-region of G bounded by a 4-cycle $abcd$, which contains no odd vertex in its interior. Then the interior of Γ has either a single diagonal, a single 4-vertex, or $Q^{(m)}$ for some $m \geq 1$, as shown in Fig. 8, where $Q^{(m)}$ consists of m paths $bu_1v_1d, bu_2v_2d, \dots, bu_mv_md$ and $m - 1$ vertices p_1, \dots, p_{m-1} of degree 6, in which $\deg(u_i) = \deg(v_i) = 4$ for each $i \in \{1, \dots, m\}$, p_j is adjacent to $b, d, u_j, v_j, u_{j+1}, v_{j+1}$ for each $j \in \{1, \dots, m - 1\}$, and $av_1, au_1, cv_m, cu_m \in E(G)$.

Proof (i) Let uvw be the boundary 3-cycle of Δ which does not bound a face of G and suppose that Δ contains no odd vertex in its interior. Take Δ to be innermost and regard it as a plane triangulation H with boundary cycle uvw . Note that any vertex of H except u, v, w has even degree. By Euler’s formula, we find that H has at least four vertices of degree at most 5. Therefore, there exists a vertex x of degree 4 in Δ with its link $x_1x_2x_3x_4$. Since x is not contractible at $\{x_i, x_{i+2}\}$ for any $i = 1, 2$, there are 3- or 4-cycles C_1 and C_2 passing through x_1x_3 and x_2x_4 , respectively. If one of C_1 and C_2 , say C_1 , passes outside of Δ , then both of x_1 and x_3 lie on uvw . In this case, x_1x_3 would be a smaller contractible 3-cycle which does not bound a face, a contradiction. Thus, both C_1 and C_2 are included in Δ , and hence they cross at x and another vertex y in Δ . Since every 3-cycle, except uvw , bounds a face by the assumption on Δ , we can find a removable octahedron (induced by

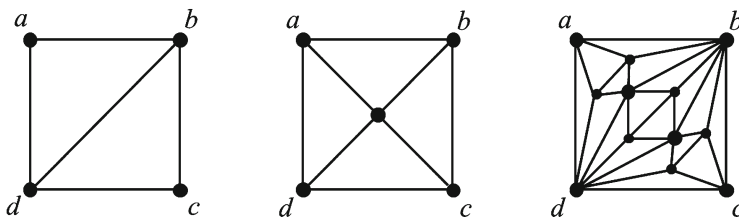


Fig. 8 The interior of Γ bounded by the 4-cycle $abcd$, a single diagonal, a single 4-vertex and $Q^{(m)}$ with $m = 3$

$\{x, x_1, x_2, x_3, x_4, y\}$) in G , which contradicts that G is minimal.

(ii) We prove the lemma by induction on the number $F \geq 2$ of faces in Γ . If $F = 2$, then we are done since Γ has a diagonal edge. So we may suppose that $F \geq 3$ and Γ has no diagonal edge. Hence, by using Euler's formula, Γ contains a 4-vertex x in its interior (cf. [11, Lemma 5]).

Let $L_x = x_1x_2x_3x_4$ be the link of x . If L_x coincides with the boundary cycle $abcd$ of Γ , then Γ is isomorphic to the center of Fig. 8. Moreover, if $x_1 = a, x_2 = b$ and $x_3 = c$, then the smaller 4-region $R = av_4cd$ is isomorphic to one of 4-regions shown in Fig. 8 by induction. (By Lemma 9(i), L_x should be put in Γ so that no edge of L_x is a diagonal of Γ .) Noting that Γ contains no odd vertex, Γ is $Q^{(1)}$ if R has a diagonal, and R contains a contractible 4-vertex in its interior if otherwise. Thus, by symmetry, L_x and the 4-cycle $abcd$ share at most two vertices.

First suppose that L_x and the 4-cycle $abcd$ share at most one vertex, say $x_1 = a$. Since x is not contractible, let C_1 and C_2 be 3- or 4-cycles passing through x_1xx_3 and x_2xx_4 , respectively. By the assumption, all vertices of C_1 and C_2 lie in the interior of Γ and both cycles are of length 4 by Lemma 9(i), that is, there exists a vertex u in Γ such that $C_1 = xx_1ux_3$ and $C_2 = xx_2ux_4$ by the planarity. Note that each edge of C_2 lies in the interior of Γ since $\{x_2, x_4\} \cap \{a, b, c, d\} = \emptyset$. If C_1 also lies in Γ , then without loss of generality, $x_1 (= a)x_4u$ bounds a 3-region containing x, x_2 and x_3 in its interior, contrary to Lemma 9(i). Thus, avoiding a non-facial 3-cycle in Γ , we may suppose that $u = c$ and that ux_1 lies outside of Γ . Hence, by symmetry, Γ is separated to three 4-regions bounded by $R_1 = x_1bcx_2$, $R_2 = x_1x_2cx_4$ and $R_3 = x_1x_4cd$, respectively. Observe that R_2 includes two adjacent 4-vertices x and x_3 and that by induction, each of R_1 and R_3 is isomorphic to one of 4-regions shown in Fig. 8. By Lemma 9(i) and the non-existence of odd inner vertices in Γ , both R_1 and R_3 are isomorphic to $Q^{(m_1)}$ and $Q^{(m_2)}$ for some $m_1, m_2 \geq 1$. Hence, Γ is isomorphic to $Q^{(m)}$ for $m = m_1 + m_2 + 1$. (By the parity of ${}^\circ_G(x_2)$ and ${}^\circ_G(x_4)$, we can uniquely embed $Q^{(m_1)}$ and $Q^{(m_2)}$ into R_1 and R_3 , respectively.)

Therefore, we finally suppose that L_x and the 4-cycle $abcd$ share exactly two vertices. By Lemma 9(i), Γ does not contain a non-facial 3-region. Thus, by symmetry, we have two cases; (1) $x_1 = a$ and $x_3 = c$ and (2) $x_1 = a$ and $x_2 = b$.

For the former case, we can apply a 4-contraction of x at $\{x_2, x_4\}$, a contradiction. So we suppose that the case (2) occurs. Similarly to the argument in the above cases, there are 3- or 4-cycles C_1 and C_2 passing through x_1xx_3 and x_2xx_4 , respectively. In particular, by Lemma 9(i), each of those cycles has length 4 since $\{x_3, x_4\} \cap \{a, b, c, d\} = \emptyset$, and then we let $C_1 = xx_1ux_3$ and $C_2 = xx_2vx_4$. If at least one of u and v is contained in the interior of Γ , then we have a 3-region in Γ containing x, x_3 and x_4 in its interior, contrary to Lemma 9(i). Thus, each of u and v coincides with c or d by the simplicity and Lemma 9(i).

By symmetry, we first consider $u = v = c$. In this case, a 4-region ax_4cd in Γ is isomorphic to one of 4-regions shown in Fig. 8. Similarly to the above arguments, we have that Γ is isomorphic to $Q^{(m)}$ for some $m \geq 2$. Next, we let $u = c$ and $v = d$. In this case, a 4-region x_3cdx_4 in Γ cannot be isomorphic to any 4-region shown in Fig. 8, for otherwise, x_3 or x_4 must be an odd vertex, contrary to the assumption. Therefore, the lemma holds. \square

Theorem 10 Every 2-odd triangulation on the sphere can be obtained from $C_3 + \bar{K}_2$ by a sequence of 4-splittings and octahedron additions, where $C_3 + \bar{K}_2$ is the 2-odd triangulation shown in Fig. 7.

Proof Let G be a minimal 2-odd triangulation, and so we shall prove $G = C_3 + \bar{K}_2$. Observe that a triangulation on the sphere has at least four vertices, and that one with exactly four vertices must be a tetrahedron. If $|V(G)| = 5$, then G is isomorphic to $C_3 + \bar{K}_2$, since it is a unique triangulation with five vertices. Hence we suppose that $|V(G)| \geq 6$.

It is known that every triangulation on the sphere (with at least four vertices) contains at least four vertices of degree at most 5. Hence, G contains a 4-vertex since G is a 2-odd triangulation. Then let v be a 4-vertex in G with link $v_1v_2v_3v_4$.

Consider the 4-contraction of v at $\{v_1, v_3\}$ in G . If the resulting graph \tilde{G} is still a 2-odd triangulation, then this contradicts the minimality of G . Hence \tilde{G} has a loop or multiple edges incident to $[v]_{v_1=v_3}$, or the number of odd vertices in \tilde{G} is not two. In the former case, v_1 and v_3 are joined by a path P of length one or two in G through neither v, v_2 nor v_4 . In the latter case, both v_1 and v_3 have odd degree in G , and \tilde{G} is an even triangulation. Similarly, for the 4-contraction of v at $\{v_2, v_4\}$, we see that G has a path P' of length at most two joining v_2 and v_4 in $G - \{v, v_2, v_4\}$, or both v_2 and v_4 have odd degree in G . If both P and P' exist in G , then they must cross at their middle vertices, by the planarity, and hence both P and P' have length exactly two.

Consequently, by symmetry, we have one of the following structures around v (see Fig. 9):

- (a) v_2 and v_4 are odd vertices, and $P = v_1v_3 \in E(G)$.
- (b) v_2 and v_4 are odd vertices, and $P = v_1xv_3$ with $x \notin \{v, v_2, v_4\}$.
- (c) $P = v_1yv_3$ and $P' = v_2yv_4$ with $y \notin \{v, v_1, v_2, v_3, v_4\}$.

Case (a) Since both v_2 and v_4 are odd vertices, two triangular regions $v_1v_2v_3$ and $v_1v_4v_3$ are faces by Lemma 9(i). So we have $G \simeq C_3 + \bar{K}_2$, contrary to $|V(G)| \geq 6$.

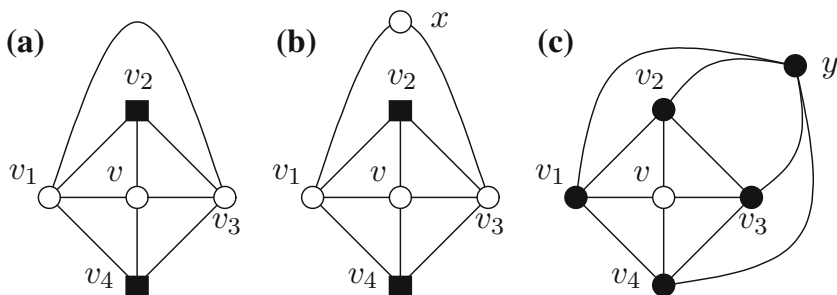


Fig. 9 The structure around v : a white circle (resp., a black square) denotes an even (resp., odd) vertex, and a black circle denotes a vertex of even or odd degree

Case (b) Let us consider the structure of the interior I_C of the 4-region bounded by $C = xv_1v_2v_3$. We can apply Lemma 9(ii) to I_C since all inner vertices are even. If $v_1v_3 \in E(G)$, then G has the structure (a), but we note $xv_2 \notin E(G)$. (For otherwise, i.e., $xv_2 \in E(G)$, then the interior of at least one of the two regions xv_1v_2 and xv_2v_3 contains a vertex, since v_2 is odd. This contradicts Lemma 9(i).) If I_C contains a single 4-vertex, then v_2 must be even, by Lemma 9(i), a contradiction. Finally, we consider the case when I_C is isomorphic to $Q^{(k)}$ for some $k \geq 1$. If two adjacent 4-vertices in $Q^{(k)}$, say s and s' , are neighbors of v_1 , then v_2 has even degree, a contradiction. On the other hand, if they are neighbors of v_2 , then the 4-vertex s is contractible at $\{s', v_1\}$ (or $\{s', v_3\}$), a contradiction. Hence this case does not happen.

Case (c) Let $D = yv_2v_3$ be the 3-cycle and let I_D be the interior of the region bounded by D . Let G_D be the subgraph of G induced by the vertices of D and those contained in I_D . By symmetry, we may suppose that G_D contains an odd vertex of G . If G_D contains two odd vertices of G , then the outer region of D contains only even vertices in G , which contradicts Lemma 9(i). Hence, by the Handshaking Lemma and Lemma 8, I_D contains exactly one odd vertex of G , and other odd vertices of G_D are contained in D but all of them have even degree in G .

If I_D has a single vertex, then the interior of two triangular regions yv_1v_2 and yv_3v_4 must contain vertices, since ${}^\circ_G(v_2)$ and ${}^\circ_G(v_3)$ are even. However, this contradicts Lemma 9(i) since G is a 2-odd triangulation. Thus we may suppose that I_D contains at least two vertices. Here we prove that I_D contains a 4-vertex. If we let p_i be the number of i -vertices in G_D , then we have $3p_3 + 2p_4 + p_5 \geq 12$, by Euler's formula, which implies that G_D has at least four vertices of degree less than 6. Observe that at least two of y, v_2 and v_3 have degree at least 4 in G_D , and that G_D contains exactly one odd vertex s in I_D . Hence, if no vertex in I_D has degree 4, then G_D has at most four vertices of degree less than 6. However, the above inequality for G_D claims that if G_D has exactly four vertices of degree less than 6, then all of them must have degree 3, contrary to the current situation.

Let u be a 4-vertex u in I_D which can be found as above, and let $u_1u_2u_3u_4$ be the link of u in G . Since at least one neighbor of u is contained in I_D and since all three vertices of D have even degree in G , the structure around u must be (c). Hence G has two paths P_1 and P_2 of length 2 which join u_1 and u_3 , and u_2 and u_4 respectively, and cross at their middle vertex z . By symmetry, we may suppose that u_2 and u_3 are contained in the triangular region u_1u_4z . If all of the three triangular regions u_1u_2z, u_2u_3z and u_3u_4z have no inner vertex, then we can apply an octahedron removal to remove u, u_2 and u_3 . For, if one of them, say u_1u_2z , contains inner vertices, then the interior of u_1u_2z must have an odd vertex, and then contains a 4-vertex again by the same argument as above. This argument does not continue since G is finite.

We have shown that unless $G = C_3 + \bar{K}_2$, G admits a 4-contraction or an octahedron removal. Hence the proof completes. \square

Now we prove Proposition 6.

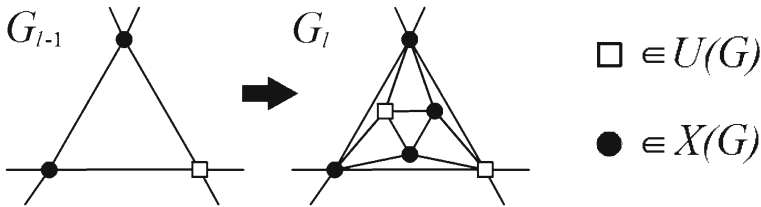


Fig. 10 An octahedron addition and a color factor

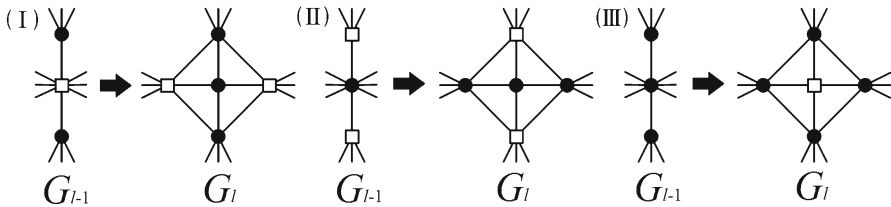


Fig. 11 4-Splittings and color factors

Proof of Proposition 6 Let G be a 2-odd triangulation. By Theorem 10, there exists a sequence of 2-odd triangulations $C_3 + \bar{K}_2 = G_0, G_1, G_2, \dots, G_k = G$ such that G_{i+1} is obtained from G_i by a single octahedron addition or a single 4-splitting. We prove the proposition by induction on k . When $k = 0, G = G_k = G_0 = C_3 + \bar{K}_2$ is the face subdivision of C_3 .

If G_l is obtained from G_{l-1} by either a single octahedron addition or a single 4-splitting, then G_{l-1} is a face subdivision with a unique color factor, by induction hypothesis. In the former case, G_l is also a face subdivision with a unique color factor, as shown in Fig. 10. In the latter case, we have three possibilities for the structure around a vertex to which a 4-splitting is applied, as shown in Fig. 11. In any case, G_l is a face subdivision with a unique color factor, and hence the proposition holds. \square

4 Lemmas and a Proof of Theorem 3

Let $L = C_3 + \bar{K}_2$, and let \tilde{L} be the 2-odd triangulation obtained from $C_3 + \bar{K}_2$ by a single octahedron addition (see Fig. 12).

In order to prove Theorems 3 and 4, we give two tools for induction on the number of vertices, one is to use an octahedron, and the other is to use two adjacent 4-vertices added on an edge. We first describe how an octahedron in 2-odd triangulations can be moved, and how an octahedron removal is related to 4-contractions. (We can find a similar result for even triangulations [5].)

Lemma 11 *Let G be a 2-odd triangulation.*

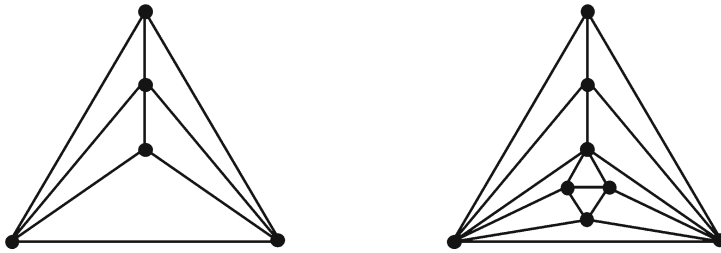


Fig. 12 The 2-odd triangulations L and \tilde{L}

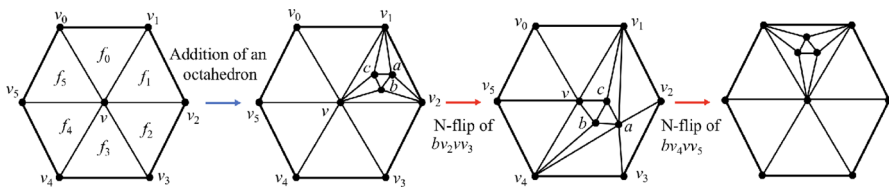


Fig. 13 Moving an octahedron

1. Let f and f' be two faces of G , and let G_f (resp., $G_{f'}$) denote the one obtained from G by applying an octahedron addition to f (resp., f'). Then G_f and $G_{f'}$ are N -equivalent.
2. Let G' denote a 2-odd triangulation obtained from G by two octahedron additions. Then G' can be reduced into G by N -flips and 4-contractions.

Proof

1. Let v be a vertex of G whose degree is $k \geq 4$. Let f_0, f_1, \dots, f_{k-1} be the faces incident to v lying around v in this cyclic order. We can see that the octahedron added in f_1 can be moved to f_0 as shown in Fig. 13. (Note that the figure shows the case when $\circ(v) = 6$, but the same can be done unless $\circ(v) = 3$, by repeatedly applying similar N -flips.) It is easy to see that repeating the above procedures, the octahedron added in f can be moved to any face f' .
2. By the result (1), we can move two added octahedra freely in G , and hence the two octahedra can be put in two adjacent faces in G . Such two octahedra can be eliminated as shown in Fig. 14 by a single N -flip and three 4-contractions. \square

By Lemma 11, we have the following proposition which follows from Theorem 10 immediately.

Proposition 12 Every 2-odd triangulation with an odd number of vertices can be reduced to L by a sequence of 4-contractions and N -flips, and one with an even number of vertices can be reduced to \tilde{L} by the two operations.

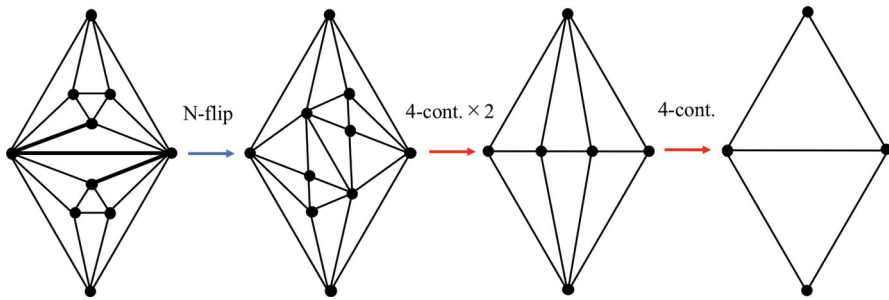


Fig. 14 Eliminating two octahedra by a single N -flip and three 4-contractions

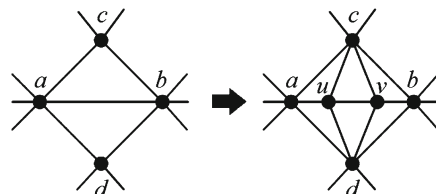
Proof Let G be a 2-odd triangulation. By Theorem 10, there exists a finite sequence of 2-odd triangulations $G = G_0, G_1, \dots, G_k = L$ such that G_i is obtained from G_{i-1} by either a 4-contraction or an octahedron removal, for $i = 1, \dots, k$. Observe that the 4-contraction preserves the parity of the number of vertices but the octahedron removal does not. Hence if $|V(G)|$ is odd (resp., even), the sequence contains an even (resp., odd) number of octahedron removals. Observe that an octahedron in a 2-odd triangulation can be moved to any face by N -flips, and that two octahedra can be eliminated by N -flips and 4-contractions, by Lemma 11. Hence, if $|V(G)|$ is odd, then G can be reduced to L by a sequence of 4-contractions. Otherwise (i.e., if $|V(G)|$ is even), G can be to \tilde{L} . \square

Secondly we deal with two adjacent 4-vertices u, v . Let G be a 2-odd triangulation and let $e = ab$ be an edge of G shared by two triangular faces abc and abd . Subdivide e by two vertices u and v to form a path $auvb$, and add edges uc, ud, vc and vd , as shown in Fig. 15. This operation is a 2-subdivision, and $\{u, v\}$ is a 2-subdividing pair.

Lemma 13 Let G be a 2-odd triangulation and let v be a 4-vertex with link $v_1v_2v_3v_4$, where v_1 is even. Let u be the common neighbor of v_1 and v_2 other than v . Suppose that the triangulation H obtained from G by a 4-contraction of v at $\{v_1, v_3\}$ is a 2-odd triangulation. Then G can be transformed into the one obtained from H by a 2-subdivision of the edge $u[v]_{v_1=v_3}$, by a sequence of N -flips, keeping the degree of v_2 .

Proof Let $vv_4a_1a_2 \dots a_kv_2$ be the link of v_1 in G , where $k \geq 4$ is an even integer and let $a_k = u$. If $\deg_G(v_1) = 4$, then $\{v_1, v\}$ can be regarded as a 2-subdividing pair lying on the edge uv_3 in G . Hence G is a 2-odd triangulation obtained from H by a

Fig. 15 The 2-subdivision of ab



2-subdivision of the edge $u[v]_{v_1=v_3}$.

So we suppose that $\circ_G(v_1) \geq 6$. Apply the N -flip to the path $vv_4v_1a_1$ in G to decrease the degree of v_1 , and let G' be the resulting graph (see Fig. 16). By the assumption, a_1, \dots, a_k are not adjacent to v_3 in G . Moreover, $v_4, a_1, a_2, \dots, a_k, v_2$ are all distinct. Hence G' is simple and we see that $\circ(v_2)$ is unchanged, and that v still has degree 4. After this, repeating the N -flip to $va_i v_1 a_{i+1}$ for all even $i \in \{2, 4, \dots, k - 3\}$, we can transform G into a 2-odd triangulation obtained from H by a 2-subdivision of the edge $u[v]_{v_1=v_3}$, keeping $\circ(v_2)$. \square

Two 2-odd triangulations G and G' are NP_2 -equivalent, denoted $G \sim_{NP_2} G'$, if G and G' can be transformed into each other by N - and P_2 -flips. For a 2-odd triangulation H , let $H + \Gamma(m)$ denote a 2-odd triangulation obtained from H by repeating a 2-subdivision of an edge m times, where we note that $H + \Gamma(m)$ has $|V(H)| + 2m$ vertices. Since we can move a 2-subdividing pair on edges freely by P_2 -flips, we see that the expression $H + \Gamma(m)$ is well-defined.

Now we prove Theorem 3.

Proof of Theorem 3 By Proposition 12, if $|V(G)|$ is even, then G can be reduced to \tilde{L} by N -flips and 4-contractions, and hence, applying Lemma 13 repeatedly, we have $G \sim_{NP_2} \tilde{L} + \Gamma(m)$ (where $m = \frac{1}{2}(|V(G)| - 8)$), since a 2-subdividing pair can be moved freely by P_2 -flips. Similarly, if $|V(G)|$ is odd, then $G \sim_{NP_2} L + \Gamma(m)$, where $m = \frac{1}{2}(|V(G)| - 5)$. Therefore, if $|V(G)| = |V(G')|$, then G and G' are NP_2 -equivalent, via the standard form $\tilde{L} + \Gamma(m)$ or $L + \Gamma(m)$, depending on the parity of the number of vertices. \square

5 Proof of Theorem 4

In this section, we establish a way to transform 2-odd triangulations without P_2 -flips. In the proof of Theorem 3, it is a key fact that a 2-subdividing pair can be moved freely by P_2 -flips, but forbidding P_2 -flips in Theorem 4, we carefully observe how a 2-subdividing pair can be moved only by N -flips.

Lemma 14 *Let H be a 2-odd triangulation, and let e_1, \dots, e_k be all edges incident to a vertex $v \in V(H)$ appearing in this cyclic order with respect to the rotation of v .*

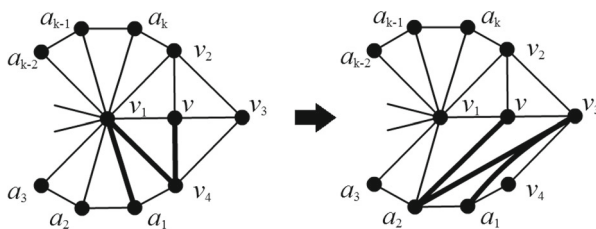


Fig. 16 The N -flip of the path $vv_4v_1a_1$

Let G (resp., G') be the 2-odd triangulation obtained from H by a 2-subdivision of e_i (resp., e_{i+2}). If $k = \text{deg}(v) \geq 4$, then G and G' are N -equivalent.

Proof Let $v_1v_2\dots v_k$ be the link of a vertex $v \in V(G)$ and we suppose $e_i = vv_i$ for each $i \in \{1, \dots, k\}$ and $k = \text{deg}(v) \geq 4$. Suppose that G is obtained from H by a 2-subdivision of vv_i with two vertices x and y to form a path $vxyv_i$. If $\text{deg}_H(v) = 4$, then x and v can be regarded as a 2-subdividing pair on vv_{i+2} in G . Therefore, we have $G = G'$. So we may suppose that $\text{deg}_H(v) \geq 5$. First apply the N -flip of the path $xv_{i+1}vv_{i+2}$, and next apply the N -flip of the path $xv_{i-1}yv_i$ (see Fig. 17). Since v_1, \dots, v_k are all distinct in G and $k \geq 5$, each N -flip preserves the simpleness. Hence G can be transformed into G' by N -flips. \square

In Lemma 14, we say that e_i and e_{i+2} are *alternate edges* incident to v . Lemma 14 tells us that a 2-subdividing pair on an edge e can be moved to an alternate edge of e incident to the same vertex. However, we note that the condition “ $\text{deg}(v) \geq 4$ ” is necessary in Lemma 14. Hence, in order to move a 2-subdividing pair freely, we introduce the notion of “flexibility” of a 2-odd triangulation, as follows.

Two edges e and e' in a triangulation are *equivalent* if there exists a sequence of edges e_1, \dots, e_m of G with $e = e_1$ and $e' = e_m$ such that for $i = 1, \dots, m - 1$, e_i and e_{i+1} are alternate edges incident to the same vertex of degree at least 4 in G . We call such a sequence of edges an *alternate sequence* between e and e' .

Let G be a 2-odd triangulation. Let G_1 (resp., G_2) be the 2-odd triangulation obtained from G by a 2-subdivision of e_1 (resp., e_2). By Lemma 14, G_1 and G_2 are N -equivalent if e_1 and e_2 are equivalent in G . It is easy to see the following hold;

- a frame edge and a spoke in G are not equivalent, and
- any two frame edges of G are equivalent (since every frame edge joins two vertices of even degree at least 4, and since frame edges incident to the same vertex appear alternately in the rotation around the vertex).

We say that a 2-odd triangulation G is *flexible* if any two spokes of G are equivalent. We give a characterization of flexible 2-odd triangulations, as in the following.

Theorem 15 *Let G be a 2-odd triangulation, and let x and x' be the two odd vertices. Then G is flexible if and only if at least one of the following two conditions holds:*

- (i) *At least one of x and x' is of degree at least five.*

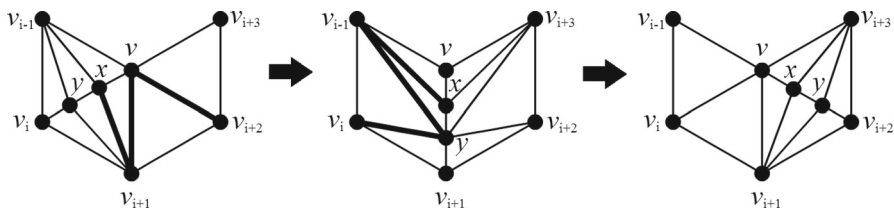


Fig. 17 Moving a 2-subdividing pair

- (ii) If both x and x' have degree three, then x and x' have at most one common neighbor.

Proof If x is an odd vertex of degree at least five, then all spokes incident to x are equivalent in G , since two alternate spokes are equivalent, by Lemma 14. It is easy to see that any spoke is equivalent to some spoke incident to x , since we can find an alternate sequence between them. Consequently, any two spokes of G are equivalent, via the spokes incident to x . So it suffices to prove the following:

A 2-odd triangulation G with two 3-vertices x and x' is flexible if and only if x and x' have at most one common neighbor.

Let H be the frame of G , which can uniquely be taken by Proposition 6. Then H has two triangular faces $X = abc$ and $X' = a'b'c'$, which have x and x' in the interior in G , respectively. Let R be the subgraph of H induced by all edges on the boundary cycle of an even face of H . That is, $R = H - D$, where $D = E(X) \cap E(X')$ which might be empty.

(“Only if” part.) Suppose that X and X' share at least two vertices, say $a = a'$ and $b = b'$. If $c = c'$, then $G = C_3 + \bar{K}_2$, in which $\{ax, ax'\}$, $\{bx, bx'\}$ and $\{cx, cx'\}$ are three equivalence classes of spokes, and we are done. Hence we may suppose that $c \neq c'$. Then $R = H - ab$, which is a bipartite plane graph with a facial cycle $acbc'$ since each face of R is even. We color the vertices of R black and white, according to the bipartition of R , so that a and c are black and white, respectively. Observe that a spoke e is equivalent to xa in G if and only if e is incident to a black vertex. Hence xa and xc are not equivalent since xc is incident to a white vertex.

(“If” part.) Suppose that two 3-cycles X and X' share at most one vertex in H . Then $R = H$ since $D = E(X) \cap E(X') = \emptyset$. Since H is connected, X and X' are connected in H by a path P . Taking P to be shortest, we let $P = p_1 \dots p_m$ with $p_1 = a$ and $p_m = a'$, in which each p_i with $i \in \{2, \dots, m-1\}$ is supposed to be contained in neither X nor X' , where we might have $a = a'$, i.e., $m = 1$. Cutting R along P and making a copy $\tilde{P} = \tilde{p}_1 \dots \tilde{p}_m$ of P , we obtain the plane graph, say R' , whose boundary cycle is $p_1 b c p_1 \dots \tilde{p}_m c' b' p_m \dots p_1$. Then R' is bipartite since each face of R' is even. In the bipartition of R' , p_1 and c have the same color, say black, and \tilde{p}_1 and b are white.

Observe that if e and e' are two spokes in G incident to vertices of R' with the same color, say black, then we can find an alternate sequence of edges in R' between e and e' without crossing P , in which every edge in the sequence is incident to a black vertex of R' . On the other hand, if e and e' are incident to vertices in R' with distinct colors, then we can take two alternate sequences in G through faces in R' from e to some edge incident to p_i , and from e' to an edge incident to \tilde{p}_i for some i , since they form a single alternate sequence of edges in G between e and e' which crosses P exactly once in G . In the sequence, we should note that p_i and \tilde{p}_i are the same vertex in R , but they are distinct vertices in R' with distinct colors. So any two spokes are equivalent in G . \square

Consider three 2-odd triangulations L_9, L_s and L_f shown in Fig. 18. We note that L_s and L_f are obtained from $L = C_3 + \bar{K}_2$ by 2-subdividing a spoke and a frame edge, respectively, and that L_9 is a flexible 2-odd triangulation obtained from L_s by

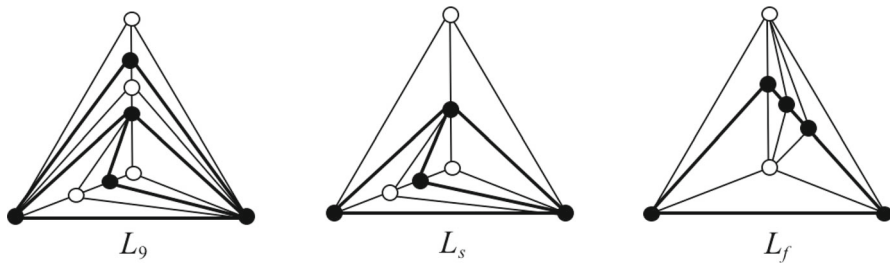


Fig. 18 The 2-odd triangulations L_9 , L_s and L_f

2-subdividing a spoke. For a triangulation G and an octahedron O which is an induced subgraph of G , $G - O$ denotes a triangulation obtained from G by an octahedron removal which removes the three 4-vertices of O .

Lemma 16 *Every flexible 2-odd triangulation G with $k \geq 9$ vertices can be reduced to a flexible 2-odd triangulation with $k - 2$ vertices by N -flips and a single 4-contraction, preserving the flexibility, unless G is isomorphic to L_9 .*

Proof Let T be a flexible 2-odd triangulation. Let T' (resp., T'') be any 2-odd triangulation obtained from T by a single 4-splitting (resp., octahedron addition). We first observe that T' and T'' are flexible, by the characterization of flexible 2-odd triangulations (Theorem 15). Moreover, moving the octahedron in T'' to another face of T by N -flips (Lemma 11(1)), we see that all 2-odd triangulations in the process are flexible.

We first establish the following claim.

CLAIM. By several N -flips, G can be transformed into a flexible 2-odd triangulation which admits a 4-contraction, preserving the flexibility.

Proof of the Claim. By Theorem 10, G has a 4-contractible 4-vertex (i.e., one to which a 4-contraction can be applied) or a removable octahedron O . If we have the former, then the claim holds with no N -flip. Hence suppose the latter, and let $G' = G - O$. Since $|V(G)| \geq 9$, G' still has a 4-contractible 4-vertex v or a removable octahedron O' , by Theorem 10 again.

If G' is flexible, then O can be moved freely in G' by N -flips, preserving the flexibility as mentioned in the first paragraph. So if G' has v , then v is a required 4-contractible 4-vertex in G with O moved somewhere. On the other hand, if G' has O' , then let $G'' = G' - O'$. If G'' is flexible, then moving O and O' suitably in G' , we can eliminate them using N -flips and 4-contractions by Lemma 11(2), preserving the flexibility. The case when G'' is not flexible will be considered in the next case, since O and O' may be dealt with equally in G'' .

If G' is not flexible, then by Theorem 15, G' has two 3-vertices x and x' with neighbor a, b, c and a', b', c' , respectively, with $a = a'$ and $b = b'$, where we note $c \neq c'$ since $|V(G')| \geq 6$. Observe that O is contained in one of the six 3-regions incident to x or x' in G . If G' has a 4-contractible 4-vertex v which is also 4-contractible in G , then we are done. Hence we may suppose that $v = c$ and O is put in the face xbc in G' . (For, since $\circ_{G'}(a) \geq 5$ and $\circ_{G'}(b) \geq 5$, v coincides with

neither a nor b .) In this case, we can move O in G to the face $x'ab$ by two N -flips, keeping at least one of $\circ(x) \geq 5$ and $\circ(x') \geq 5$, and after that, we can apply a 4-contraction of $v = c$, preserving the flexibility. If G' has the octahedron O' , then O' is incident to neither x nor x' , since G' is not flexible. In this case, we can move O' in G to a face neighboring to the 3-region including O , and apply the procedures in Lemma 11(2). Then we can apply a single 4-contraction after an N -flip, keeping at least one of $\circ(x) \geq 5$ and $\circ(x') \geq 5$. \square

By the claim, we may suppose that G has a 4-contractible 4-vertex v with link $v_1v_2v_3v_4$, and let G' be the resulting graph by the 4-contraction at $\{v_1, v_3\}$. If G' is flexible, then the 4-contraction is a required one, and hence we suppose that G' is not flexible. By Theorem 15, G' has two 3-vertices x and x' with neighbors a, b, c and a', b', c' , respectively, with $a = a'$ and $b = b'$, where we note $c \neq c'$ since $|V(G')| \geq 7$.

We first suppose that $\circ_G(x) \geq 5$ or $\circ_G(x') \geq 5$, and that they decrease to 3 by the 4-contraction of v in G' . Then we may suppose that $v_2 = x$ in G' by symmetry. That is, v_2 has degree 5 in G with link $v_1v_2v_3bc$, and by the 4-contraction of v at $\{v_1, v_3\}$, the link of v_2 is deformed into $bc[v]_{v_1=v_3}$ in G' . (Since the case when the link of v_2 is $v_1v_2v_3ab$ can be similarly proved, we prove only the case when it is $v_1v_2v_3bc$.) By Lemma 13, G can be transformed into G' with a 2-subdivision of the edge $c[v]_{v_1=v_3}$ by N -flips, keeping $\circ(v_2) = 5$, and hence preserving the flexibility, by Theorem 15. Moreover, the 2-subdividing pair on the link of x or x' can be moved to the edge ab by two N -flips, keeping at least one of $\circ(x) \geq 5$ and $\circ(x') \geq 5$. By Theorem 10, G' can still be reduced by a sequence of 4-contractions and octahedron removals, since $|V(G')| \geq 7$. These reductions in G' must be done in the 4-region $xax'b$ containing c and c' , since no face incident to ab is eliminated in G' . (Note that a or b may be identified to another vertex in the sequence of the above reductions. Even if this occurs, then we keep the 4-region $xax'b$ with this labeling.) Hence all graphs arisen in the process must be flexible, since both x and x' always have degree five. (If the sequence contains two octahedron removals, then we apply Lemma 11(2). If the sequence contains a single octahedron removal but no 4-contraction, then $G' = \tilde{L}$, contrary to G' being non-flexible.)

If G has two 3-vertices x and x' sharing at most one common neighbor, but they have at least two common neighbors in G' by the 4-contraction of v . We may let $pb'qb$ be the link of v in G , and suppose that the 4-contraction of v at $\{b, b'\}$ transforms G into G' , where p is contained in the interior of the 4-region bounded by $abv b'$ in G . (See the left-hand of Fig. 19.) Since the 4-contraction of v at $\{b, b'\}$ is

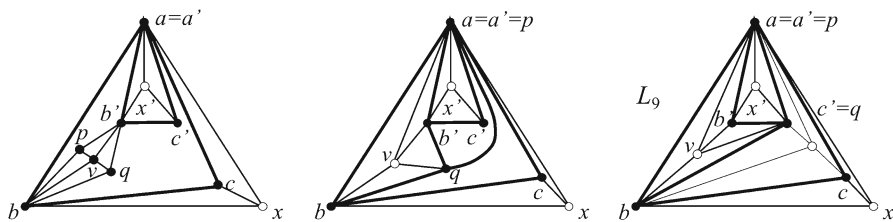


Fig. 19 Structures around v

applicable in G , we have $a = a' = p$. If the 4-contraction of v at $\{p, q\}$ is applicable in G , then the two 3-vertices x and x' share only one neighbor $a = a'$ in the resulting graph, which is a required flexible 2-odd triangulation with fewer vertices, by Theorem 15. Hence we prove that if the 4-contraction of v at $\{p, q\}$ is not applicable, then G is isomorphic to L_9 , as follows.

We first suppose that q coincides with c' or c , say the former by symmetry. Then apply Lemma 9(ii) to the 4-region $ac'bc$. (If we have an octahedron removal for reducing the 4-region, then we move the octahedron to a face xbc . Then the 4-contraction of v at $\{b, b'\}$ is applicable preserving the flexibility, since x has degree five in the current graph. So we may suppose that any 4-region can be reduced only by 4-contractions. This trick will be used in this proof several times.) Since the 4-region $acbc$ is bounded by only frame edges, the 4-region must have exactly one vertex, as in the right-hand of Fig. 19. Then the graph is isomorphic to L_9 . (If the region $ac'bc$ is isomorphic to $Q^{(m)}$ for some $m \geq 1$, then at least one of a and c has odd degree, a contradiction.)

Hence G has the 6-region D bounded by a cycle $C = qb'c'pcb$. Since all edges of C are frame edges, the frame edges divide the interior of C into regions bounded by an even cycle. If D has a chord of C which must be a frame edge, then only an admissible chord is pq , as in the center of Fig. 19, since other chords are not obstructions of the 4-contraction of v at $\{p, q\}$. In this case, apply Lemma 9(ii) to two 4-regions $pqb'c'$ and $pcbq$. Similarly to the above case, each of two 4-regions must have exactly one vertex, and hence, we can apply the 4-contraction of the 4-vertex in $pqb'c'$ at $\{c', q\}$ keeping $c \neq c'$, contrary to the resulting graph being non-flexible.

Finally we consider the case when D has no chord. If D contains exactly one inner vertex u , then u has degree 6 and is adjacent to all vertices of C . So, applying the 4-contraction of c' at $\{x', u\}$, we have a flexible 2-odd triangulation with fewer vertices, a contradiction. Thus, we may suppose that D contains at least two vertices. By the result in [2, Theorem 3.1], we know that the number of vertices of degree at least 6 in a 6-region is at most 1 if all vertices in the region are of degree at least 6, and hence D contains an inner 4-vertex u with link $L_u = u_1u_2u_3u_4$. By the minimality of G with respect to the flexibility, the 4-contraction of u at $\{u_1, u_3\}$ and that at $\{u_2, u_4\}$ are not applicable, and so there are two 3- or 4-cycles passing through u_1uu_3 and u_2uu_4 . By a similar argument in the proof of Theorem 10 and the assumption for octahedra in D , at least one of u_1, u_3 , and at least one of u_2, u_4 , say u_1 and u_2 by symmetry, are contained in C . Since D has no chord, then u_1u_2 coincides with an edge of C . Moreover, by the same reason, we may suppose that u_3 is an inner vertex of D . Observe that if u_4 is also inner vertex in D , then u_4 has degree at least 6. (For otherwise, a 4-contraction of u at $\{u_2, u_4\}$ is possible.) Since D can have a diagonal, a 4-contraction of u at $\{u_1, u_3\}$ is always possible in D , a contradiction. \square

Let G be a 2-odd triangulation and let H be the frame of G . Consider an N -flip of a 3-path $P = abcd$ connecting two vertices a and d of the color factor of G . Then the edge bc must be contained in H . If we let $a_1 \dots a_mcb$ and $bcd_1 \dots d_l$ be the links of a and d in G in the same cyclic order, then the N -flip of P in G corresponds to a

generalized diagonal flip of the edge bc in H to replace it with a_1d_1 in the 2-cell region of H bounded by $ba_1 \dots a_mcd_1 \dots d_1$, as shown in the left-hand of Fig. 20. (Observe that any generalized diagonal flip preserves the size of faces of H , and hence, the number of odd faces of H is always preserved in the process of generalized diagonal flips.) This interpretation will be useful in the following lemma.

Lemma 17 *Let G be a non-flexible 2-odd triangulation with $|V(G)| \geq 9$. Then G can be transformed into a flexible 2-odd triangulation by a sequence of N -flips.*

Proof Let H be the frame of G . Since $|V(G)| \geq 9$, we have $H \neq C_3$. By Theorem 15, G has two 3-vertices x and x' sharing two common neighbors. Hence H has two triangular faces abc and abc' sharing the edge ab . Let D be the 4-region bounded by $C = acbc'$ not containing the two triangular faces. Since $|V(G)| \geq 9$, the interior of D has at least one edge of H .

First suppose that $\circ_H(c) \geq 3$ or $\circ_H(c') \geq 3$, say the former. Let $c'' \neq a$ be the neighbor of c in H which is consecutive to b with respect to the rotation of c . (Note that $c' \neq c''$ since otherwise at least three 3-regions in H , which means that at least three odd vertices, a contradiction.) Then we can apply a generalized diagonal flip of bc to ac'' in H , as shown in the right-hand of Fig. 20. The resulting plane graph, denoted by H' , has exactly two triangular faces abc' and acc'' . By Theorem 15, the face subdivision of H' is flexible.

Secondly we suppose that $\circ_H(c) = \circ_H(c') = 2$. Since D has an inner edge, we have $\circ_H(a) \geq 4$ and $\circ_H(b) \geq 4$. Now apply a generalized diagonal flip of ab to cc' . The resulting graph can be dealt as the one in the previous case. \square

Let K be a flexible 2-odd triangulation, and let $K + (s, t)$ denote a 2-odd triangulation obtained from K by 2-subdivisions of spokes s times and frame edges t times, respectively, where $s, t \geq 0$ and $K = K + (0, 0)$. Since K is flexible, every 2-subdividing pair added on a spoke can be moved to any spoke, and so can be any 2-subdividing pair on a frame edge. Therefore, the notation $K + (s, t)$ is well-defined, that is, any two graphs expressed by $K + (s, t)$ are N -equivalent for any $s, t \geq 0$. In Lemma 13, if G is transformed into G' by a single 4-contraction so that $|U(G)| = |U(G')| + 1$, then $G \sim_N K + (1, 0)$. On the other hand, if $|U(G)| = |U(G')|$, then $G \sim_N K + (0, 1)$. Therefore, if G can be reduced to K by a sequence of N -flips and 4-contractions, preserving the flexibility, we have $G \sim_N K + (s, t)$ by applying Lemma 13 repeatedly, where $s = |U(G)| - |U(K)|$ and

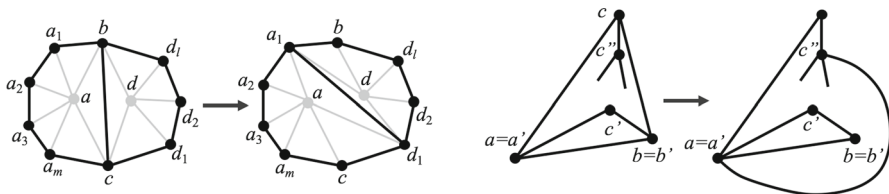


Fig. 20 Generalized diagonal flips in the frame H

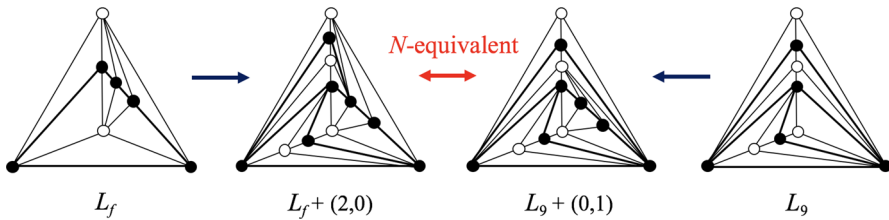


Fig. 21 $L_f + (2, 0) \sim_N L_9 + (0, 1)$

$2(s + t) = |V(G)| - |V(K)|$. This fact plays an important role for our inductive argument in the following proof.

Proof of Theorem 4 Let G and G' be 2-odd triangulations with $|V(G)| = |V(G')|$ and $|U(G)| = |U(G')|$. By Theorem 10, if $|V(G)| = 5$, then $G = L$. If $|V(G)| = 7$, then G is either L_s or L_f , which can be distinguished by $|U(G)|$. If $|V(G)| = 8$, then G is isomorphic to \tilde{L} . Hence suppose $|V(G)| = |V(G')| \geq 9$. If G or G' is not flexible, then we can apply Lemma 17 to transform it to a flexible one. So we may suppose that both G and G' are flexible.

If $|V(G)|$ is even, then both G and G' can be reduced to \tilde{L} by N -flips and 4-contractions, preserving the flexibility, by Lemma 16. Since $|V(G)| = |V(G')|$ and $|U(G)| = |U(G')|$, we have $G \sim_N \tilde{L} + (s, t) \sim_N G'$, where $s = |U(G)| - 3$ and $t = \frac{1}{2}(|V(G)| - 8) - s$.

Suppose that $|V(G)|$ is odd. Applying 4-contractions and N -flips to G , we obtain a minimal flexible 2-odd triangulation H , preserving the flexibility. By Lemma 16, H is either L_f or L_9 . Observe that there exist exactly two 2-odd triangulations L_f and L_s with seven vertices, but L_s is not flexible. If both G and G' can be reduced to either of L_f and L_9 by 4-contractions and N -flips, then G and G' are N -equivalent, via the form $L_f + (s, t)$ or $L_9 + (s, t)$, since L_f and L_9 are flexible. So suppose that G is reduced to L_f , but G' is to L_9 . In this case, we have $G \sim_N L_f + (s, t)$ and $G' \sim_N L_9 + (s', t')$, by Lemma 13. Note that $s = s' + 2 \geq 2$ and $t' = t + 1 \geq 1$, since $|V(L_f)| = 7$, $|U(L_f)| = 2$, $|V(L_9)| = 9$ and $|U(L_9)| = 4$. Seeing that $L_f + (2, 0) \sim_N L_9 + (0, 1)$ by Fig. 21, we have

$$G \sim_N L_f + (s, t) \sim_N (L_f + (2, 0)) + (s - 2, t) \sim_N (L_9 + (0, 1)) + (s', t' - 1) \sim_N L_9 + (s', t') \sim_N G'$$

This completes the proof of Theorem 4. □

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