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ORIGINAL PAPER

Long Paths in Bipartite Graphs and Path-Bistar Bipartite Ramsey Numbers

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Abstract

In this paper, we focus on a so-called Fan-type condition assuring us the existence of long paths in bipartite graphs. As a consequence of our main result, we completely determine the bipartite Ramsey numbers $b(P_s, B_{t_1,t_2})$, where B_{t_1,t_2} is the graph obtained from a t_1-star and a t_2-star by joining their centers.

Keywords Fan-type condition · Bipartite Ramsey number · Bistar · Bipartite graph

Mathematics Subject Classification 05C55 · 05C38 · 05C07

1 Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph. We let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $x \in V(G)$, we let $N_G(x)$ and $d_G(x)$ denote the *neighborhood* and the *degree* of x, respectively; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and $d_G(x) = |N_G(x)|$. For

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$X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X. For two graphs G and H, we write $H \subset G$ if G contains H as a subgraph. Let P_n and K_{n_1,n_2} denote the *path* of order n and the *complete bipartite graph* with partite sets having cardinalities n_1 and n_2, respectively. For terms and symbols not defined here, we refer the reader to [3].

Our main target in this paper is the bipartite Ramsey number. Let H^r and H^b be bipartite graphs. The following fact is obtained by similar argument in the original Ramsey's theorem: there exists a positive integer N such that for any edge-disjoint spanning subgraphs G^r and G^b of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$, $H^r \subset G^r$ or $H^b \subset G^b$. The smallest value of N satisfying the above property is called the *bipartite Ramsey number* with respect to H^r and H^b and denoted by $b(H^r, H^b)$. Note that $b(H^r, H^b) = b(H^b, H^r)$.

If H^b is a star, then the determination problem of $b(H^r, H^b)$ is reduced to a problem of finding H^r under a high minimum degree condition. Thus the bipartite Ramsey numbers involving stars tend to be simply determined. For example, Harary et al. [6] proved that $b(K_{1,s}, K_{1,t}) = s + t - 1$ and Hattingh and Henning [7] completely determined the value $b(P_s, K_{1,t})$ for $s \geq 2$ and $t \geq 2$. Further results for the bipartite Ramsey number related to stars were given in [2, 12].

In Graph Theory, many types of degree conditions were studied for some important properties. We explain it with the Hamiltonicity of graphs as an example. Dirac [4] proved that if a graph G of order $n \geq 3$ satisfies $d_G(x) \geq \frac{n}{2}$ for all $x \in V(G)$, then G is Hamiltonian. This result influenced sufficient conditions for the existence of a Hamiltonian cycle with many extensions, for example, degree-sum condition, neighborhood-union condition, and so on (see a survey [9]). One of important extensions is a so-called Fan-type condition. Fan [5] proved that if a 2-connected graph G of order n satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2} \quad \text{for all } x, y \in V(G) \text{ with } \mathrm{dist}_G(x, y) = 2,$$

where $\mathrm{dist}_G(x, y)$ is the distance between x and y, then G is Hamiltonian, and the result straightforward leads to Dirac's result. In Graph Theory, similar situations occur, i.e., a minimum degree condition is frequently replaced by a Fan-type condition, that is a condition concerning $\max\{d_G(x), d_G(y)\}$ for non-adjacent vertices x and y (see, for example [10, 11, 13]).

We carry the concept to bipartite graphs. As we mentioned above, some bipartite Ramsey numbers involving stars are determined using a high minimum degree condition problem. We will later show that a Fan-type condition gives manageable objects which can be replaced by stars. From such a motivation, we study a Fan-type condition for long paths in bipartite graphs. The following is one of our main results.

Theorem 1 *Let m and n be positive integers with $n \geq m$. Let G be a bipartite graph having partite sets X_1 and X_2 with $|X_1| = |X_2| = n$. If*

(D1) $\max\{d_G(x_1), d_G(x_2)\} \geq m$ or
(D2) $\min\{d_G(x_1), d_G(x_2)\} \geq \frac{n+1}{2}$

for all vertices $x_1 \in X_1$ *and* $x_2 \in X_2$ *with* $x_1x_2 \notin E(G)$, *then* G *contains a path* P with $|V(P)| \geq 2m$.

The condition (D1) in Theorem 1is best possible because $G = K_{n,n} - E(K_{m-1,m-1} \cup K_{n-m+1,n-m+1})$ satisfies $\max\{d_G(x_1), d_G(x_2)\} \geq m-1$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G)$, and any paths of G have at most $2m-1$ vertices.

Let n_1 and n_2 be non-negative integers, and let S_1 and S_2 be two vertex-disjoint stars having n_1+1 vertices and n_2+1 vertices, respectively. The (n_1, n_2) *-bistar*, denoted by B_{n_1,n_2}, is the graph obtained from S_1 and S_2 by joining their centers. Note that the $(n_1, 0)$-bistar is the star having n_1+2 vertices and the (0, 0)-bistar is the connected graph of order two. Recently, Hattingh and Joubert [8] proved that $b(B_{s,s}, B_{t,t}) = s+t+1$, and Alm et al. [1] extended the result as $b(B_{s_1,s_2}, B_{t_1,t_2}) = s_1+t_1+1$ for $s_1 \geq s_2$ and $t_1 \geq t_2$. In particular, we obtain $b(K_{1,s}, K_{1,t}) = b(B_{s-1,s-1}, B_{t-1,t-1})$. Hence the bipartite Ramsey number involving bistars seems to be related to one involving stars.

Recall that $b(P_s, K_{1,t+1}) (= b(P_s, B_{t,0}))$ was determined by Hattingh and Henning [7]. In this paper, using Theorem 1, we extend their result and determine the value $b(P_s, B_{t_1,t_2})$ as following.

Theorem 2 *Let* s, t_1 *and* t_2 *be integers with* $s \geq 2$ *and* $t_1 \geq t_2 \geq 0$. *Then the following hold.*

(i) *If* $t_1 = t_2$, *then* $b(P_s, B_{t_1,t_2}) = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$.

(ii) *Assume that* $t_1 > t_2$.

(ii-a) *If* $t_1 \geq \lfloor \frac{s-1}{2} \rfloor$, *then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} \lfloor \frac{s-1}{2} \rfloor + t_1 + 1 & \left(s \text{ is even, or } s \text{ is odd and } t_1 \equiv 0 \left(\bmod \frac{s-1}{2} \right) \right) \\ \lfloor \frac{s-1}{2} \rfloor + t_1 & (\text{ otherwise }). \end{cases}$$

(ii-b) *If* $t_1 < \lfloor \frac{s-1}{2} \rfloor$, *then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} 2t_1 + 1 & \left(2t_1 - t_2 \geq \lfloor \frac{s-1}{2} \rfloor \right) \\ \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 & (\text{ otherwise }). \end{cases}$$

2 Proof of Theorem 1

We start with two lemmas. The following lemma is well-known (see, for example [7]).

Lemma 1 *Let m be a positive integer, and let G be a bipartite graph. If* $d_G(x) \geq m$ *for all* $x \in V(G)$*, then G contains a path P such that* $|V(P)| \geq 2m$.

Lemma 2 *Let m be a positive integer. Let G be a connected bipartite graph having partite sets* X_1 *and* X_2 *with* $|X_1| \geq |X_2|$*, and let* $x_1 \in X_1$*. If* $d_G(x) \geq m$ *for all* $x \in X_1$*, then G contains a path P such that* x_1 *is an end-vertex of P and* $|V(P)| \geq 2m$.

Proof We proceed by induction on m. It is clear that the lemma holds for $m = 1$. Thus we may assume that $m \geq 2$.

Let $H_0 = G - \{x_1, y : y \in N_G(x_1), d_G(y) = 1\}$. Since $|V(H_0)| \geq |X_1 - \{x_1\}| \geq |X_2| - 1 \geq d_G(x_1) - 1 \geq m - 1 \geq 1$, H_0 is non-empty. Since $|V(H_0) \cap X_1| = |X_1| - 1 \geq |X_2| - 1 \geq |V(H_0) \cap X_2| - 1$, there exists a component H_1 of H_0 such that $|V(H_1) \cap X_1| \geq |V(H_1) \cap X_2| - 1$. Since G is connected, it follows from the definition of H_0 that there exists a vertex $x_2 \in N_G(x_1) \cap V(H_1)$ and $|V(H_1)| \geq 2$.

Since $|V(H_1 - x_2) \cap X_1| = |V(H_1) \cap X_1| \geq |V(H_1) \cap X_2| - 1 = (|V(H_1 - x_2) \cap X_2| + 1) - 1$, there exists a component H_2 of $H_1 - x_2$ such that $|V(H_2) \cap X_1| \geq |V(H_2) \cap X_2|$. Since $d_G(x_2) \geq 2$, there exists a vertex $x_3 \in N_G(x_2) \cap V(H_2)$. Note that $x_3 \in X_1$ and $d_{H_2}(x) = d_G(x) - |N_G(x) - V(H_2)| \geq m - |N_G(x) \cap \{x_2\}| \geq m - 1$ for all $x \in V(H_2) \cap X_1$. By the induction hypothesis, H_2 contains a path Q such that x_3 is an end-vertex of Q and $|V(Q)| \geq 2(m-1)$. Then the path $P = x_1x_2x_3Q$ is a desired path. □

Proof of Theorem 1 Let m, n, G, X_1 and X_2 be as in Theorem 1. By way of contradiction, suppose that every path of G has at most $2m - 1$ vertices. Let $P = y_1y_2 \cdots y_l$ be a longest path of G. Then $l \leq 2m - 1$. Note that $V(G) - V(P) \neq \emptyset$ because $|V(G)| = 2n \geq 2m$. Without loss of generality, we may assume that $y_1 \in X_1$.

Since P is a longest path, all neighbors of y_1 are contained in $V(P) \cap X_2$. So, if $d_G(y_1) \geq m$, then $|V(P)| = |V(P) \cap X_1| + |V(P) \cap X_2| \geq 2|V(P) \cap X_2| \geq 2d_G(y_1) \geq 2m$, a contradiction. Thus, we have $d_G(y_1) \leq m - 1$.

Suppose that there exists a vertex $u \in X_2 - V(P)$ such that (D2) $\min\{d_G(y_1), d_G(u)\} \geq \frac{n+1}{2}$ holds. Let $I_1 = \{1 \leq i \leq \frac{l}{2} : y_1y_{2i} \in E(G)\}$ and $I_2 = \{1 \leq i \leq \frac{l}{2} : uy_{2i-1} \in E(G)\}$. Note that $|I_1| = d_G(y_1) \geq \frac{n+1}{2}$ and since y_l is not a neighbor of u, $|I_2| = d_G(u) - d_{G-V(P)}(u) \geq \frac{n+1}{2} - |X_1 - V(P)|$. Thus,

$$\begin{aligned} n - |X_1 - V(P)| =& |X_1 \cap V(P)| \geq \frac{l}{2} \geq |I_1 \cup I_2| \\ =& |I_1| + |I_2| - |I_1 \cap I_2| \geq n \\ & + 1 - |X_1 - V(P)| - |I_1 \cap I_2|. \end{aligned}$$

This implies $I_1 \cap I_2 \neq \emptyset$, say $i \in I_1 \cap I_2$. Then $y_ly_{l-1} \cdots y_{2i}y_1y_2 \cdots y_{2i-1}u$ is a path longer than P, a contradiction.

Therefore, for $u \in X_2 - V(P)$, (D1) $\max\{d_G(y_1), d_G(u)\} \geq m$ holds. Since $d_G(y_1) \leq m-1$, we have $d_G(u) \geq m$ for $u \in X_2 - V(P)$. Since $|X_1| = |X_2|$ and $|V(P) \cap X_1| \geq |V(P) \cap X_2|$, there exists a component H_0 of $G - V(P)$ such that $|V(H_0) \cap X_2| \geq |V(H_0) \cap X_1|$. Let $h = \max\left\{|N_G(u) \cap V(P)| : u \in V(H_0) \cap X_2\right\}$. Take a vertex $u^* \in V(H_0) \cap X_2$ so that $|N_G(u^*) \cap V(P)| = h$. Since $|V(P) \cap X_1| \leq \frac{l+1}{2} \leq \frac{2m}{2}$ and $u^* y_1 \notin E(G)$, we have $0 \leq h \leq m-1$. For $u \in V(H_0) \cap X_2$, since $d_G(u) \geq m$,

$$d_{H_0}(u) = d_G(u) - |N_G(u) \cap V(P)| \geq m - h(\geq 1).$$

Then by Lemma 2, there exists a path P' of H_0 such that u^* is an end-vertex of P' and $|V(P')| \geq 2(m-h)$. If $h = 0$, then $|V(P')| \geq 2m$, which is a contradiction. Thus $h \geq 1$.

Note that $N_G(u^*) \cap V(P) \subseteq V(P) \cap (X_1 - \{y_1\})(= \{y_{2j-1} : j \geq 2\})$. Let j be the maximum integer satisfying $u^* y_{2j-1} \in E(G)$. Since $|N_G(u^*) \cap V(P)| = h$, we have $j \geq h+1$. Let P'' be the path as $P'' = y_1 P y_{2j-1} u^* P'$. Then $|V(P'')| \geq (2j-1) + 2(m-h) \geq (2(h+1)-1) + 2(m-h) > 2m$, which is a contradiction. This completes the proof of Theorem 1. □

3 Proof of Theorem 2

In this section, we prove Theorem 2. We first give several supporting lemmas.

Lemma 3 *Let N be a positive integer, and let t_1 and t_2 be non-negative integers with $N \geq t_1 \geq t_2$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. If $B_{t_1,t_2} \not\subset G^b$, then*

(N1) $\max\{d_{G^r}(x_1), d_{G^r}(x_2)\} \geq N - t_2$ *or*
(N2) $\min\{d_{G^r}(x_1), d_{G^r}(x_2)\} \geq N - t_1$

for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 x_2 \notin E(G^r)$.

Proof Let $x_1 \in X_1$ and $x_2 \in X_2$ be vertices such that $x_1 x_2 \notin E(G^r)$. Since $B_{t_1,t_2} \not\subset G^b$, $d_{G^b}(x_1) \leq t_j$ or $d_{G^b}(x_2) \leq t_{3-j}$ for each $j \in \{1,2\}$. Since $d_{G^r}(x_i) + d_{G^b}(x_i) = N$, this implies that

$$d_{G^r}(x_1) \geq N - t_j \text{or} d_{G^r}(x_2) \geq N - t_{3-j} \quad \text{for each} j \in \{1,2\}. \tag{1}$$

If $d_{G^r}(x_1) \geq N - t_2$ or $d_{G^r}(x_2) \geq N - t_2$, then (N1) holds. Thus we may assume that $d_{G^r}(x_1) < N - t_2$ and $d_{G^r}(x_2) < N - t_2$. Then by (1), we have $d_{G^r}(x_1) \geq N - t_1$ and $d_{G^r}(x_2) \geq N - t_1$, which implies (N2). □

Lemma 4 *Let s be an integer with $s \geq 2$, and let t_1 and t_2 be non-negative integers with $t_1 \geq t_2$. Then $b(P_s, B_{t_1,t_2}) \leq \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$.*

Proof Let $N = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$.

Suppose that $B_{t_1,t_2} \not\subset G^b$. It suffices to show that $P_s \subset G^r$. Since $t_1 \geq t_2$, it follows from Lemma 3 that $\max\{d_{G^r}(x_1), d_{G^r}(x_2)\} \geq N - t_1 = \lfloor \frac{s-1}{2} \rfloor + 1$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$. Since $N \geq \lfloor \frac{s-1}{2} \rfloor + 1$, applying Theorem 1 with $n = N$ and $m = \lfloor \frac{s-1}{2} \rfloor + 1$, we obtain a path P in G^r with

$$|V(P)| \geq 2\left(\left\lfloor \frac{s-1}{2} \right\rfloor + 1\right) \geq 2\left(\frac{s-2}{2} + 1\right) = s,$$

as desired. □

Lemma 5 *Let s be an odd integer with $s \geq 3$, and let t_1 and t_2 be non-negative integers such that $t_1 > t_2$ and $t_1 \not\equiv 0(\bmod \frac{s-1}{2})$. Then $b(P_s, B_{t_1,t_2}) \leq \frac{s-1}{2} + t_1$.*

Proof Let $N = \frac{s-1}{2} + t_1$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. By way of contradiction, suppose that $P_s \not\subset G^r$ and $B_{t_1,t_2} \not\subset G^b$. Since $t_1 > t_2$, it follows from Lemma 3 that $\max\{d_{G^r}(x_1), d_{G^r}(x_2)\} \geq N - t_1 = \frac{s-1}{2}$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$.

Claim 1 *If a component H of G^r contains a path of order $s-1$, then $|V(H)| = s-1$.*

Proof Suppose that H contains a path $P = y_1y_2\cdots y_{s-1}$. Without loss of generality, we may assume that $y_1 \in X_1$. Note that $y_{s-1} \in X_2$. Since H contains no path of order s, $N_H(y_1) \subseteq V(P) \cap X_2$ and $N_H(y_{s-1}) \subseteq V(P) \cap X_1$. If $y_1y_{s-1} \notin E(H)$, then $d_H(y_1) \leq |V(P) \cap (X_2 - \{y_{s-1}\})| = \frac{s-3}{2}$, $d_H(y_{s-1}) \leq |V(P) \cap (X_1 - \{y_1\})| = \frac{s-3}{2}$, which contradicts the fact that $\max\{d_H(y_1), d_H(y_{s-1})\} \geq \frac{s-1}{2}$. Thus $y_1y_{s-1} \in E(H)$. In particular, $y_1y_2\cdots y_{s-1}y_1$ is a cycle of H. Since G^r contains no path of order s, it follows that $N_H(y_i) \subseteq V(P)$ for all $i (1 \leq i \leq s-1)$. In particular, $H[V(P)] = H$. □

Since $N = \frac{s-1}{2} + t_1 \geq \frac{s-1}{2}$, applying Theorem 1 with $n = N$ and $m = \frac{s-1}{2}$, we obtain a path P in G^r with $|V(P)| \geq 2 \cdot \frac{s-1}{2} = s-1$. It follows from Claim 1 that $G^r[V(P)]$ is a component of G^r. In particular, $d_{G^r[V(P)]}(x) \leq \frac{s-1}{2} = N - t_1 < N - t_2$ for all $x \in V(P)$. This together with Lemma 3 implies that $d_{G^r}(u) \geq N - t_1$ for all $u \in V(G^r) - V(P)$.

Since $N - \frac{s-1}{2} = t_1 \geq 1$, $V(G^r) - V(P) \neq \emptyset$. Let H be a component of G^r other than $G^r[V(P)]$. Since $d_{G^r}(u) \geq N - t_1 = \frac{s-1}{2}$ for every $u \in V(H)$, it follows from Lemma 1 that H contains a path of order $s-1$. Then by Claim 1, $|V(H)| = s-1$ (i.e., $|V(H) \cap X_1| = \frac{s-1}{2}$). Since H is arbitrary, $N(= |X_1|)$ is a multiple of $\frac{s-1}{2}$, which contradicts the assumption that $t_1 \not\equiv 0(\bmod \frac{s-1}{2})$. □

Lemma 6 *Let s be an integer with $s \geq 2$, and let t_1 and t_2 be non-negative integers with $\lfloor \frac{s-1}{2} \rfloor > t_1 > t_2$. Then*

$$b(P_s, B_{t_1,t_2}) \le \begin{cases} 2t_1 + 1 & \left(2t_1 - t_2 \ge \lfloor \frac{s-1}{2} \rfloor\right) \\ \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 & (\text{ otherwise }). \end{cases}$$

In other words, Lemma 6 concludes that $b(P_s, B_{t_1,t_2}) \le \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\}$.

Proof of Lemma 6 Let $N = \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\}$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. Suppose that $B_{t_1,t_2} \not\subset G^b$ as a subgraph. It suffices to show that $P_s \subset G^r$. Note that $N - t_2 \ge (\lfloor \frac{s-1}{2} \rfloor + t_2 + 1) - t_2 = \lfloor \frac{s-1}{2} \rfloor + 1$ and $N - t_1 \ge \frac{N+1}{2}$ because

$$\begin{aligned} 2(N - t_1) - (N + 1) &= N - 2t_1 - 1 \\ &= \left\lfloor \frac{s-1}{2} \right\rfloor + t_2 + \max\left\{2t_1 - t_2 - \left\lfloor \frac{s-1}{2} \right\rfloor, 0\right\} - 2t_1 \\ &= \begin{cases} 0 & \left(2t_1 - t_2 \ge \lfloor \frac{s-1}{2} \rfloor\right) \\ \lfloor \frac{s-1}{2} \rfloor - (2t_1 - t_2) > 0 & (\text{ otherwise }). \end{cases} \end{aligned}$$

This together with Lemma 3 implies that, for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$,

- $\max\{d_{G^r}(x_1), d_{G^r}(x_2)\} \ge N - t_2 \ge \lfloor \frac{s-1}{2} \rfloor + 1$ or
- $\min\{d_{G^r}(x_1), d_{G^r}(x_2)\} \ge N - t_1 \ge \frac{N+1}{2}$.

Since $N = \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\} \ge \lfloor \frac{s-1}{2} \rfloor + 1$, applying Theorem 1 with $n = N$ and $m = \lfloor \frac{s-1}{2} \rfloor + 1$, we obtain a path P in G^r with

$$|V(P)| \ge 2\left(\left\lfloor \frac{s-1}{2} \right\rfloor + 1\right) \ge 2\left(\frac{s-2}{2} + 1\right) = s,$$

as desired. □

Proof of Theorem 2 Let s, t_1 and t_2 be as in Theorem 2. We first prove the theorem for the case where $s = 2$, i.e., $b(P_2, B_{t_1,t_2}) = t_1 + 1$. By Lemma 4, we have $b(P_2, B_{t_1,t_2}) \le t_1 + 1$. Now we prove that $b(P_2, B_{t_1,t_2}) \ge t_1 + 1$. Let X_1 and X_2 be the partite sets of K_{t_1,t_1}. Let G^r be the graph obtained from K_{t_1,t_1} by deleting all edges, and let $G^b = K_{t_1,t_1}$. Then it is clear that $P_2 \not\subset G^r$ and $B_{t_1,t_2} \not\subset G^b$, and so $b(P_2, B_{t_1,t_2}) \ge t_1 + 1$. Thus we may assume that $s \ge 3$. Let $q \in \mathbb{N} \cup \{0\}$ and $r (0 \le r \le \lfloor \frac{s-1}{2} \rfloor - 1)$ be the integers satisfying $t_1 = \lfloor \frac{s-1}{2} \rfloor q + r$.

(i) Suppose that $t_1 = t_2$. Let $N = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$. By Lemma 4, we have $b(P_s, B_{t_1,t_2}) \le N$. Now we prove that $b(P_s, B_{t_1,t_2}) \ge N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$. We partition X_i into $q+2$ sets $X_i^0, X_i^1, \ldots, X_i^{q+1}$ with $|X_i^0| = |X_i^1| = \cdots = |X_i^q| = \lfloor \frac{s-1}{2} \rfloor$ and $|X_i^{q+1}| = r$. Note that $X_i^{q+1} = \emptyset$ if and only if $t_1 \equiv 0(\bmod \lfloor \frac{s-1}{2} \rfloor)$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ such that

$$E(G^r) = \bigcup_{0 \le j \le q+1} \{x_1x_2 : x_1 \in X_1^j, x_2 \in X_2^j\},$$

and let $G^b = K_{N-1,N-1} - E(G^r)$. Then the order of longest paths of G^r is at most $2\lfloor \frac{s-1}{2} \rfloor (\le s-1)$. Furthermore, since $\min\{d_{G^b}(x_1), d_{G^b}(x_2)\} \le (N-1) - \lfloor \frac{s-1}{2} \rfloor = t_1 (= t_2)$ for every edge $x_1x_2 \in E(G^b)$, we see that $B_{t_1,t_2} \not\subseteq G^b$. Therefore $b(P_s, B_{t_1,t_2}) \ge N$.

(ii-a) Suppose that $t_1 > t_2$ and $t_1 \ge \lfloor \frac{s-1}{2} \rfloor$. Note that $q \ge 1$. Let

$$N = \begin{cases} \lfloor \frac{s-1}{2} \rfloor + t_1 + 1 & \left(s \text{ is even, or } s \text{ is odd and } t_1 \equiv 0\left(\bmod \frac{s-1}{2}\right)\right) \\ \lfloor \frac{s-1}{2} \rfloor + t_1 & (\text{ otherwise }). \end{cases}$$

By Lemmas 4 and 5, we have $b(P_s, B_{t_1,t_2}) \le N$. Now we prove that $b(P_s, B_{t_1,t_2}) \ge N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$. If s is even, or s is odd and $t_1 \equiv 0(\bmod \frac{s-1}{2})$, we partition X_i into $q+2$ sets $X_i^0, X_i^1, \ldots, X_i^{q+1}$ with $|X_i^0| = |X_i^1| = \cdots = |X_i^q| = \lfloor \frac{s-1}{2} \rfloor$ and $|X_i^{q+1}| = r$; otherwise, we partition X_i into $q+2$ sets $X_i^0, X_i^1, \ldots, X_i^{q+1}$ with

- $|X_i^j| = \frac{s-1}{2}$ for $i \in \{1,2\}$ and $j (0 \le j \le q)$ with $(i,j) \notin \{(1,0),(2,1)\}$,
- $|X_1^0| = |X_2^1| = \frac{s-3}{2}$ and
- $|X_1^{q+1}| = |X_2^{q+1}| = r$.

Note that $X_i^{q+1} = \emptyset$ if and only if $t_1 \equiv 0(\bmod \lfloor \frac{s-1}{2} \rfloor)$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ obtained by

- joining all vertices in X_1^0 to all vertices in $X_2^0 \cup X_2^{q+1}$,
- joining all vertices in X_2^1 to all vertices in $X_1^1 \cup X_1^{q+1}$ and
- for each $j (2 \le j \le q)$, joining all vertices in X_1^j to all vertices in X_2^j,

and let $G^b = K_{N-1,N-1} - E(G^r)$. If s is even, then the order of longest paths of G^r is at most $2\lfloor \frac{s-1}{2} \rfloor + 1 = 2 \cdot \frac{s-2}{2} + 1 = s-1$; if s is odd and $t_1 \equiv 0(\bmod \frac{s-1}{2})$, then the order of longest paths of G^r is $2\lfloor \frac{s-1}{2} \rfloor = 2 \cdot \frac{s-1}{2} = s-1$; if s is odd and $t_1 \not\equiv 0(\bmod \frac{s-1}{2})$, then the order of longest paths of G^r is at most

$$\max\left\{2\cdot\frac{s-1}{2}, 2\cdot\frac{s-3}{2}+1\right\} = s-1.$$

Furthermore, since we easily check that $d_{G^b}(x) \le t_1$ for all $x \in V(G^b)$, $B_{t_1,t_2} \not\subseteq G^b$. Therefore $b(P_s, B_{t_1,t_2}) \ge N$.

(ii-b) Suppose that $t_1 > t_2$ and $t_1 < \lfloor\frac{s-1}{2}\rfloor$. Let $N = \lfloor\frac{s-1}{2}\rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor\frac{s-1}{2}\rfloor, 0\}$. By Lemma 6, we have $b(P_s, B_{t_1,t_2}) \le N$. Now we prove that $b(P_s, B_{t_1,t_2}) \ge N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$. If $2t_1 - t_2 \ge \lfloor\frac{s-1}{2}\rfloor$ (i.e., $N-1 = 2t_1$), we partition X_i into two sets X_i^1 and X_i^2 with $|X_i^1| = |X_i^2| = t_1$; otherwise (i.e., $N = \lfloor\frac{s-1}{2}\rfloor + t_2$), we partition X_i into two sets X_i^1 and X_i^2 with $|X_i^1| = \lfloor\frac{s-1}{2}\rfloor$ and $|X_i^2| = t_2$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ such that

$$E(G^r) = \bigcup_{j\in\{1,2\}} \{x_1x_2 : x_1 \in X_1^j, x_2 \in X_2^j\},$$

and let $G^b = K_{N-1,N-1} - E(G^r)$. Since $t_2 < t_1 < \lfloor\frac{s-1}{2}\rfloor$, the order of longest paths of G^r is at most $2\lfloor\frac{s-1}{2}\rfloor (\le 2\cdot\frac{s-1}{2} = s-1)$. Furthermore, if $2t_1 - t_2 \ge \lfloor\frac{s-1}{2}\rfloor$, then $d_{G^b}(x) = (N-1) - t_1 = t_1$ for all $x \in V(G^b)$; if $2t_1 - t_2 < \lfloor\frac{s-1}{2}\rfloor$, then $\min\{d_{G^b}(x_1), d_{G^b}(x_2)\} = t_2$ for every edge $x_1x_2 \in E(G^b)$. In either case, $B_{t_1,t_2} \not\subseteq G^b$. Therefore $b(P_s, B_{t_1,t_2}) \ge N$.

This completes the proof of Theorem 2. □

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