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The Edge-Connectivity of Strongly 3-Walk-Regular Graphs

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Abstract
E.R. van Dam and G.R. Omidi generalized the concept of strongly regular graphs as follows. If for any two vertices the number of ℓ-walks (walks of length ℓ) from one vertex to the other is the same which depends only on whether the two vertices are the same, adjacent or non-adjacent, then G is called a strongly ℓ-walk-regular graph. The existence of strongly ℓ-walk-regular graphs which are not strongly 3-walk-regular graphs is unknown. In this paper, we prove that the edge-connectivity of a connected strongly 3-walk-regular graph G of degree $k \geq 3$ is equal to k. Moreover, if G is not the graph formed by adding a perfect matching between two copies of K_4, then each edge cut set of size k is precisely the set of edges incident with a vertex of G. For a regular graph G in general, we also give a sufficient and tight condition such that G is 1-extendable.

Keywords Edge-connectivity · Eigenvalue · Strongly 3-walk-regular graph · Perfect matching · 1-Extendable

Mathematics Subject Classification 05C40 · 05C50 · 05C70

1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. Let B be a subset of the vertex set $V(G)$ of G. The graph $G - B$ is derived from G by deleting the vertices of B and the edges incident with a vertex of B. If $B = \{u\}$, we denote $G - B$ by $G - u$ for convenience. The graph $G[B]$ is the subgraph of G induced by B. Let E be a subset of the edge set $E(G)$. The graph $G - E$ is defined with vertex set $V(G - E) = V(G)$ and edge set $E(G) - E$. Let S_1 and S_2 be two disjoint subsets of $V(G)$ (or two vertex-disjoint subgraphs of G). Denote the set of edges between S_1

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and S_2 by $E(S_1, S_2)$. Let G be a connected graph. A subset S of $V(G)$ is said to be a *cut set* if $G - S$ is not connected. The *connectivity* of a non-complete graph G is the minimum size of a cut set in G. We always admit that the connectivity of K_{k+1} is k. A subset N of the edge set $E(G)$ is said to be an *edge cut set* of G if there are two disjoint subsets A and B of $V(G)$ such that $V(G) = A \bigcup B$ and $E(A, B) = N$. An edge cut set $N = E(A, B)$ is called a *cyclic edge cut set* if both $G[A]$ and $G[B]$ contain a cycle. The *(cyclic) edge-connectivity* of G is the minimum size of a (cyclic) edge cut set in G.

A *matching* M of G is a subset of $E(G)$ such that the edges are independent in G. If M covers each vertex of G, then M is called a *perfect matching* or *a one-factor*. The graph G is 1-*extendable* if each edge of G is contained in a perfect matching of G, and G is *factor-critical* if $G - u$ has a perfect matching for any vertex u of G. Tutte Theorem (see Theorem 2.4) is a fundamental result in matching theory, referring to [1,14].

E.R. van Dam and G.R. Omidi generalized the concept of strongly regular graphs to strongly ℓ-walk-regular graphs in [9]. For an integer $\ell \geq 2$, a (non-complete) graph G is called *a strongly ℓ-walk-regular graph*, if for any two vertices of G the number of walks of length ℓ from one vertex to the other is the same which depends only on whether the two vertices are the same, adjacent or non-adjacent. In [9], it was shown that a strongly ℓ-walk-regular graph, which is not a strongly regular graph, is either a strongly ℓ-walk-regular graph for any odd integer $\ell \geq 3$, or a strongly ℓ-walk-regular graph for a unique odd integer $\ell \geq 3$. There are many examples of strongly 3-walk-regular graphs which are not strongly regular graphs. However, the existence of strongly ℓ-walk-regular graphs which are not strongly 3-walk-regular graphs is unknown.

Let $1, 2, \ldots,$ and $v = |V(G)|$ be the vertices of G. The adjacency matrix of G is a square matrix of order $|V(G)|$, which is denoted by $A = (a_{ij})$, where $a_{ij} = 1$ if the vertex i and the vertex j are adjacent, and $a_{ij} = 0$ otherwise. The eigenvalues of G are the eigenvalues of A. In the sense of adjacency matrix, G is a strongly regular graph if there exist three numbers k, a and b such that $A^2 = kI + aA + b(J - I - A)$, where I is the identity matrix and J is the all-one matrix. Similarly, for an integer $\ell \geq 2$, a graph G is called a strongly ℓ-walk-regular graph if $A^\ell = aI + bA + c(J - I - A)$ for some numbers a, b and c. It is obvious that a strongly regular graph is also a strongly ℓ-walk-regular graph for any integer $\ell \geq 2$ by induction on ℓ as A^2 is a linear combination of A, I and J.

In general, a strongly ℓ-walk-regular graph can be not regular, and can be connected even for $c = 0$. It determined the strongly ℓ-walk-regular graphs with $c = 0$ in Proposition 5.1 from [9]. These graphs are precisely either a disjoint union of complete graphs of the same order, or a disjoint union of complete bipartite graphs of the same size and possibly some isolated vertices when ℓ is odd. If $c > 0$, then J can be expressed as a polynomial of A and thus $AJ = JA$, which implies that G is regular. Moreover, since for any two non-adjacent vertices there is a walk of length ℓ connecting them when $c > 0$, the diameter of G is at most ℓ, and thus G is connected.

Let G be a connected strongly regular graph of degree $k \geq 3$. In [3] it was shown that the connectivity of G is equal to k, and each cut set of size k is precisely the set of neighbours of a vertex in G. Later, the minimum size of the cut sets which disconnect G into non-singleton components was partially studied in [4,5]. For the cyclic edge-connectivity of G, it was studied in [15].

In this paper, we prove that the edge-connectivity of a connected strongly 3-walk-regular graph G of degree $k \geq 3$ is equal to k. Moreover, if G is not the graph formed by adding a perfect matching between two copies of K_4, which has parameters $v = 8, k = 4, a = 6, b = 10$ and $c = 6$, then each edge cut set of size k is precisely the set of edges incident with a vertex of G. For a regular graph G in general, we also give a sufficient and tight condition such that G is 1-extendable. Our proofs are mainly based on spectrum technique and Tutte Theorem (see Theorem 2.4). In this paper, for any terminology used but not defined, one may refer to [11,13].

2 Main Tools

For a connected k-regular graph with four distinct eigenvalues, a characterization of the spectrum is given in [6, Theorem 2.6]. The characterization is as follows.

Lemma 2.1 *Let G be a connected k-regular graph on v vertices with eigenvalues with multiplicities $k^1, \theta_1^{m_1}, \theta_2^{m_2}$ and $\theta_3^{m_3}$, and let $m = \frac{v-1}{3}$. Then $m_1 = m_2 = m_3 = m$ and $k = m$ or $k = 2m$, or G has two or four integral eigenvalues. Moreover, if G has exactly two integral eigenvalues, then the other two have the same multiplicities and are of the form $\frac{p \pm \sqrt{q}}{2}$, where p, q are integers.*

The following conclusion is from Lemma 3.3 and Proposition 4.1 in [9].

Lemma 2.2 *Let G be a connected strongly 3-walk-regular graph which is not a strongly regular graph. Then G has four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$ and $\theta_1 + \theta_2 + \theta_3 = 0$, where k is the degree of G.*

The following conclusion is a result based on interlacing technique, referring to Corollary 4.8.4 in [2].

Lemma 2.3 *Let G be a connected k-regular graph on v vertices. Suppose that S is a subset of $V(G)$. Then $|E(S, \overline{S})| \geq \frac{(k-\theta)|S|(v-|S|)}{v}$, where θ is the second largest eigenvalue of G and $\overline{S} = V(G) - S$.*

Let G be a graph and S be a subset of $V(G)$. The number of odd components of $G - S$ is denoted by $o(G - S)$. Tutte [14] (or [1]) proved the following conclusion.

Theorem 2.4 *A graph G has a perfect matching if and only if for any subset S of $V(G)$, it satisfies that $o(G - S) \leq |S|$.*

3 Main Results

Let G be a strongly regular graph of degree $k \geq 3$. Since the edge-connectivity of a graph is not less than its connectivity, thus we have that the edge-connectivity of G is equal to k by the result of [3], and each edge cut set of size k is precisely the set of edges incident with a vertex in G as any two non-adjacent vertices of G has a common neighbour. (This conclusion can be also derived directly from the result in [15].)

Note 1 Let G be a connected strongly 3-walk-regular graph on v vertices of degree $k \geq 3$. We can suppose that G is not a strongly regular graph by the above discussion when we study the edge-connectivity of G. Thus by Lemma 2.2, G has four distinct eigenvalues and the spectrum can be denoted by $k^1, \theta_1^{m_1}, \theta_2^{m_2}$ and $\theta_3^{m_3}$. Let A be the adjacency matrix of G. We can suppose that $A^3 = aI + bA + c(J - I - A)$ for some numbers (integers) a, b and c. Since there are precisely a 3-walks from each vertex to itself, thus each vertex is in $\frac{a}{2}$ triangles. Since for any two adjacent vertices there are precisely b 3-walks from one vertex to the other, thus each edge is in $b - 2k + 1$ 4-cycles in G. In the following discussions on strongly 3-walk-regular graphs, we always have this assumption.

Theorem 3.1 *Let G be a connected strongly 3-walk-regular graph defined as Note 1. If G has non-integral eigenvalues, then the spectrum is $k^1, \left(\frac{m+\sqrt{n}}{2}\right)^f, (-m)^g, \left(\frac{m-\sqrt{n}}{2}\right)^f$, where $m, n \in Z$, $m \neq 0$, $f = \frac{v-1-\frac{k}{m}}{3}$ and $g = \frac{v-1+\frac{2k}{m}}{3}$. Moreover, we have the following conclusions.*

- (i) *For $m = 1$, then $k \geq 8$ is even, and the four distinct eigenvalues of G are $k, -1$ and $\frac{1\pm\sqrt{6k-3}}{2}$. Moreover, we have $a - c = \frac{3k-2}{2}$, $b - c = \frac{3k}{2}$ and $2cv = (k+1)(k-2)(2k-1)$.*
- (ii) *For $m \neq 1$, if $k \geq 3$ is odd, then $\lambda_2(G) \leq \frac{\sqrt{18k-6}}{3}$; if $k \geq 4$ is even, then $\lambda_2(G) \leq \frac{\sqrt{18k-12}}{3}$.*
- (iii) *We must have $k \neq 3$ and $k \neq 4$, which means that if G has non-integral eigenvalues, then $k \geq 5$.*

Proof By Lemma 2.2, we have $\theta_1 + \theta_2 + \theta_3 = 0$. If $m_1 = m_2 = m_3 = t$, then $0 = tr(A) = k + t(\theta_1 + \theta_2 + \theta_3) = k$, a contradiction. Therefore, G has precisely two distinct non-integral eigenvalues by Lemma 2.1. Moreover, the two distinct non-integral eigenvalues have the same multiplicity and are of the form $\frac{m\pm\sqrt{n}}{2}$, with $m, n \in Z$. Then another eigenvalue is $-m$ as $\theta_1 + \theta_2 + \theta_3 = 0$. Now we can suppose that the eigenvalues with multiplicities are $k^1, \left(\frac{m+\sqrt{n}}{2}\right)^f, (-m)^g, \left(\frac{m-\sqrt{n}}{2}\right)^f$. If $m = 0$, then $0 = tr(A) = k + (f - g)m = k$, a contradiction. Now suppose $m \neq 0$. Combining $v = 1 + 2f + g$ with $0 = tr(A) = k + (f - g)m$, we have $f = \frac{v-1-\frac{k}{m}}{3}$ and $g = \frac{v-1+\frac{2k}{m}}{3}$.

Since all eigenvalues of A except k are the roots of equation $x^3 + (c-b)x - (a-c) = 0$, we have $(-m)^3 + (c - b)(-m) - (a - c) = 0$ (1),

$\left(\frac{m+\sqrt{n}}{2}\right)^3 + (c - b)\left(\frac{m+\sqrt{n}}{2}\right) - (a - c) = 0$ (2)

and $\left(\frac{m-\sqrt{n}}{2}\right)^3 + (c - b)\left(\frac{m-\sqrt{n}}{2}\right) - (a - c) = 0$ (3).

By equality (1) we have $m^3 = (b - c)m - (a - c)$ (4).

By (2)–(3) we have $3m^2 + n = 4(b - c)$ (5).

By (2) and (5) we have $-m^3 + mn = 4(a - c)$ (6).

Notice that $kv = tr(A^2) = k^2 + m^2 g + f\left[\left(\frac{m+\sqrt{n}}{2}\right)^2 + \left(\frac{m-\sqrt{n}}{2}\right)^2\right] = k^2 + \left(\frac{f}{2} + g\right)m^2 + \frac{f}{2}n$ (7).

Substituting the values of f and g in (7), we have $6k(v - k - \frac{m}{2}) = 3m^2(v-1) + n(v - 1 - \frac{k}{m})$ (8).

Notice that $k^3 = a + bk + c(v - 1 - k) = a - c + (b - c)k + cv$ (9).

Combining (5) and (6) with (9), we have $k^3 = \frac{3m^2k - m^3}{4} + \frac{m+k}{4}n + cv$ (10).

(i) Suppose $m = 1$. By (8) we have $n = 6k - 3$. By (5) and (6) we have $a - c = \frac{3k-2}{2}$ and $b - c = \frac{3k}{2}$. By (10) we have $2cv = (k+1)(k-2)(2k-1)$. Moreover, the four distinct eigenvalues of G are k, -1 and $\frac{1\pm\sqrt{6k-3}}{2}$.

We see that k is even as $a - c = \frac{3k-2}{2}$ is an integer. If $k = 4$, then by $2cv = (k+1)(k-2)(2k-1)$ we have $cv = 5 * 7$. Therefore, we have $v = 7, a = 10, b = 11$ and $c = 5$, or $v = 35, a = 6, b = 7$ and $c = 1$. It is easy to check that there are no graphs with such parameters (see the penultimate sentence of Note 1). If $k = 6$, then by $2cv = (k+1)(k-2)(2k-1)$ we have $cv = 2 * 7 * 11$. Since $f = \frac{v-1-\frac{k}{m}}{3} = \frac{v-7}{3}$, thus $v \equiv 1(mod3)$. Therefore, we have $v = 22, a = 15$ or $v = 154, a = 9$, which is impossible as a is even (see Note 1). Thus we have $k \geq 8$.

(ii) Suppose $m \neq 1$. If $m \geq 2$, then by (8) we have $6k(v - k - 1) \geq 6k(v - k - \frac{m}{2}) = 3m^2(v-1) + n(v - 1 - \frac{k}{m}) \geq 3m^2(v-1) + n(v - 1 - \frac{k}{2})$, which implies $3m^2 + n < 6k$. If $-k \leq m \leq -1$, then by (8) we have $6k(v - \frac{k}{2}) \geq 6k(v - k - \frac{m}{2}) = 3m^2(v-1) + n(v - 1 - \frac{k}{m}) \geq 3m^2(v-1) + nv$, which implies $3m^2 + n < 6k$.

By the above discussion we have $3m^2 + n < 6k$. By (5) we see that $4|(3m^2 + n)$. Thus $3m^2 + n \leq 6k - 2$ if $k \geq 3$ is odd, and $3m^2 + n \leq 6k - 4$ if $k \geq 4$ is even. By elementary inequality we have $(|m| + \sqrt{n})^2 \leq (\frac{1}{\sqrt{3}}\sqrt{3}|m| + \sqrt{n})^2 \leq (\frac{1}{3} + 1)(3m^2 + n) = \frac{4}{3}(3m^2 + n)$. Thus we have $\frac{|m|+\sqrt{n}}{2} \leq \frac{\sqrt{3}}{3}\sqrt{3m^2 + n}$. Notice that n is not the square of an integer, and thus $n \geq 2$. If $k \geq 3$ is odd, then $\frac{|m|+\sqrt{n}}{2} \leq \frac{\sqrt{18k-6}}{3}$. Moreover, by $3m^2 + n \leq 6k - 2$ we have $|m| \leq \sqrt{2k-2} \leq \frac{\sqrt{18k-6}}{3}$. Thus $\lambda_2(G) \leq \frac{\sqrt{18k-6}}{3}$. If $k \geq 4$ is even, then $\frac{|m|+\sqrt{n}}{2} \leq \frac{\sqrt{18k-12}}{3}$. Moreover, by $3m^2 + n \leq 6k - 4$ we have $|m| \leq \sqrt{2k-2} \leq \frac{\sqrt{18k-12}}{3}$. Thus $\lambda_2(G) \leq \frac{\sqrt{18k-12}}{3}$.

(iii) For $k = 3$, by $f = \frac{v-1-\frac{k}{m}}{3}$ we have $m = \pm 1$. By (i) of Theorem 3.1 we have $m = -1$. By the proof of (ii) of Theorem 3.1, which says $3m^2 + n \leq 6k - 2 = 16$, we have $n \leq 13$. By (5) we have $4|(3m^2 + n)$ and thus $n \equiv 1(mod4)$, which implies $n = 5$ or 13. (Notice that n is not the square of an integer, since G has non-integral eigenvalues.) If $n = 5$, by (10) we have $cv = 24$, which implies $v = 6, 8, 12$ or 24. By $f = \frac{v+2}{3}$ we have $v \equiv 1(mod3)$, a contradiction. If $n = 13$, by (10) we have $cv = 18$, which implies $v = 6, 9$ or 18. By $f = \frac{v+2}{3}$ we have $v \equiv 1(mod3)$, a contradiction.

For $k = 4$, by $f = \frac{v-1-\frac{k}{m}}{3}$ we have $m = \pm 1$ or ± 2. By (i) of Theorem 3.1 we have $m = -1$ or ± 2. By the proof of (ii) of Theorem 3.1, we have $3m^2 + n \leq 6k - 4 = 20$. By (5) we have $4|(3m^2 + n)$.

If $m = -1$, then we have $n \leq 17$ and $n \equiv 1 (mod 4)$, which implies $n = 5, 13$ or 17. If $n = 5$, then by (8) we have $v = 6$, a contradiction. If $n = 13$, then by (8) and (10) we have $v = 15$ and $cv = 51$, a contradiction. If $n = 17$, then by (8) and (10) we have $v = 33$ and $cv = 48$, a contradiction.

If $m = \pm 2$, then we have $n \leq 8$ and $4|n$, which implies $n = 8$. For $m = -2$, by (8) and (10) we have $v = 17$ and $cv = 46$, a contradiction. For $m = 2$, by (8) and (10) we have $v = 21$ and $cv = 42$, which implies $a = 4, b = 7$ and $c = 2$. It is not difficult to show that there is no such graphs with these parameters. (If G has parameters $v = 21, k = 4, a = 4, b = 7$ and $c = 2$, then by Note 1 we have that each vertex is in 2 triangles, and there is no 4-cycles. Let u be a vertex, and $N_i(u)$ be the set of vertices from which to u the distance is i, where $0 \leq i \leq 3$ (diameter≤ 3). Then the subgraph induced by $N_1(u)$ (the neighbours of u) is the disjoint union of two copies of K_2, and the subgraph induced by $N_2(u)$ is 2-regular with 8 vertices, and has a perfect matching (Notice that for any two non-adjacent vertices there are precisely $c = 2$ 3-walks from one vertex to the other). Moreover, each vertex in $N_2(u)$ has precisely one neighbour in $N_1(u)$. Since each vertex is in 2 triangles, thus we must have $|N_3(u)| = 4$, which implies $v = 1 + 4 + 8 + 4 = 17$, a contradiction. We complete the proof. □

Theorem 3.2 *Let G be a connected strongly 3-walk-regular graph defined as Note 1. If G has integral eigenvalues (each eigenvalue is an integer), then its second largest eigenvalue is at most $k - 2$. Moreover, we have $k \geq 4$.*

Proof Without loss of generality, we can suppose $k > \theta_1 > \theta_2 > \theta_3$. We prove the conclusion by contradiction. Suppose $\theta_1 = k - 1$.

Since all eigenvalues of A except k are the roots of equation $x^3 + (c - b)x - (a - c) = 0$, we have $\theta_2 + \theta_3 = 1 - k$, $\theta_2\theta_3 + (k-1)(\theta_2 + \theta_3) = c - b$ and $(k-1)\theta_2\theta_3 = a - c$. If $\theta_3 = -k$, then G is a bipartite graph, and thus $c = 0$. By Proposition 5.1 of [9] we have that G is a complete bipartite graph and is thus a strongly regular graph, a contradiction. Thus $\theta_3 \geq 1 - k$, and then $\theta_2 \leq 0$. If $\theta_2 = 0$ and thus $\theta_3 = 1 - k$, then we have $(k-1)|k$ as $tr(A) = 0$, which is impossible. Thus we have $-1 \geq \theta_2 > \theta_3 \geq 2 - k$. As $a \leq k(k-1)$, $c > 0$, we have $(k-1)\theta_2\theta_3 \leq k(k-1) - 1$, which implies $\theta_2\theta_3 \leq k - 1$.

Since $\theta_2\theta_3 \leq k - 1$ and $\theta_2 + \theta_3 = 1 - k$, we have $\theta_2 = -1$ and $\theta_3 = 2 - k$ for $-1 \geq \theta_2 > \theta_3 \geq 2 - k$. Thus $a - c = (k-1)(k-2)$ and $b - c = k^2 - 3k + 3$. Since a is even, thus c is even. By $k^3 = a + bk + c(v - 1 - k)$, we have $cv = 2(k+1)(k-1)$, which implies $v|(k-1)(k+1)$ as c is even. Combining $tr(A) = 0$ and $tr(A^2) = vk$ with $1 + m_1 + m_2 + m_3 = v$, by substitution of eigenvalues we have $m_1 = \frac{2(k-1)(v-1-k)}{2k^2-3k}$. Since $(k-1, 2k^2 - 3k) = 1$, we have $(2k^2 - 3k)|(2v - 2(k+1))$. Since $v|(k-1)(k+1)$, we have $2v - 2(k+1) \leq 2k^2 - 2k - 4 < 2(2k^2 - 3k)$. Then by $(2k^2 - 3k)|(2v - 2(k+1))$ we have $2k^2 - 3k = 2v - 2(k+1)$, which implies $2v = 2k^2 - k + 2$. Then $(2k^2 - k + 2)|2(k+1)(k-1)$, which implies $2k^2 - k + 2 = 2(k+1)(k-1)$ or $k = 4$. Then G has parameters $v = 15, k = 4, a = 8, b = 9$ and $c = 2$. It is not difficult to show that it is impossible, since each vertex is in 4 triangles and each edge is in 2 4-cycles by Note 1. Then we complete the proof of the first part.

Now suppose that the case $k = 3$ is possible. By the above discussion we have that the second largest eigenvalue of G is at most $k - 2 = 1$. Thus the four distinct eigenvalues are $3, 1, 0$ and -1 ($\theta_1 + \theta_2 + \theta_3 = 0$), which is impossible as $m_2 = 8 - 2v < 0$. We complete the proof. □

Remark 1 By Theorems 3.1 and 3.2 (see Note 1) we have that each connected cubic strongly 3-walk-regular graph is a strongly regular graph.

Lemma 3.3 *Let G be a connected strongly 3-walk-regular graph defined as Note 1. Then G contains no k-clique (a clique of size k) except the graph formed by adding a perfect matching between two copies of K_4, which has parameters $v = 8, k = 4, a = 6, b = 10$ and $c = 6$.*

Proof By Theorems 3.1 and 3.2 we can suppose $k \geq 4$. Now suppose that G contains a k-clique W. Thus $a \geq (k-1)(k-2)$. Let x be a vertex of $\overline{W}$ such that x is adjacent to a vertex y of W. Then we have $|E(x, W)| \mid k$, since each vertex is in the same number of triangles. Since both the vertex x and the vertex y are in $\frac{a}{2}$ triangles, thus the vertex x has the the unique neighbour in W which is the vertex y for $k \geq 4$. It implies $a = (k-1)(k-2)$. Consequently, for each vertex u in $\overline{W}$, if u is adjacent to W, then it has precisely one neighbour in W. Then each vertex is in a k-clique as G is connected.

It is easy to see that two distinct k-cliques are vertex-disjoint by the above discussion. Therefore, the vertices of G can be partitioned into some k-cliques. For any two adjacent k-cliques W and H, the number of edges between them is at least 2, since both an edge between them and an edge in W are in the same number of 4-cycles. Thus, we can suppose that there are $p \geq 2$ edges between them, and thus there are two vertices u_1, w_1 in W which are adjacent to x_1, y_1 in H, respectively. Since both the edge u_1w_1 and the edge u_1x_1 are in the same number of 4-cycles, we have $p - 1 = 1 + (k-2)(k-3)$, which implies $p = k = 4$. Moreover, G is the graph formed by adding a perfect matching between two copies of the complete graph K_4, which has parameters $v = 8, k = 4, a = 6, b = 10$ and $c = 6$. We complete the proof. □

Theorem 3.4 *Let G be a connected strongly 3-walk-regular graph of degree $k \geq 3$. Then the edge-connectivity of G is equal to k. Moreover, if G is not the graph formed by adding a perfect matching between two copies of K_4, which has parameters $v = 8, k = 4, a = 6, b = 10$ and $c = 6$, then each edge cut set of size k is precisely the set of edges incident with a vertex of G.*

Proof By Note 1 we need only to consider that G is not a strongly regular graph. Let M be an edge cut of minimum size in G. Then there exists a subset S of $V(G)$ such that $|S| \leq \frac{v}{2}$ and $M = E(S, \overline{S})$, where $\overline{S} = V(G) - S$. The two induced subgraphs $G[S]$ and $G[\overline{S}]$ are both connected by the choice of M.

If $|S| = 1$, then the conclusion is easy to see.

If $1 < |S| \leq k$, then $|M| \geq |S|(k + 1 - |S|) \geq k$ by regularity. If equality holds, then $G[S]$ is a clique of size k, which is impossible by Lemma 3.3 except the graph formed by adding a perfect matching between two copies of K_4 with parameters $v = 8, k = 4, a = 6, b = 10$ and $c = 6$.

Now consider the remained case $k + 1 \leq |S| \leq \frac{v}{2}$. Suppose that G has non-integral eigenvalues. Then we are in the case of Theorem 3.1. As in Theorem 3.1, for $m = 1$, we have $k \geq 8$, and $|M| \geq \frac{(k-\frac{1+\sqrt{6k-3}}{2})|S|(v-|S|)}{v}$ by Lemma 2.3, which implies $|M| \geq \left(k - \frac{1+\sqrt{6k-3}}{2}\right)\frac{k+1}{2} > k$ for $k \geq 8$, since the function $h(x) = 2x^2 - 3x - (x+1)\sqrt{6x-3} - 1$ is monotonously increasing when $x \geq 3$ and $h(5) > 0$.

For $m \neq 1$, then by (iii) of Theorem 3.1, we have $k \geq 5$.

Suppose that $k \geq 5$ is odd. By Lemma 2.3 and Theorem 3.1 we have $|M| \geq \frac{(k-\frac{\sqrt{18k-6}}{3})|S|(v-|S|)}{v} \geq \frac{(k+1)(k-\frac{\sqrt{18k-6}}{3})}{2} > k$ if and only if $3k^4 - 12k^3 - 7k^2 - 2k + 2 > 0$. It is obviously true if $k \geq 5$.

Suppose that $k \geq 6$ is even. By Lemma 2.3 and Theorem 3.1 we have $|M| \geq \frac{(k-\frac{\sqrt{18k-12}}{3})|S|(v-|S|)}{v} \geq \frac{(k+1)(k-\frac{\sqrt{18k-12}}{3})}{2} > k$ if and only if $3k^4 - 12k^3 - 5k^2 + 2k + 4 > 0$. It is obviously true if $k \geq 6$.

Now we can suppose that G has integral eigenvalues. By Theorem 3.2 we have $k \geq 4$ and $\lambda_2(G) \leq k-2$. For $k+1 \leq |S| \leq \frac{v}{2}$, we have $|M| \geq \frac{(k-(k-2))|S|(v-|S|)}{v} \geq \frac{2(k+1)}{2} > k$ by Theorem 3.2. We complete the proof. □

Remark 2 By Theorem 2.4 it is not difficult to see that for each connected k-regular graph G on v vertices with edge-connectivity at least $k-1$, G is 1-extendable for even v and is factor-critical for odd v (see [14]). Let G be a connected strongly 3-walk-regular graph on v vertices. Then by Theorem 3.4 we see that G is 1-extendable for even v and is factor-critical for odd v.

Considering the 3-walks (walks of length 3) in a regular graph G in general, we give a sufficient and tight condition such that G is 1-extendable as follows.

Theorem 3.5 *Let G be a k-regular ($k \geq 3$) graph on even number of vertices. If for any two non-adjacent vertices of G the number of 3-walks from one vertex to the other is at least $[\frac{k-1}{2}]$, then G is 1-extendable.*

Proof Set $d = [\frac{k-1}{2}]$. We prove the conclusion by contradiction. Suppose that G is not 1-extendable. By Theorem 2.4, there is a subset S of $V(G)$ such that $o(G-S) \geq |S|$ and $G[S]$ contains an edge e.

Let $P_1, P_2, \ldots, P_{|S|}$ be the odd components of $G-S$. By regularity, S can accept at most $k|S|-2$ edges from $G-S$ as the edge e is inside S. If each vertex in P_i has a neighbour in S for $1 \leq i \leq |S|$, then we have $|E(P_i, S)| \geq |P_i| \geq k$ for $|P_i| \geq k$ and $|E(P_i, S)| \geq |P_i|(k+1-|P_i|) \geq k$ for $|P_i| \leq k-1$. It implies that $|E(G-S, S)| \geq k|S|$, which is a contradiction as S can accept at most $k|S|-2$ edges from $G-S$. Thus, without loss of generality, we can suppose that there exists a vertex u in P_1 such that u has no neighbour in S and thus $|P_1| > k$. It is easy to see that $|E(P_1, S)| \geq d$ as the number of 3-walks is at least d from the vertex u to each vertex in P_2. For any component P_i for $i > 1$, each vertex w in P_i has a neighbour in S as there is a 3-walk from the vertex u to w. Similarly, we have $|E(S, P_i)| \geq k$.

If there exists an odd component which is a singleton set. Then we have $|S| \geq k$. Let S_1 be the subset of S of which each vertex has a neighbour in P_1. Then each vertex w in $S - S_1$ has a neighbour in S_1 as there is a 3-walk from the vertex u to w. Thus $|E(S)| + |E(S, P_1)| \geq k$, where $E(S)$ is the set of edges in S. We have $2|E(S)| + |E(S, P_1)| > k$ as $|E(S)| > 0$. Therefore, S can accept at most $k(|S|-1)-1$

edges from $G - S - P_1$, which is a contradiction to the fact that $|E(S, P_i)| \geq k$ for $1 < i \leq |S|$.

Now we can suppose that there is no singleton set among the odd components of $G-S$. Similarly, if $|P_i| < k$ for some i, then $|E(S, P_i)| \geq |P_i|(k-|P_i|+1) \geq 2(k-1)$. Thus we have $|E(G - S, S)| \geq 2k - 2 + k(|S| - 2) + d = k|S| - 2 + d$, which is impossible as S accepts at most $k|S| - 2$ edges by regularity. Thus we have $|P_i| \geq k$ for any $1 \leq i \leq |S|$. For any vertex w in S, there are at most $k - 1$ neighbours in $G - S - P_1$ as there is a 3-walk from the vertex u to w. Similarly, the number of neighbours of $S - w$ in $G - S - P_1$ is at most $(k - 1)(|S| - 1)$ which is less than $|G - S - P_1|$ as $|P_i| \geq k$ for $2 \leq i \leq |S|$. Thus there is a vertex x in $G - S - P_1$ such that x has only one neighbour w in S, which implies $|E(w, P_1)| \geq d$ as there are d 3-walks from the vertex u to x. Consequently, we have $|E(w, P_1)| \geq d$ for any vertex w in S.

If k is odd, then $|E(P_1, S)| \geq d|S| \geq 2d$ is odd as P_1 is an odd component, which implies $|E(P_1, S)| \geq 2d+1$. Since there is an edge in S and $|E(P_i, S)| \geq k$ for $i > 1$, we have $|E(P_1, S)| \leq k|S| - k(|S| - 1) - 2 = k - 2$. Thus we have $d \leq \frac{k-3}{2}$, which is a contradiction.

If k is even, then $|P_2| > k$ as $|P_2| \geq k$ is odd and thus $|E(P_2, S)| > k$ as each vertex in P_2 has a neighbour in S, which implies $|E(P_2, S)| \geq k + 2$ by parity. Thus we have $2d \leq |E(P_1, S)| \leq k|S| - k(|S| - 2) - (k + 2) - 2$ and thus $d \leq \frac{k-4}{2}$, which is a contradiction. We complete the proof. □

The lower bound in Theorem 3.5 is tight. Indeed, if $k \geq 9$ is an odd integer, we can construct a k-regular graph G on $2k + 4$ vertices such that for any two non-adjacent vertices of G, the number of 3-walks from one vertex to the other is at least $\frac{k-3}{2}$, but G is not 1-extendable. In detail, let A be the graph derived from $K_{\frac{k-1}{2}}$ by deleting the edges of a hamiltonian cycle of $K_{\frac{k-1}{2}}$ and B be the graph derived from $K_{\frac{k-3}{2}}$ by deleting the edges of a hamiltonian cycle of $K_{\frac{k-3}{2}}$. Let C be a 4-cycle, $D = K_2 = xy$ be an edge and E be K_k. Now G is formed from the five parts as follows. For any two distinct parts of A, B and C, connect each vertex in one part to each vertex in the other part. Also, add an edge from the vertex x to each vertex of A and to each vertex of (any) $\frac{k-1}{2}$ vertices of E, and add an edge from the vertex y to each vertex of B and to each vertex of the remained $\frac{k+1}{2}$ vertices of E, as required.

If $k \geq 8$ is an even integer, we can construct a k-regular graph G on $2k+4$ vertices such that for any two non-adjacent vertices of G, the number of 3-walks from one vertex to the other is at least $\frac{k-4}{2}$, but G is not 1-extendable. In detail, let A be a $\frac{k-4}{2}$-clique $K_{\frac{k-4}{2}}$ and B be a $\frac{k-4}{2}$-clique $K_{\frac{k-4}{2}}$. Let C be a 5-clique, $D = K_2 = xy$ be an edge and E be a graph on $k + 1$ vertices such that there is one vertex, say u, of degree $k-2$ and the other k vertices of degree $k - 1$ (such a graph can be derived from K_{k+1} by deleting the edges of a path of length 2 and $\frac{k-2}{2}$ independent edges). Now G is formed from the five parts as follows. Connect each vertex in C to each vertex in A and B. Add all the edges except a perfect matching between A and B. Also, add two edges ux and uy, and add an edge from the vertex x to each vertex of A and to each vertex of (any) $\frac{k}{2}$ vertices of degree $k - 1$ in E, and add an edge from the vertex y to each vertex of B and to each vertex of the remained $\frac{k}{2}$ vertices of degree $k-1$ in E, as required.

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