Graphs and Combinatorics (2020) 36:27–39
https://doi.org/10.1007/s00373-019-02115-1

ORIGINAL PAPER

Classification of Maximum Hittings by Large Families

Candida Bowtell[1] · Richard Mycroft[2]

Received: 3 October 2018 / Revised: 27 August 2019 / Published online: 16 November 2019

Abstract
For integers r and n, where n is sufficiently large, and for every set $X \subseteq [n]$ we determine the maximal left-compressed intersecting families $\mathcal{A} \subseteq \binom{[n]}{r}$ which achieve maximum hitting with X (i.e. have the most members which intersect X). This answers a question of Barber, who extended previous results by Borg to characterise those sets X for which maximum hitting is achieved by the star.

Keywords Set systems · Intersecting families · Compressions

1 Introduction

The celebrated Erdős–Ko–Rado Theorem [4] states that for all integers $r \leq n/2$ and every family $\mathcal{A} \subseteq \binom{[n]}{r}$, if $\mathcal{A}$ is intersecting (meaning that no two members of $\mathcal{A}$ are disjoint), then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. To see that this bound is tight, fix any $a \in [n]$ and consider the family $\mathcal{S}_a := \{A \in \binom{[n]}{r} : a \in A\}$. We refer to $\mathcal{S}_a$ as the *star at a*, and we denote the star at 1 simply by $\mathcal{S}$ (note that $\mathcal{S}_a$ and $\mathcal{S}$ both depend on the values of n and r, but this will always be clear from the context). For $r > n/2$ the family $\binom{[n]}{r}$ itself is intersecting, so the Erdős–Ko–Rado Theorem determines the maximum size of an intersecting family on $\binom{[n]}{r}$ for all integers r and n.

One natural extension of this result is to find the maximum size of an intersecting family $\mathcal{A} \subseteq \binom{[n]}{r}$ which is *non-trivial*, that is, which is not a subfamily of a star. Hilton and Milner [7] demonstrated that in fact such families must be significantly smaller

This research originated in a summer research placement funded by an award from the LMS Undergraduate Bursary Scheme.

✉ Richard Mycroft
r.mycroft@bham.ac.uk

Candida Bowtell
bowtell@maths.ox.ac.uk

1 Mathematical Institute, University of Oxford, Radcliffe Observatory Quarter, Woodstock Rd, Oxford OX2 6GG, UK

2 School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK

than stars. More precisely, they proved that for all $1 < r < n/2$, every non-trivial intersecting family $\mathcal{A} \subseteq \binom{[n]}{r}$ has $|\mathcal{A}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. This bound is also tight, as demonstrated by the *Hilton–Milner family* $\mathcal{HM} := \{A \in \mathcal{S} : A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$, and Hilton and Milner additionally proved that (up to isomorphism) $\mathcal{HM}$ is the unique non-trivial intersecting family of this size for $r = 2$ and $r \geq 4$, and the families $\mathcal{HM}$ and $\mathcal{A}_{2,3} := \{A \in \binom{[n]}{r} : |A \cap \{1, 2, 3\}| \geq 2\}$ are the only two non-trivial intersecting families of this size for $r = 3$. The logical next step is to ask for the maximum size of an intersecting family $\mathcal{A} \subseteq \binom{[n]}{r}$ which is neither a subfamily of the star nor of the Hilton–Milner family. For $r \geq 4$ this was solved implicitly by Hilton and Milner [7], and very recently Han and Kohayakawa [6] gave a simpler proof which also includes the case $r = 3$.

The method of *compression* (also known as *shifting*), is a key technique in proving each of the results stated above. Given $i, j \in [n]$ and a family $\mathcal{A} \subseteq \binom{[n]}{r}$, the ij-*shift* $S_{ij}(\mathcal{A})$ of $\mathcal{A}$ is the family obtained by the following change: for each set $A \in \mathcal{A}$ for which $i \in A$ and $j \notin A$, replace A by $B := (A \setminus \{i\}) \cup \{j\}$ in $\mathcal{A}$ if $B \notin \mathcal{A}$. We say that a family is *left-compressed* if $S_{ij}(\mathcal{A}) = \mathcal{A}$ for every $i > j$. The following equivalent form of this definition is convenient. For sets $A, B \in \binom{[n]}{r}$, write $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ with $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$. We say that $A \leq B$ if $a_i \leq b_i$ for every $i \in [r]$. A family $\mathcal{A} \subseteq \binom{[n]}{r}$ is then left-compressed if for every $A, B \in \binom{[n]}{r}$ with $A \in \mathcal{A}$ and $B \leq A$ we have $B \in \mathcal{A}$. For a wide-ranging overview of compressions of set systems, see the survey by Frankl [5]. The relevance to intersecting families arises through the well-known fact that if $\mathcal{A} \subseteq \binom{[n]}{r}$ is intersecting then for every $i, j \in [n]$ the family $S_{ij}(\mathcal{A})$ is also intersecting, so when seeking the maximum size of an intersecting family we can restrict our attention solely to left-compressed families. In particular, it is easily observed that the families $\mathcal{S}$, $\mathcal{HM}$ and $\mathcal{A}_{2,3}$ are each left-compressed.

Another natural extension of the Erdős–Ko–Rado Theorem is to ask for the maximum size of an intersecting family $\mathcal{A} \subseteq \binom{[n]}{r}$ if we only count those sets $A \in \mathcal{A}$ which intersect a fixed subset $X \subseteq [n]$. Without further restriction this problem is a trivial consequence of the Erdős–Ko–Rado Theorem (we can fix any $a \in X$ and take $\mathcal{A} = \mathcal{S}_a$), but Borg [3] observed that the 'correct' interpretation of the problem is to consider only left-compressed families $\mathcal{A}$. Using his terminology, we say that a set A *hits* a set X if $A \cap X \neq \emptyset$, and the *hitting* of a family $\mathcal{A}$ with a set X is

$$\mathrm{hit}_X(\mathcal{A}) := |\{A \in \mathcal{A} : A \cap X \neq \emptyset\}|,$$

that is, the number of members of $\mathcal{A}$ which hit X. So we seek to identify, for each n, r and $X \subseteq [n]$, the left-compressed intersecting families $\mathcal{A} \in \binom{[n]}{r}$ which maximise $\mathrm{hit}_X(\mathcal{A})$. Clearly we need only consider maximal left-compressed intersecting families (MLCIFs),[1] and we say that an MLCIF is *optimal* for X if it achieves this maximum.

[1] It is important to note that the order of conditions here is irrelevant, in that a maximal left-compressed intersecting family is precisely a maximal intersecting family which is left-compressed. Indeed, if an MLCIF $\mathcal{A}$ is not maximal with respect to the intersecting property, then there is a larger intersecting family $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}'$, and by repeated shifts of $\mathcal{A}'$ we obtain a left-compressed intersecting family $\mathcal{A}''$ with $|\mathcal{A}''| = |\mathcal{A}'|$ and $\mathcal{A} \subseteq \mathcal{A}''$, contradicting the maximality of $\mathcal{A}$.

Fix any $1 \leq r \leq n$. If $r > n/2$ then the family $\binom{[n]}{r}$ is the only MLCIF, so vacuously is the unique optimal MLCIF for every $X \subseteq [n]$. We therefore assume henceforth that $n \geq 2r$. Likewise, in the case $r = 1$ the family $\{\{1\}\}$ is vacuously the unique optimal MLCIF for every $X \subseteq [n]$. For $r = 2$ there exist two MLCIFs, namely $\mathcal{S} = \{\{1, x\} : x \in [2, n]\}$ and $\mathcal{A}_{2,3} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, and a straightforward case analysis shows that $\mathcal{A}_{2,3}$ is the unique optimal MLCIF for $X \in \{\{2\}, \{3\}, \{2, 3\}\}$, that both $\mathcal{A}_{2,3}$ and $\mathcal{S}$ are optimal MLCIFs for $X = \{2, 3, x\}$ with $x \in [4, n]$ and $X = \{y, z\}$ with $y \in \{2, 3\}$ and $z \in [4, n]$, and that for every other non-empty X the family $\mathcal{S}$ is the unique optimal MLCIF. (Actually, if $n = 4$ there are a few more cases in which both families are optimal MLCIFs, but we omit the details since our primary interest is the case where n is large.) Unfortunately, for $r \geq 3$ the number of MLCIFs grows rapidly, so case analyses quickly prove intractable. Observe, however, that if $1 \in X$ then the Erdős–Ko–Rado Theorem implies that $\mathcal{S}$ is the unique optimal MLCIF for X, and if X is empty then trivially every MLCIF is optimal for X. We therefore restrict our attention henceforth to non-empty sets $X \subseteq [2, n]$. Borg considered for which such sets the star is optimal, and gave both general sufficient conditions under which this occurs (Theorem 1), as well as a precise characterisation for the case $|X| = r$ (Theorem 2).

Theorem 1 (Borg [3]) *Suppose that $r \geq 2$ and $n \geq 2r$. Then $\mathcal{S}$ is optimal for every $X \subseteq [2, n]$ satisfying at least one of the following:*

(i) $|X| > r$;
(ii) $X \geq X'$, *where $\mathcal{S}$ is known to be optimal for X';*
(iii) $X = \{2k, 2k + 2, \ldots, 2r\}$ *for any $k \leq r$.*

Theorem 2 (Borg [3]) *Suppose that $r \geq 2$ and $n \geq 2r$, and fix $X \subseteq [2, n]$ with $|X| = r$. If $n = 2r$, then $\mathcal{S}$ is optimal for X if and only if $X \geq \{2, 4, \ldots, 2r\}$, whilst if $n > 2r$ then $\mathcal{S}$ is optimal for X if and only if one of the following statements holds:*

(i) $r = 2$ *and* $X \neq \{2, 3\}$;
(ii) $r = 3$ *and* $|X \cap \{2, 3\}| \leq 1$;
(iii) $r \geq 4$ *and* $X \neq [2, r + 1]$.

More recently Barber [2] generalised these results by precisely characterising the cases for which the star is optimal for sufficiently large n. Observe for this that if $X \subseteq [2, r + 1]$ is non-empty then $\text{hit}_X(\mathcal{HM}) = \text{hit}_X(\mathcal{S}) + 1$, so $\mathcal{S}$ is certainly not optimal for such X.

Theorem 3 (Barber [2]) *Suppose that $r \geq 3$ and that n is sufficiently large, and fix non-empty $X \subseteq [2, n]$. Then $\mathcal{S}$ is optimal for X if and only if $X \nsubseteq [2, r + 1]$ and one of the following statements holds:*

(i) $|X| = 1$;
(ii) $|X| = 2$ *and* $X \cap \{2, 3\} = \emptyset$;
(iii) $|X| = 3$ *and* $|X \cap \{2, 3\}| \leq 1$;
(iv) $|X| \geq 4$.

Addressing the cases where $\mathcal{S}$ is not optimal for X, Barber posed the following question.

Question 4 *Is there a short list of families, one of which is optimal for every X?*

That is, can we find a small collection of MLCIFs $\mathcal{F}$, such that for every $X \subseteq [n]$, there exists $\mathcal{A} \in \mathcal{F}$ such that $\mathcal{A}$ is optimal for X? The main result of this paper answers this question in the affirmative for sufficiently large n; our list consists of the star $\mathcal{S}$ along with a class of families $\mathcal{AHM}_t$ for $3 \leq t \leq r+1$ which includes the families $\mathcal{A}_{2,3}$ and $\mathcal{HM}$ introduced previously. Specifically, (in the context of fixed integers $n \geq r$,) for each $t \in [n]$ we define

$$\mathcal{AHM}_t := \{A \in \mathcal{S} : A \cap [2,t] \neq \emptyset\} \cup \left\{A \in \binom{[n]}{r} : [2,t] \subseteq A\right\}$$

and call $\mathcal{AHM}_t$ the *t-adjusted Hilton–Milner* family. It is easy to check that $\mathcal{AHM}_t$ is a left-compressed intersecting family for every $t \geq 3$. Furthermore, for $3 \leq t \leq r+1$ the family $\mathcal{AHM}_t$ is in fact an MLCIF (see Proposition 10). Observe in particular that $\mathcal{HM} = \mathcal{AHM}_{r+1}$ and that $\mathcal{A}_{2,3} = \mathcal{AHM}_3$. We can now formally state our main result.

Theorem 5 *Suppose that $r \geq 3$ and that n is sufficiently large, and fix a non-empty subset $X \subseteq [2,n]$.*

(a) *If $X = \{2\}$ then $\mathcal{AHM}_3$ is optimal for X.*
(b) *If $|X| = 2$ and $X \cap \{2,3\} \neq \emptyset$ then $\mathcal{AHM}_3$ is optimal for X. Furthermore, if also $4 \in X$, then $\mathcal{AHM}_4$ is simultaneously optimal for X.*
(c) *If $|X| = 3$ and $\{2,3\} \subseteq X$ then $\mathcal{AHM}_3$ is optimal for X. Furthermore, if also $X = \{2,3,4\}$, then $\mathcal{AHM}_4$ is simultaneously optimal for X.*
(d) *If $X \subseteq [2, r+1]$ and X is not as in (a)–(c), then $\mathcal{AHM}_m$ is optimal for X, where $m := \max X$.*

No other MLCIFs are optimal for X as in (a)–(d), and for every other $X \subseteq [2,n]$ the star $\mathcal{S}$ is the unique optimal MLCIF for X.

In particular, the only non-empty sets $X \subseteq [n]$ for which there is not a unique optimal MLCIF are $\{2,4\}$, $\{3,4\}$ and $\{2,3,4\}$. Our proof of Theorem 5 follows the approach of Barber, which in turn developed the work of Ahlswede and Khachatrian [1] on generating families. In particular we use Barber's key observation that every MLCIF can be generated by a collection of subsets of $[2r]$ to narrow down the possible MLCIFs for a set X to a collection small enough to compare against each other. We introduce generating families and this key result in the next section, before giving the proof of Theorem 5 in Sect. 3 and concluding with some further remarks and questions in Sect. 4.

1.1 Notation

For integers $r \leq n$, we write $[n] := \{1, \ldots, n\}$ and $[r,n] := \{r, r+1, \ldots, n\}$; for $r > n$ we consider $[r,n]$ to be empty. Given a set X we use $\binom{X}{r}$ to denote the family of all subsets of X of size r and $\mathcal{P}(X)$ to denote the set of all subsets of X.

2 Generating Families

Fix integers $1 \le r \le n$, and let $\mathcal{G}$ be a collection of subsets of $[n]$. Then the family $\langle \mathcal{G} \rangle_{n,r}$ *generated by* $\mathcal{G}$ *with respect to* n *and* r is given by $\langle \mathcal{G} \rangle_{n,r} := \{A \in \binom{[n]}{r} : A \supseteq G$ for some $G \in \mathcal{G}\}$; we omit the subscripts and write simply $\langle \mathcal{G} \rangle$ when n and r are clear from the context. We call $\mathcal{G}$ a *generating family* of $\langle \mathcal{G} \rangle$. Observe that members of $\mathcal{G}$ of size greater than r do not contribute to $\langle \mathcal{G} \rangle$. Every family $\mathcal{A} \in \binom{[n]}{r}$ is a generating family of itself, but many families $\mathcal{A}$ admit more concise generating families. For example, we have $\mathcal{S} = \langle \{\{1\}\} \rangle$, $\mathcal{A}_{2,3} = \langle \{\{1,2\}, \{1,3\}, \{2,3\}\} \rangle$, $\mathcal{HM} = \langle \{\{1,i\} : 2 \le i \le r+1\} \cup \{[2, r+1]\} \rangle$ and $\mathcal{AHM}_t = \langle \{\{1,i\} : 2 \le i \le t\} \cup \{[2,t]\} \rangle$.

The following key observation of Ahlswede and Khachatrian motivates this definition for working with intersecting families.

Theorem 6 (Ahlswede–Khachatrian [1]) *Suppose that $n \ge 2r$ and that $\mathcal{G} \subseteq \mathcal{P}([n])$ has $|G| \le r$ for every $G \in \mathcal{G}$. Then $\mathcal{G}$ is intersecting if and only if $\langle \mathcal{G} \rangle$ is intersecting.*

Since there may be many different generating families for an MLCIF on $\binom{[n]}{r}$, it is helpful to define a single canonical generating family of each such family. For an MLCIF $\mathcal{A} \subseteq \binom{[n]}{r}$ we do this as follows. First, we say that a set $G \subseteq [n]$ is a *potential generator* of $\mathcal{A}$ if for every $A \in \binom{[n]}{r}$ with $G \subseteq A$ we have $A \in \mathcal{A}$. We then define the *canonical generating family* $\mathcal{G}$ of $\mathcal{A}$ to be the set of all minimal potential generators of $\mathcal{A}$ (where minimality is with respect to inclusion), and we call the elements of $\mathcal{G}$ the *generators* of $\mathcal{A}$. Observe that since every element of $\mathcal{A}$ is a potential generator of $\mathcal{A}$, the canonical generating family $\mathcal{G}$ of $\mathcal{A}$ is indeed a generating family of $\mathcal{A}$. Also note that by definition $\mathcal{G}$ must be an *antichain*, meaning that no element of $\mathcal{G}$ is a proper subset of another element of $\mathcal{G}$. Our next proposition establishes the key property that $\mathcal{G}$ is supported on the first $2r$ elements of $[n]$, and is in fact essentially unique in having this property (the existence of a generating family with this property can also be obtained from results of Barber [2]; see Lemma 8 and the discussion preceding it).

Lemma 7 *Fix integers $n \ge r$, let $\mathcal{A}$ be an MLCIF on $\binom{[n]}{r}$, and let $\mathcal{G}$ be the canonical generating family of $\mathcal{A}$. Then $G \subseteq [2r]$ for every $G \in \mathcal{G}$. Furthermore, if $n \ge 3r$ then $\mathcal{G}$ is the only generating family of $\mathcal{A}$ which is an antichain each of whose members is a subset of $[2r]$.*

Proof To prove the first statement, suppose for a contradiction that there exists $G \in \mathcal{G}$ with $G \nsubseteq [2r]$. Then the set $X := G \cap [2r]$ is a proper subset of G. Let A be the set consisting of the elements of G and the $r - |G|$ largest elements of $[n] \setminus G$, so $|A| = r$ and we have $A \in \mathcal{A}$ since $G \subseteq A$ and G is a generator of $\mathcal{A}$. Furthermore, since G is a minimal potential generator of $\mathcal{A}$, the set X is not a potential generator of $\mathcal{A}$, meaning that there exists a set $B \in \binom{[n]}{r} \setminus \mathcal{A}$ with $X \subseteq B$. Now, since $\mathcal{A}$ is a maximal intersecting family, there must exist a set $C \in \mathcal{A}$ with $C \cap B = \emptyset$ (as otherwise we could add B to $\mathcal{A}$). It follows that $C \cap X = \emptyset$. Choose any set $Z \subseteq [2r] \setminus (X \cup C)$ with $|Z| = r - |X|$ (this is possible since $|C| = r$ so $[2r] \setminus (X \cup C)$ has size at least $r - |X|$). Then $D := X \cup Z$ is a set of size r. Moreover our choices of A and Z ensure that $D \le A$, so the fact that $A \in \mathcal{A}$ and $\mathcal{A}$ is left-compressed implies $D \in \mathcal{A}$. However, $D \cap C = \emptyset$, contradicting the fact that $\mathcal{A}$ is intersecting.

For the second statement, since $\mathcal{G}$ is an antichain, it suffices to prove that for $n \geq 3r$ there do not exist two distinct generating families $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{A}$ which are both antichains such that every $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ has $G \subseteq [2r]$. Suppose for a contradiction that such families exist, and let i be minimal such that $\mathcal{G}_1 \cap \binom{[2r]}{i} \neq \mathcal{G}_2 \cap \binom{[2r]}{i}$. Assume without loss of generality that there exists $A \in \mathcal{G}_1 \cap \binom{[2r]}{i}$ with $A \notin \mathcal{G}_2$. Since $A \in \mathcal{G}_1$ we have $T := A \cup \{n - r + i + 1, \ldots, n\} \in \mathcal{A}$, so there must exist $B \in \mathcal{G}_2$ with $B \subseteq T$. However, since $A \notin \mathcal{G}_2$ we have $B \neq A$, whilst by minimality of i and the fact that $\mathcal{G}_1$ is an antichain we cannot have $B \subsetneq A$. It follows that $B \not\subseteq A$, that is, $B \cap \{n - r + i + 1, \ldots, n\} \neq \emptyset$. However, for $n \geq 3r$ we then have $B \not\subseteq [2r]$, contradicting our assumption on $\mathcal{G}_2$. □

We define the *rank* of an MLCIF $\mathcal{A} \subseteq \binom{[n]}{r}$ to be the smallest size of a generator of $\mathcal{A}$ (this is well-defined since the generators are the members of the canonical generating family). Clearly $\mathcal{S}$ is the unique MLCIF of rank one. The following proposition plays a key role in the proof of our main theorem by showing that when identifying optimal MLCIFs for a non-empty set X we need only consider MLCIFs of rank one or two; MLCIFs of larger rank simply cannot generate enough sets to be optimal.

Proposition 8 *Fix r and let n be sufficiently large. For every non-empty $X \subseteq [2, n]$, every MLCIF which is optimal for X has rank one or two.*

Proof Fix a non-empty set $X \subseteq [2, n]$ and let $\mathcal{A}$ be an MLCIF of rank at least three. Then by Lemma 7 there are at most 2^{2r} generators in $\mathcal{G}$, each of which generates at most $\binom{n}{r-3}$ elements of $\mathcal{A}$, so $\text{hit}_X(\mathcal{A}) \leq 2^{2r}\binom{n}{r-3}$. On the other hand, the star $\mathcal{S}$ is an MLCIF with $\text{hit}_X(\mathcal{S}) \geq \binom{n-2}{r-2}$, so for n sufficiently large $\mathcal{A}$ is not optimal for X. □

Our next lemma shows that the canonical generating family of an MLCIF $\mathcal{A}$ partially inherits the property of being left-compressed, in the sense that the family of generators of smallest size must be left-compressed. Combined with Theorem 6 this shows that in fact these generators form a left-compressed intersecting family, though not necessarily an MLCIF, as shown e.g. by $\mathcal{AHM}_4$.

Lemma 9 *Fix $n \geq 2r$, let $\mathcal{A}$ be an MLCIF on $\binom{[n]}{r}$, let $\mathcal{G}$ be the canonical generating family of $\mathcal{A}$, and let k be the rank of $\mathcal{A}$. Then the subfamily $\mathcal{G} \cap \binom{[n]}{k}$ is left-compressed.*

Proof Suppose for a contradiction that there exist $A \in \mathcal{G} \cap \binom{[n]}{k}$ and $B \in \binom{[n]}{k} \setminus \mathcal{G}$ with $B \leq A$. Let C be the set of the $r - k$ largest elements of $[n] \setminus A$, and let D be the set of the $r - k$ largest elements of $[n] \setminus B$. Then $S := A \cup C$ and $T := B \cup D$ are both elements of $\binom{[n]}{r}$, and the fact that $B \leq A$ implies that $T \leq S$. Since $A \subseteq S$ and $A \in \mathcal{G}$ we have $S \in \mathcal{A}$, and since $\mathcal{A}$ is left-compressed it follows that $T \in \mathcal{A}$. However, the fact that $\mathcal{A}$ is left-compressed then implies that $B \cup F \in \mathcal{A}$ for every set $F \in \binom{[n]\setminus B}{r-k}$, and so B is a potential generator of $\mathcal{G}$. This gives a contradiction, since $B \notin \mathcal{G}$ and $\mathcal{A}$ has no generators of size less than $k = |B|$. □

We now justify our assertion made in the introduction that the family $\mathcal{AHM}_t$ is indeed an MLCIF.

Proposition 10 *For $n \geq 2r$ and $3 \leq t \leq r + 1$, the family $\mathcal{AHM}_t$ is an MLCIF.*

Proof For $t \geq 3$, the fact that $\mathcal{AHM}_t$ is left-compressed follows immediately from the definition, and Theorem 6 implies that $\mathcal{AHM}_t$ is intersecting. It remains to show that $\mathcal{AHM}_t$ is maximal with these properties. For $t = r + 1$ this follows from the Hilton–Milner Theorem [7] which states that $\mathcal{HM} = \mathcal{AHM}_{r+1}$ is the largest intersecting family which is not a subfamily of a star. So suppose for a contradiction that $t \leq r$ and $\mathcal{AHM}_t$ is not an MLCIF. Then $\mathcal{AHM}_t$ is a proper subset of an MLCIF $\mathcal{AHM}_t^*$, so we may choose a set $A \in \mathcal{AHM}_t^* \setminus \mathcal{AHM}_t$. It follows from the definition of $\mathcal{AHM}_t$ that if $1 \in A$ then $\{1, t+1, t+2, \ldots, t+r-1\} \leq A$, and if $1 \notin A$ then $\{2, 3, \ldots, t-1, t+1, t+2, \ldots, r+2\} \leq A$. Since $\mathcal{AHM}_t^*$ is left-compressed this implies that either $\{1, t+1, t+2, \ldots, t+r-1\} \in \mathcal{AHM}_t^*$ or $\{2, 3, \ldots, t-1, t+1, t+2, \ldots, r+2\} \in \mathcal{AHM}_t^*$. Observe that $\{2, 3, \ldots, t, t+r, t+r+1, \ldots, 2r\} \in \mathcal{AHM}_t$ and $\{1, t, r+3, r+4, \ldots, 2r\} \in \mathcal{AHM}_t$ but $\{1, t+1, t+2, \ldots, t+r-1\} \cap \{2, 3, \ldots, t, t+r, t+r+1, \ldots, 2r\} = \emptyset$, and $\{2, 3, \ldots, t-1, t+1, t+2, \ldots, r+2\} \cap \{1, t, r+3, r+4, \ldots, 2r\} = \emptyset$. So in either case the family $\mathcal{AHM}_t^*$ is not intersecting, a contradiction. □

Observe that if the set $\{2, 3\}$ is a generator of an MLCIF $\mathcal{A}$, then $\mathcal{AHM}_3 \subseteq \mathcal{A}$, and it then follows from the maximality of $\mathcal{AHM}_3$ that $\mathcal{A} = \mathcal{AHM}_3$. This establishes the following corollary.

Corollary 11 *For $n \geq 2r$, if $\{2, 3\}$ is a generator of an MLCIF $\mathcal{A}$, then $\mathcal{A} = \mathcal{AHM}_3$.*

Using this, we can establish a more detailed understanding of MLCIFs of rank 2. For this we define $\mathcal{I}_j$ for $j \geq 2$ to be the set of all MLCIFs $\mathcal{A} \subseteq \binom{[n]}{r}$ of rank two whose generators of size two are precisely the sets $\{1, 2\}, \ldots, \{1, j\}$. Observe that $\mathcal{AHM}_m \in \mathcal{I}_m$ for every $m \geq 4$, but that $\mathcal{AHM}_3 \notin \mathcal{I}_3$.

Proposition 12 *Let $n \geq 2r$ and suppose that $\mathcal{A} \subseteq \binom{[n]}{r}$ is an MLCIF of rank 2. Then either $\mathcal{A} = \mathcal{AHM}_3$ or $\mathcal{A} \in \bigcup_{j=2}^{r+1} \mathcal{I}_j$.*

Proof Let $\mathcal{F}$ be the set of all generators of $\mathcal{A}$ of size two. If $\{2, 3\} \in \mathcal{F}$ then $\mathcal{A} = \mathcal{AHM}_3$ by Corollary 11, so we may assume $\{2, 3\} \notin \mathcal{F}$. Since $\mathcal{F}$ is left-compressed by Lemma 9 it follows that $\mathcal{F} = \{\{1, i\} : 2 \leq i \leq j\}$ for some integer $j \geq 2$, that is, that $\mathcal{A} \in \mathcal{I}_j$. If $j > r + 1$ then the fact that $\mathcal{A}$ is intersecting implies that $1 \in A$ for every $A \in \mathcal{A}$, so $\mathcal{A}$ is a subfamily of the star $\mathcal{S}$, contradicting the fact that $\mathcal{A}$ is an MLCIF of rank 2. So we must have $j \leq r + 1$ as required. □

3 Proof of Theorem 5

Proposition 8 tells us that every MLCIF which is optimal for any non-empty $X \subseteq [2, n]$ must have rank one or two. Before proceeding to the proof of Theorem 5 we now further narrow down these possibilities to just two MLCIFs for each such set $X \neq \{2\}$. These possibilities are given in Corollary 14, which follows directly from our next proposition stating that almost all members of $\bigcup_{j=2}^{r+1} \mathcal{I}_j$ cannot be optimal. Similar statements can be made for the case $X = \{2\}$, but due to the fact that $\mathcal{AHM}_2$ is not an MLCIF it is convenient instead to defer this case to the proof of Theorem 5.

Proposition 13 *Fix $r \geq 3$, let n be sufficiently large, let $X \subseteq [2, n]$ be non-empty and write $m := \max X$.*

(i) *If $X \nsubseteq [2, r+1]$ then $\mathrm{hit}_X(\mathcal{S}) > \mathrm{hit}_X(\mathcal{A})$ for every $\mathcal{A} \in \bigcup_{j=2}^{r+1} \mathcal{I}_j$. That is, $\mathcal{S}$ hits X more than any family in $\bigcup_{j=2}^{r+1} \mathcal{I}_j$.*

(ii) *If $X \subseteq [2, r+1]$ and $X \neq \{2\}$, then $\mathrm{hit}_X(\mathcal{AHM}_m) > \mathrm{hit}_X(\mathcal{A})$ for every $\mathcal{A} \in \bigcup_{j=2}^{r+1} \mathcal{I}_j \setminus \{\mathcal{AHM}_m\}$. That is, $\mathcal{AHM}_m$ hits X more than any other family in $\bigcup_{j=2}^{r+1} \mathcal{I}_j$.*

Proof For (i), fix an MLCIF $\mathcal{A} \in \bigcup_{j=2}^{r+1} \mathcal{I}_j$ and let $\mathcal{G}$ be the canonical generating family of $\mathcal{A}$. By Lemma 7 we have $|\mathcal{G}| \leq 2^{2r}$. Define

$$\mathcal{S}' := \{S \in \mathcal{S} : \{1, m\} \subseteq S \text{ and } [2, r+1] \cap S = \emptyset\},$$

and

$$\mathcal{A}^\star := \{A \in \mathcal{A} : \{1, j\} \subseteq A \text{ for some } j \in [2, r+1]\}.$$

Then (for sufficiently large n) we have $|\mathcal{S}'| = \binom{n-r-2}{r-2} > 2^{2r}\binom{n}{r-3} \geq |\mathcal{A} \setminus \mathcal{A}^\star|$; the final inequality holds since every set in $\mathcal{A} \setminus \mathcal{A}^\star$ is generated by one of the at most 2^{2r} generators of size at least 3, each of which generates at most $\binom{n}{r-3}$ sets. Observe also that $\mathcal{A}^\star \subseteq \mathcal{S}$, that $\mathcal{S}' \cap \mathcal{A}^\star = \emptyset$, and that every element of $\mathcal{S}'$ is an element of $\mathcal{S}$ which hits X. It follows that

$$\mathrm{hit}_X(\mathcal{S}) \geq \mathrm{hit}_X(\mathcal{A}^\star) + |\mathcal{S}'| > \mathrm{hit}_X(\mathcal{A}^\star) + |\mathcal{A} \setminus \mathcal{A}^\star| \geq \mathrm{hit}_X(\mathcal{A}),$$

as required.

For (ii) we introduce the following notation: for any MLCIF $\mathcal{A}$, write

$$\mathcal{A}^\circ := \{A \in \mathcal{A} : \{1, j\} \subseteq A \text{ for some } j \in [2, m]\} \text{ and } \mathcal{A}^+ = \mathcal{A} \setminus \mathcal{A}^\circ.$$

Assume $X \subseteq [2, r+1]$, and observe that since $m = \max X$, for each $x \in X$ the set $\{1, x\}$ is a generator of $\mathcal{AHM}_m$. It follows that $\mathrm{hit}_X(\mathcal{AHM}_m) \geq |X|\binom{n-r-1}{r-2}$. Consider any MLCIF $\mathcal{A} \in \bigcup_{j=2}^{r+1} \mathcal{I}_j$ with $\mathcal{A} \neq \mathcal{AHM}_m$, and let $\mathcal{G}$ be the canonical generating family of $\mathcal{A}$, so $|\mathcal{G}| \leq 2^{2r}$ by Lemma 7. Suppose first that $\mathcal{A} \in \bigcup_{j=2}^{m-1} \mathcal{I}_j$. Then $\mathcal{G}$ contains at most $|X| - 1$ generators of the form $\{1, x\}$ with $x \in X$, each of which generates at most $\binom{n}{r-2}$ members of $\mathcal{A}$, whilst each of the at most 2^{2r} remaining generators $G \in \mathcal{G}$ generates at most $r\binom{n}{r-3}$ members of $\mathcal{A}$ which hit X. (In fact, generators G satisfying $|G| = 2$ but not hitting X generate at most $r\binom{n}{r-3}$ members of $\mathcal{A}$ which hit X, namely those sets which contain both G and one of the at most r elements of X. All other generators have size at least 3, and thus generate at most $\binom{n}{r-3}$ members of $\mathcal{A}$.) So we have $\mathrm{hit}_X(\mathcal{A}) \leq (|X| - 1)\binom{n}{r-2} + 2^{2r} r \binom{n}{r-3}$, and so (for n sufficiently large) $\mathrm{hit}_X(\mathcal{AHM}_m) > \mathrm{hit}_X(\mathcal{A})$, as required.

We may therefore assume that $\mathcal{A} \in \bigcup_{j=m}^{r+1} \mathcal{I}_j$, and in particular that $\{1, j\}$ is a generator of $\mathcal{A}$ for every $j \in [2, m]$. Observe that we then have $\mathcal{A}^\circ = \mathcal{AHM}_m^\circ$, so

$\text{hit}_X(\mathcal{A}^\circ) = \text{hit}_X(\mathcal{AHM}_m^\circ)$. We now compare $\text{hit}_X(\mathcal{A}^+)$ and $\text{hit}_X(\mathcal{AHM}_m^+)$; observe for this that $\mathcal{AHM}_m^+$ contains precisely those sets $S \in \binom{[n]}{r}$ with $1 \notin S$ and $[2, m] \subseteq S$, so we have $\text{hit}_X(\mathcal{AHM}_m^+) = \binom{n-m}{r-m+1}$. On the other hand, since $\mathcal{A}$ is intersecting, Theorem 6 tells us that $\mathcal{G}$ is intersecting also; since $\mathcal{G}$ includes $\{1, j\}$ for every $j \in [2, m]$, it follows that every generator $G \in \mathcal{G}$ must satisfy either $1 \in G$ or $[2, m] \subseteq G$. However, every set $A \in \mathcal{A}$ with $1 \in A$ which hits X is an element of $\mathcal{A}^\circ$, so the sets generated by generators G with $1 \in G$ do not contribute to $\text{hit}_X(\mathcal{A}^+)$. Also, since $\mathcal{AHM}_m$ is an MLCIF whose generators are $[2, m]$ and $\{1, j\}$ for $j \in [2, m]$, and $\mathcal{A} \neq \mathcal{AHM}_m$, we must have $[2, m] \notin \mathcal{G}$. So every generator $G \in \mathcal{G}$ with $[2, m] \subseteq G$ has size at least m, and so the number of sets generated by generators of this form is at most $2^{2r}\binom{n}{r-m}$. We conclude that (for sufficiently large n) we have $\text{hit}_X(\mathcal{A}^+) \leq 2^{2r}\binom{n}{r-m} < \text{hit}_X(\mathcal{AHM}_m^+)$, and so

$$\begin{aligned}\text{hit}_X(\mathcal{A}) &= \text{hit}_X(\mathcal{A}^\circ) + \text{hit}_X(\mathcal{A}^+) < \text{hit}_X(\mathcal{AHM}_m^\circ) + \text{hit}_X(\mathcal{AHM}_m^+)\\ &= \text{hit}_X(\mathcal{AHM}_m),\end{aligned}$$

as required. □

Recall that if $X \subseteq [2, r+1]$ is non-empty then the star $\mathcal{S}$ is not optimal for X since $\text{hit}_X(\mathcal{AHM}_{r+1}) = \text{hit}_X(\mathcal{S}) + 1$. This fact, together with Propositions 8, 12 and 13, immediately implies the following important corollary, narrowing down the list of potential optimal families for a set X to just two possibilities.

Corollary 14 *For every $r \geq 3$ the following statements hold for sufficiently large n.*

(1) *For every non-empty $X \subseteq [2, r+1]$ with $X \neq \{2\}$, if $\mathcal{A}$ is an MLCIF which is optimal for X then $\mathcal{A} \in \{\mathcal{AHM}_3, \mathcal{AHM}_m\}$, where $m = \max X$.*
(2) *For every $X \nsubseteq [2, r+1]$, if $\mathcal{A}$ is an MLCIF which is optimal for X then $\mathcal{A} \in \{\mathcal{S}, \mathcal{AHM}_3\}$.*

Finally, to prove Theorem 5 we simply need to compare, for each set X, the hitting of these two potential optimal families with X. We do this on a case-by-case basis.

Proof of Theorem 5 Throughout this proof we will use our assumption that n is sufficiently large relative to r without further comment. We begin with case (a), where $X = \{2\}$. By Proposition 8 and Proposition 12 the only possible optimal MLCIFs for X are $\mathcal{S}$, $\mathcal{AHM}_3$ and the members of $\mathcal{I}_j$ for $2 \leq j \leq r+1$. Observe that $\text{hit}_X(\mathcal{AHM}_3) = \binom{n-2}{r-2} + \binom{n-3}{r-2}$, whilst $\text{hit}_X(\mathcal{S}) = \binom{n-2}{r-2}$. Furthermore, for each $\mathcal{A} \in \bigcup_{j=2}^{r+1}\mathcal{I}_j$ we have $\text{hit}_X(\mathcal{A}) \leq \binom{n-2}{r-2} + 2^{2r}\binom{n}{r-3}$, since $\mathcal{A}$ has at most 2^{2r} generators by Lemma 7. It follows that $\mathcal{AHM}_3$ is the unique optimal MLCIF for X.

We next turn to case (b), where $|X| = 2$ and $X \cap \{2, 3\} \neq \emptyset$. If $X = \{2, 3\}$ then Corollary 14 implies that $\mathcal{AHM}_3$ is the unique optimal MLCIF for X. Assume therefore that either $X = \{2, m\}$ or $X = \{3, m\}$ for some $m \geq 4$. In each case we have

$$\begin{aligned}\mathrm{hit}_X(\mathcal{S}) &= \binom{n-2}{r-2} + \binom{n-3}{r-2},\\ \mathrm{hit}_X(\mathcal{AHM}_3) &= \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-3}, \text{ and, if } m \in [4, r+1], \text{ then}\\ \mathrm{hit}_X(\mathcal{AHM}_m) &= \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-m}{r-m+1}.\end{aligned}$$

Note that $\binom{n-4}{r-3} \geq \binom{n-m}{r-m+1}$ for all $m \geq 4$ with equality if and only if $m = 4$. By Corollary 14 it follows that $\mathcal{AHM}_3$ is the unique optimal MLCIF for X for $m > 4$, whilst $\mathcal{AHM}_3$ and $\mathcal{AHM}_4$ are the only two optimal MLCIFs for X if $m = 4$.

Now we consider case (c), where $X = \{2, 3, m\}$ for some $m \geq 4$. We then have

$$\begin{aligned}\mathrm{hit}_X(\mathcal{S}) &= \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-2},\\ \mathrm{hit}_X(\mathcal{AHM}_3) &= \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-3}{r-2}, \text{ and, if } m \in [4, r+1], \text{ then}\\ \mathrm{hit}_X(\mathcal{AHM}_m) &= \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-2} + \binom{n-m}{r-m+1}.\end{aligned}$$

Observe that $\binom{n-3}{r-2} \geq \binom{n-4}{r-2} + \binom{n-m}{r-m+1}$ for all $m \geq 4$ with equality holding if and only if $m = 4$. By Corollary 14 it follows that $\mathcal{AHM}_3$ is the unique optimal MLCIF for X for $m > 4$ whilst $\mathcal{AHM}_3$ and $\mathcal{AHM}_4$ are the only two optimal MLCIFs for X if $m = 4$.

Lastly, in case (d) we have that $X \subseteq [2, r+1]$ and that X does not meet the conditions of cases (a)–(c). Define $m := \max X$. Then by Corollary 14 the only two possibilities for optimal MLCIFs for X are $\mathcal{AHM}_3$ and $\mathcal{AHM}_m$. Observe that $\mathcal{AHM}_m$ has $|X|$ generators of size 2 which intersect X, so $\mathrm{hit}_X(\mathcal{AHM}_m) \geq |X|\binom{n-r-1}{r-2}$. If $\{2, 3\} \subseteq X$, then $|X| \geq 4$ (otherwise we have case (b) or (c)), so we have

$$\mathrm{hit}_X(\mathcal{AHM}_3) \leq 3\binom{n-2}{r-2} < |X|\binom{n-r-1}{r-2} \leq \mathrm{hit}_X(\mathcal{AHM}_m).$$

Similarly, if $|\{2, 3\} \cap X| = 1$ then $X = \{3\}$ or $|X| \geq 3$ (otherwise we have case (a) or (b)). If $X = \{3\}$ then $\mathcal{AHM}_3 = \mathcal{AHM}_m$, whilst if $|X| \geq 3$ then we have

$$\mathrm{hit}_X(\mathcal{AHM}_3) \leq 2\binom{n-2}{r-2} + |X|\binom{n}{r-3} < |X|\binom{n-r-1}{r-2} \leq \mathrm{hit}_X(\mathcal{AHM}_m).$$

When $X \cap \{2, 3\} = \emptyset$, the family $\mathcal{AHM}_3$ has no rank 2 generators hitting X, whilst $\mathcal{AHM}_m$ has $|X|$ such generators, so certainly $\mathrm{hit}_X(\mathcal{AHM}_3) < \mathrm{hit}_X(\mathcal{AHM}_m)$. In each case it follows that $\mathcal{AHM}_m$ is the unique optimal MLCIF for X.

Finally, it remains to prove that the star is the unique optimal MLCIF for every set $X \subseteq [2, n]$ which is not covered by cases (a)–(d). Any such X has $X \nsubseteq [2, r+1]$, so by Corollary 14 it suffices for this to show that $\mathrm{hit}_X(\mathcal{AHM}_3) < \mathrm{hit}_X(\mathcal{S})$. Moreover, any such X satisfies either

(i) $|X| = 1$,
(ii) $|X| = 2$ and $X \cap \{2, 3\} = \emptyset$,
(iii) $|X| = 3$ and $|X \cap \{2, 3\}| \leq 1$, or
(iv) $|X| \geq 4$.

Observe that in cases (i), (ii) and (iii) we have $\mathrm{hit}_X(\mathcal{S}) \geq |X|\binom{n-4}{r-2}$. However, in cases (i) and (ii) we also have $\mathrm{hit}_X(\mathcal{AHM}_3) \leq 6\binom{n}{r-3}$, and in case (iii) we have $\mathrm{hit}_X(\mathcal{AHM}_3) \leq 2\binom{n}{r-2} + 2\binom{n}{r-3}$. Similarly in case (iv) we have $\mathrm{hit}_X(\mathcal{S}) \geq 4\binom{n-5}{r-2}$ and $\mathrm{hit}_X(\mathcal{AHM}_3) \leq 3\binom{n}{r-2}$. So in all cases we have $\mathrm{hit}_X(\mathcal{AHM}_3) < \mathrm{hit}_X(\mathcal{S})$, as required. □

We finish this section by returning to the question of which left-compressed intersecting families (LCIFs) have maximum hitting with a fixed non-empty set $X \subseteq [n]$. For this we extend the definition of optimality to LCIFs in the natural way, saying that an LCIF $\mathcal{A} \subseteq \binom{[n]}{r}$ is optimal for X if $\mathrm{hit}_X(\mathcal{A}) \geq \mathrm{hit}_X(\mathcal{F})$ for every LCIF $\mathcal{F} \subseteq \binom{[n]}{r}$. As for MLCIFs, if $1 \in X$ then $\mathcal{S}$ is the unique optimal LCIF, so again we consider only $X \subseteq [2, n]$. Since every LCIF is a subfamily of an MLCIF, the optimal LCIFs for X are precisely the left-compressed subfamilies of optimal MLCIFs which can be formed by removing sets which do not hit X. From this observation we obtain the following corollary (which should be read in conjunction with Theorem 5).

Corollary 15 *Let $r \geq 3$ and n be sufficiently large. Suppose that $X \subseteq [2, n]$ is non-empty and let $m := \max X$.*

(i) *If $\mathcal{S}$ is not an optimal MLCIF for X then the optimal LCIFs for X are precisely the optimal MLCIFs for X.*
(ii) *If $\mathcal{S}$ is an optimal MLCIF for X then the optimal LCIFs for X are precisely the LCIFs $\mathcal{A}$ with $\mathcal{AHM}_m \subseteq \mathcal{A} \subseteq \mathcal{S}$.*

Proof Suppose first that $\mathcal{S}$ is not an optimal MLCIF for X. Then by Theorem 5 every optimal MLCIF for X has the form $\mathcal{AHM}_t$ for some $t \in [3, r+1]$, and moreover we have $t \in X$ in every case except when $X = \{2\}$ or $X = \{2, x\}$ with $x \in [4, n]$, in which case $t = 3$. (When $X = \{2, 4\}$ both $\mathcal{AHM}_3$ and $\mathcal{AHM}_4$ are optimal for X; for the former we have $t = 3$ and for the latter we have $t \in X$.) Observe that every set $A \in \mathcal{AHM}_t$ has either $A \leq B := \{1, t, n-r+3, \ldots, n\}$ or $A \leq C := \{2, 3, \ldots, t, n-r+t, \ldots, n\}$, and furthermore that for $t = 3$ every set $A \in \mathcal{AHM}_3$ has $A \leq C$. Since C hits X in all cases, and B hits X if $t \in X$, it follows that every LCIF $\mathcal{A}$ which is a proper subfamily of $\mathcal{AHM}_t$ has $\mathrm{hit}_X(\mathcal{A}) < \mathrm{hit}_X(\mathcal{AHM}_t)$, and so is not optimal, proving (i).

Now suppose that $\mathcal{S}$ is an optimal MLCIF for X. Then $\mathcal{S}$ is the unique optimal MLCIF for X by Theorem 5, so every optimal LCIF $\mathcal{A}$ for X has $\mathcal{A} \subseteq \mathcal{S}$. Furthermore we have $m > r+1$, so $\mathcal{AHM}_m$ consists precisely of those sets $A \in \binom{[n]}{r}$ with $A \leq D$, where D is the set formed by adding the $r-2$ largest elements of $[n] \setminus \{m\}$ to $\{1, m\}$.

Since D hits X it follows that every optimal LCIF $\mathcal{A}$ has $\mathcal{AHM}_m \subseteq \mathcal{A}$, and (ii) follows since $\text{hit}_X(\mathcal{AHM}_m) = \text{hit}_X(\mathcal{S})$.[2] □

4 Further Directions

It would be interesting to know how large n must be to satisfy Theorem 5 (Barber previously asked the analogous question following his proof of Theorem 3). Following our proofs directly gives a bound on n which is exponential in r, but we suspect that more careful arguments would yield a polynomial bound.

Recall that, for sufficiently large n, Theorem 3 identified all $X \subseteq [n]$ for which an MLCIF of rank 1 (that is, $\mathcal{S}$) is optimal, and Theorem 5 shows that in all other cases every optimal MLCIF for X has rank 2. In the spirit of the Hilton–Milner Theorem, it would also be interesting to consider the optimal MLCIF among all families other than the star $\mathcal{S}$, giving the following question.

Question 16 *For each $n \geq 2r$ and $X \subseteq [n]$, which MLCIFs $\mathcal{T} \neq \mathcal{S}$ satisfy* $\text{hit}_X(\mathcal{T}) \geq \text{hit}_X(\mathcal{A})$ *for every MLCIF $\mathcal{A} \neq \mathcal{S}$?*

To answer Question 16 we must certainly consider MLCIFs of rank greater than 2. Indeed, by Proposition 12 every MLCIF of rank 2 has no size 2 generators hitting any element $x \in X$ such that $x > r + 1$. So, for example, when $r = 3$ and $X = \{5\}$, no generator of size 2 in a canonical generating family can hit X. Observe that the family $\mathcal{A}_{3,4,5} := \langle \{\{a, b, c\} : 1 \leq a < b < c \leq 5\}\rangle$ has 6 generators of size 3 hitting X. Every other MLCIF (excluding the star) has at most 5 generators of size 3 hitting X, and thus for sufficiently large n the family $\mathcal{A}_{3,4,5}$ is unique in achieving maximum hitting with X among all MLCIFs excluding the star. The problem appears to become significantly harder for larger values of r, for which it seems difficult just to enumerate all the MLCIFs which exist. In fact it seems to be non-trivial to resolve even the apparently-simpler question of identifying, for every $X \subseteq [n]$, the MLCIFs $\mathcal{A}$ which maximise $\text{hit}_X(\mathcal{A})$ among all MLCIFs of rank two.

Acknowledgements We thank Ben Barber for helpful discussions, and two anonymous reviewers for their helpful suggestions for improving the presentation of this manuscript.

Funding Candida Bowtell was funded by London Mathematical Society (No. URB 14-30).

References

1. Ahlswede, R., Khachatrian, L.: The complete intersection theorem for systems of finite sets. Eur. J. Combin. **18**, 125–136 (1997)
2. Barber, B.: Maximum hitting for *n* sufficiently large. Graphs Combin. **30**, 267–274 (2014)
3. Borg, P.: Maximum hitting of a set by compressed intersecting families. Graphs Combin. **27**, 785–797 (2011)
4. Erdős, P., Ko, C., Rado, R.: Intersection theorems for systems of finite sets. Q. J. Math. Oxf. **12**, 313–320 (1961)

[2] Since here we have $m > r + 1$, the family $\mathcal{AHM}_m$ is not an MLCIF, so this conclusion does not contradict our assertion that $\mathcal{S}$ is the unique optimal MLCIF for X.

5. Frankl, P.: The Shifting Technique in Extremal Set Theory, Surveys in Combinatorics, pp. 81–110. Cambridge University Press, Cambridge (1987)
6. Han, J., Kohayakawa, Y.: The maximum size of a non-trivial intersecting uniform family that is not a subfamily of the Hilton–Milner family. Proc. Am. Math. Soc. **145**, 73–87 (2017)
7. Hilton, A., Milner, E.: Some intersection theorems for systems of finite sets. Q. J. Math. Oxf. **18**, 369–384 (1967)