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ORIGINAL PAPER

On Classification of 2-Arc Transitive Cayley Graphs of the Dicyclic Group

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Abstract
In this paper we first determine all possible connected core-free 2-arc transitive Cayley graphs of the dicyclic group, B_{4n}, and then show that this can be used to classify all connected 2-arc transitive Cayley graphs of this group in terms of regular cyclic covers, provided that we also know connected core-free 2-arc transitive Cayley graphs of the dihedral group.

Keywords Cayley graph · 2-Arc transitive Cayley graph · Core-free Cayley graph · Dicyclic group · Classification of 2-arc transitive Cayley graphs

Mathematics Subject Classification 05E18 · 20D60 · 05C25 · 20B25

1 Introduction

In this paper all graphs are finite, undirected and simple, i.e. without loops or multiple edges. Let Γ be a graph. The set of vertices of Γ is denoted by $V(\Gamma)$ and for two vertices u and v, we write $u \sim v$ to denote u is adjacent to v. The set of all vertices adjacent to u is denoted by $\Gamma(u)$. For each integer $s \geq 0$ an s-arc in Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices such that for each $0 \leq i \leq s-1$, $v_i \sim v_{i+1}$, and for each $1 \leq i \leq s-1$, $v_{i-1} \neq v_{i+1}$. For $X \leq Aut(\Gamma)$ we say Γ is (X, s)-arc transitive if X is transitive on $V(\Gamma)$ and also on the set of s-arcs of Γ. Γ is said to be s-arc transitive

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if it is (X, s)-arc transitive for $X = Aut(\Gamma)$. It is easily shown that for $s \geq 1$, an (X, s)-arc transitive graph is also $(X, s-1)$-arc transitive. For $s = 2$ a graph Γ is $(X, 2)$-arc transitive if and only if X is transitive on $V(\Gamma)$ and for each vertex v, the stabilizer X_v acts doubly transitively on $\Gamma(v)$. An $(X, 1)$-arc transitive graph is often called X-arc transitive or X-symmetric.

Let G be a group and $\emptyset \neq S \subset G$ be such that $S^{-1} = S$ and $1 \notin S$. Then the undirected *Cayley graph* of G with respect to S, $\Gamma = Cay(G, S)$, is defined as the simple graph whose vertices are the elements of G and where two vertices x and y are adjacent if and only if $xy^{-1} \in S$. The graph Γ is connected if and only if S is a generating set for G. For each $g \in G$ the mapping $\rho_g : G \to G$, defined by $\rho_g(x) = xg^{-1}$ for all $x \in G$, is a graph automorphism of Γ and $R(G) = \{\rho_g | g \in G\}$ is a subgroup of $Aut(\Gamma)$ isomorphic to G which acts regularly on $V(\Gamma)$. Also $Aut(G, S) = \{\sigma \in Aut(G) | \sigma(S) = S\}$ is a subgroup of $Aut(\Gamma)$ and $N_{Aut(\Gamma)}(R(G)) = R(G) \rtimes Aut(G, S)$, where $N_{Aut(\Gamma)}(R(G))$ denotes the normalizer of $R(G)$ in $Aut(\Gamma)$ and '$\rtimes$' is the notation for semidirect product. Γ is called a *normal* Cayley graph if $R(G) \trianglelefteq Aut(\Gamma)$. If Γ is normal, then $N_{Aut(\Gamma)}(R(G)) = Aut(\Gamma)$ and this implies $(Aut(\Gamma))_1 = Aut(G, S)$. For a given graph Γ and a given group G, Γ is a Cayley graph on G if and only if $Aut(\Gamma)$ has a subgroup isomorphic to G which acts regularly on $V(\Gamma)$. If H is a subgroup of a group G, then the *core* of H in G is defined as $Core_G(H) = \bigcap_{g \in G} gHg^{-1}$ and is the largest normal subgroup of G contained in H. A Cayley graph $\Gamma = Cay(G, S)$ is called *core-free* if $Core_{Aut(\Gamma)}(R(G)) = 1$.

Studying 2-arc transitive graphs and in particular, 2-arc transitive Cayley graphs, has been a subject of much interest in the literature [1,5,9,13–17]. In [1] all connected 2-arc transitive Cayley graphs of the cyclic group were determined. In [14] the author obtained a classification of all connected 2-arc transitive Cayley graphs of the dihedral group in terms of regular cyclic covers. Later in [5], the authors proved that some covers do not really happen as 2-arc transitive Cayley graphs of the dihedral group. It was shown in [17] that there are more 2-arc transitive dihedrants than those given in [5].

In this paper we consider the dicyclic group, B_{4n}. The family of dicyclic groups is an important family of groups which contains generalized quaternion groups of order a power of 2 as a subfamily. They are also a subfamily of generalized dicyclic groups. A Cayley graph $\Gamma = Cay(G, S)$ is called a GRR (Graphical Regular Representation) if $Aut(\Gamma) = R(G)$. Godsil has shown that abelian groups and generalized dicyclic groups are the only two infinite families of finite groups that do not admit GRRs [8]. As a special case, for any $n > 1$, the group B_{4n} has no GRRs. This special behavior is another aspect of dicyclic groups that makes studying their Cayley graphs interesting.

We coin the term *dicirculant* for a Cayley graph of a dicyclic group, as Cayley graphs on cyclic and dihedral groups have respectively been called circulants [1] and dihedrants [14]. Our goal is to classify connected 2-arc transitive dicirculants. We will use the fact that cyclic groups of composite order, as well as dicyclic groups, are all B-groups. We first determine all possible core-free 2-arc transitive connected dicirculants, and then show that each connected 2-arc transitive dicirculant is a regular cyclic cover of a connected core-free 2-arc transitive dicirculant or of a connected core-free 2-arc transitive dihedrant.

The rest of this paper is organized as follows. In Sect. 2 we clarify some notations to prevent ambiguity. Also in this section some notions and theorems that will be used in the rest of the paper, are presented as a reminder. In Sect. 3 the dicyclic group and some of its important features are discussed. Then in Sect. 4, core-free connected 2-arc transitive dicirculants are classified. Finally in Sect. 5 it is proved that if all connected core-free 2-arc transitive dihedrants are also known, then we will have a classification of all connected 2-arc transitive dicirculants, in terms of regular cyclic covers, following the important result of Sect. 4. In Sect. 5, by applying our results we also obtain a full classification of 2-arc transitive dicirculants of order $4p$, p odd prime.

2 Preliminaries

In this paper, a function f acts on its argument from the left, i.e. we write $f(x)$. The composition, fg, of two functions f and g, is defined as $(fg)(x) = f(g(x))$. The complete graph on n vertices is denoted by K_n. The graph $K_{n,n} - nK_2$ is obtained by deleting the edges of a perfect matching from the complete bipartite graph, $K_{n,n}$. The cardinality of a finite set A, is denoted by $|A|$, and the order of an element a of a group is denoted by $o(a)$. If G is a group and $H \leq G$, then G', $C_G(H)$, $N_G(H)$ and $[G : H]$, denote respectively the commutator subgroup of G, the centralizer, the normalizer and the index of H in G. Also for an integer d we define $H^{(d)} = \{g^d : g \in H\}$. If H is a characteristic subgroup of G, we write $H \trianglelefteq^c G$. For a group G and a nonempty set Ω, an action of G on Ω is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to Ω, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$, for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of ω in G is defined as $G_\omega = \{g \in G : g\omega = \omega\}$, and for $\Delta \subset \Omega$, $g\Delta = \{g\delta : \delta \in \Delta\}$. The action of G on Ω is called *semiregular* if the stabilizer of each element in Ω is trivial; it is called *regular* if it is semiregular and transitive. The *kernel* of the action of G on Ω is defined as $\{g \in G : g\omega = \omega, \forall\omega \in \Omega\}$. If this kernel is trivial, then we say $(G|\Omega)$ is a *permutation group*.

If G acts on Ω, then a partition $\Sigma = \{P_1, \dots, P_n\}$ for Ω is called a *G-invariant* partition if $gP_i \in \Sigma$ for each $g \in G$ and each $i = 1, \dots, n$. The action is called *imprimitive* if it is transitive and there is a subset $\Delta \subset \Omega$ with $\Delta \neq \Omega$ and $|\Delta| \geq 2$, called an *imprimitivity block* or simply a *block*, such that for every $g \in G$ either $g\Delta = \Delta$ or $(g\Delta) \cap \Delta = \emptyset$. A transitive action which is not imprimitive, is called *primitive*. If the action of G on Ω is transitive, then it is imprimitive if and only if there is a G-invariant partition $\{P_1, \dots, P_n\}$ for Ω with $n \geq 2$ such that at least one P_i has more than one element, and in this case each P_i would be an imprimitivity block. If Δ is a block, then for every $g \in G$, $g\Delta$ is also a block and the set $\{g\Delta | g \in G\}$ is called an imprimitivity block system for the action. If we delete repeated sets from an imprimitivity block system, a G-invariant partition is obtained. Finally we note that if K is the kernel of the action of G on Ω, then a permutation group $(\frac{G}{K}|\Omega)$ is obtained with essentially the same action, and this is imprimitive if and only if the action of G on Ω is imprimitive.

If Γ is a connected G-arc transitive graph, where $G \leq Aut(\Gamma)$, and $\mathcal{B}$ is a G-invariant partition for $V(\Gamma)$ with at least two elements, then it is not hard to prove that each block $B \in \mathcal{B}$ is an independent set, i.e. there is no edge in Γ between two vertices from B.

Let Γ be a graph and $\Sigma = \{P_1, \ldots, P_n\}$ a partition for $V(\Gamma)$. Then the *quotient graph* of Γ with respect to Σ is a simple undirected graph Γ_Σ whose vertex set is Σ and for $i \neq j$, P_i is adjacent to P_j if and only if there is a $u \in P_i$ and a $v \in P_j$, with u adjacent to v in Γ. Often Σ is the set of orbits of a subgroup N acting on $V(\Gamma)$, where $N \trianglelefteq X \leq Aut(\Gamma)$. In this case if X is fixed in the discussion and causes no ambiguity, the quotient graph will be denoted by Γ_N.

Let Γ_c and Γ be two graphs. Then Γ_c is said to be a *covering graph* for Γ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. The function f is called a *covering projection*. Clearly, if Γ is bipartite, then so is Γ_c. For each $u \in V(\Gamma)$, the *fibre* on u is defined as $fib_u = f^{-1}(u)$. The set of automorphisms of Γ_c which take any fibre to a fibre, is a subgroup of $Aut(\Gamma_c)$ and is called the group of *fibre preserving* automorphisms. The following important set is also a subgroup of $Aut(\Gamma_c)$ and is called the *group of covering transformations* for f:

$$CT(f) = \{\sigma \in Aut(\Gamma_c) \,|\, \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}$$

It is known that $K = CT(f)$ acts semiregularly on each fibre [12]. If this action is regular, then Γ_c is said to be a *regular* K*-cover* of Γ. Especially if K is cyclic, then Γ_c is called a *regular cyclic cover* of Γ.

For a graph Γ and a group K, If $A(\Gamma)$ denotes the set of 1-arcs of Γ, then a function $f : A(\Gamma) \to K$ satisfying $f(u, v) = f(v, u)^{-1}$ for all $u, v \in V(\Gamma)$, is called a K-voltage function. Corresponding to Γ and a K-voltage function f assigned to Γ, a graph $\Gamma \times_f K$ is defined with $V(\Gamma) \times K$ as its vertex set and $(u, g) \sim (v, h)$ if and only if $f(u, v) = g^{-1}h$. It can be proved that $\Gamma \times_f K$ is a regular K-cover of Γ and that every regular K-cover of Γ can be obtained using a suitable K-voltage assignment to Γ.

The following is an important theorem from [16]:

Theorem 2.1 *Suppose Γ is a connected $(X, 2)$-arc transitive graph, where $X \leq Aut(\Gamma)$. Suppose $N \trianglelefteq X$ and the number of orbits of $(N|V(\Gamma))$ is at least 3. Then:*

(i) $(N|V(\Gamma))$ *is semiregular.*
(ii) $Aut(\Gamma_N)$ *has a subgroup isomorphic to* $\frac{X}{N}$ *and* Γ_N *is* $(\frac{X}{N}, 2)$*-arc transitive.*
(iii) Γ *is a cover of* Γ_N.

Parts of the following theorem immediately follow from Theorem 2.1, but are stated clearly in the context of Cayley graphs. That Γ_H is a Cayley graph, follows by showing that $Aut(\Gamma_H)$ has a subgroup isomorphic to $\frac{R(G)}{H}$, regular on its vertices. The third part will follow by showing that the group of covering transformations equals H, and by noting that fibres are the orbits of H.

Theorem 2.2 *Let $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of an arbitrary finite group G, where $R(G) \leq X \leq Aut(\Gamma)$. If $H = Core_X(R(G))$ and $[R(G) : H] \geq 3$, then*

(i) Γ_H *is isomorphic to a core-free Cayley graph of the group* $\frac{R(G)}{H}$.
(ii) Γ_H *is* $(\frac{X}{H}, 2)$*-arc transitive.*
(iii) Γ *is a regular* H*-cover of* Γ_H.

For $\lambda \geq 1$ and $2 \leq k \leq v-1$, a 2-(v, k, λ) *design* is an ordered pair $D = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ of cardinality v is called the point set and where $\mathcal{B}$ consists of some subsets of $\mathcal{P}$ of cardinality k called blocks, with the property that every 2-element subset of $\mathcal{P}$ is contained in exactly λ blocks. If b is the number of blocks of D, then $b = \frac{\lambda v(v-1)}{k(k-1)}$. The 2-design D is called *symmetric* if $b = v$. So for symmetric D, the relation $\lambda(v-1) = k(k-1)$ holds. An *automorphism* of D is a permutation $f : \mathcal{P} \longrightarrow \mathcal{P}$ such that for each $B \subset \mathcal{P}$ of cardinality k, $B \in \mathcal{B}$ if and only if $f(B) \in \mathcal{B}$. If $D = (\mathcal{P}, \mathcal{B})$ is a 2-(v, k, λ) design, then its *complement* is defined as $D' = (\mathcal{P}, \mathcal{B}')$ where elements of $\mathcal{B}'$ are complements of the elements of $\mathcal{B}$ with respect to $\mathcal{P}$. One can verify that D' is a 2-$(v, v-k, \lambda')$ design for some λ', provided $k \leq v-2$. Clearly $Aut(D') = Aut(D)$. The *incidence graph* (*non-incidence graph*) of a design $D = (\mathcal{P}, \mathcal{B})$, is a bipartite graph whose vertex set is $\mathcal{P} \cup \mathcal{B}$, where $p \in \mathcal{P}$ is adjacent to $B \in \mathcal{B}$ if and only if $p \in B$ ($p \notin B$). Symmetric 2-transitive designs have been classified in the following theorem from [11].

Theorem 2.3 *Let D be a symmetric 2-(v, k, λ) design with $k < \frac{v}{2}$, such that $G \leq Aut(D)$ is 2-transitive on the set of points. Then D is one of the followings:*

(i) *a projective space;*
(ii) *the unique hadamard design with $v = 11$ and $k = 5$;*
(iii) *a unique design with $v = 176$, $k = 50$ and $\lambda = 14$;*
(iv) *a design with $v = 2^{2m}$, $k = 2^{m-1}(2^m - 1)$ and $\lambda = 2^{m-1}(2^{m-1} - 1)$, of which there is exactly one for each $m \geq 2$.*

An abstract group G is called a *B-group*, if every permutation group $(H|\Omega)$ with $G \leq H$, is either imprimitive or 2-transitive, provided that $(G|\Omega)$ is regular. Finite cyclic groups with composite order and all dihedral groups are B-groups [19]. Also every dicyclic group is a B-group [19].

We will also need the following theorem:

Theorem 2.4 [18] *If H is a subgroup of a group G, then $C_G(H) \trianglelefteq N_G(H)$ and $\frac{N_G(H)}{C_G(H)}$ is isomorphic to a subgroup of $Aut(H)$.*

Theorem 2.5 [1] *A connected 2-arc transitive circulant of order $n \geq 3$ is one of the following graphs:*

(i) K_n;
(ii) $K_{\frac{n}{2},\frac{n}{2}}$ *for* $\frac{n}{2} \geq 3$;
(iii) $K_{\frac{n}{2},\frac{n}{2}} - (\frac{n}{2})K_2$ *for* $\frac{n}{2} \geq 5$ *odd;*
(iv) *the cycle of length n.*

2-arc transitive regular covers of K_n where the group of covering transformations is either cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, p prime, have been classified in [6]. We will need only the following partial result:

Theorem 2.6 [6] *Let Γ be a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of K_n, $n \geq 4$. If the fibre-preserving subgroup of automorphisms of Γ acts 2-arc transitively on Γ, then $n = q + 1$ where $q \geq 5$ is a prime power and $q \equiv 1$ (mod 4).*

Also for prime p, 2-arc transitive regular $\mathbb{Z}_p \times \mathbb{Z}_p$-covers of $K_{n,n} - nK_2$ have been classified in [20] We will need only the following partial result:

Theorem 2.7 [20] *Let Γ be a connected regular $\mathbb{Z}_p \times \mathbb{Z}_p$-cover of $K_{n,n} - nK_2$, $n \geq 3$ and p prime, whose fibre-preserving subgroup of automorphisms acts 2-arc transitively on Γ. Then $n = 4$.*

Also for prime p, 2-arc transitive regular $\mathbb{Z}_p \times \mathbb{Z}_p$-covers of $K_{n,n}$ have been classified in [7] where they define three types of graphs using voltage assignments. We briefly touch on the one we need for the following partial result. For each prime p, any integer $r \geq 2$ and any monic irreducible polynomial $\varphi(x)$ over the Galois field of order p, whose degree is an integer $d \geq 2$ dividing r, a special voltage is assigned to the graph K_{p^r,p^r} using some matrices associated to $\varphi(x)$, to obtain the covering graph $X(r, p, \varphi(x))$. Refer to [7] for details of this construction.

Theorem 2.8 [7] *Let Γ be a connected regular $\mathbb{Z}_p \times \mathbb{Z}_p$-cover of $K_{n,n}$, p prime, whose fibre-preserving subgroup of automorphisms acts 2-arc transitively on Γ. Then one of the following occurs:*

(i) $n = 3$;
(ii) $n = p \geq 5$;
(iii) $n = p^r \geq 4$, $r \geq 2$ *and* $\Gamma \cong X(r, p, \varphi(x))$ *for some* $\varphi(x)$ *as specified before.*

A useful summary of 3-transitive permutation groups is stated in the following theorem:

Theorem 2.9 [6] *Let G be a 3-transitive permutation group of degree at least 4. Then one of the following occurs:*

(i) *The socle of G is 3-transitive; or*
(ii) $PSL_2(q) \leq G \leq P\Gamma L_2(q)$ *with natural action on the projective line of degree $q + 1$, for odd $q \geq 5$ and with the socle of G being isomorphic to $PSL_2(q)$ and not 3-transitive; or*
(iii) $G = AGL(m, 2)$, $m \geq 3$; *or*
(iv) $G = \mathbb{Z}_2^4 : A_7$; *or*
(v) $G = S_4$ *(of degree 4).*

3 The Dicyclic Group

For each $n \geq 1$, the *dicyclic group* of order $4n$ is defined as

$$B_{4n} = \left\langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \right\rangle.$$

The well-known *generalized quaternion group* of order 2^{k+2} is a dicyclic group for $n = 2^k$ for every $k \geq 1$. We will also mention the *dihedral group* in this article, so let's recall that the dihedral group of order $2n$ is defined as $D_{2n} = \langle a, b | a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

Every element of B_{4n} is of the form a^i or $a^i b$ for some $0 \le i < 2n$ and $|B_{4n}| = 4n$. For each i, $(a^i b)^2 = b^2$ and $o(a^i b) = 4$. The only element of order 2 is $b^2 = a^n$.

We have $B_4 \simeq \mathbb{Z}_4$ and for $n \ge 2$, B_{4n} is nonabelian. An important note is that for $n \ge 2$, if $Cay(B_{4n}, S)$ is a connected Cayley graph of B_{4n}, then $|S| \ge 4$, an immediate consequence of which is that unlike dihedral groups, no connected Cayley graph of a dicyclic group is a cycle or a cover of a cycle. To see this, note that if $|S| = 1$, then $S = \{a^n\}$, and if $|S| = 2$, then $S = \left\{x, x^{-1}\right\}$ for some x of order greater than 2, and in both cases S doesn't generate B_{4n}, as $\langle S \rangle$ is cyclic. If $|S| = 3$, then $S = \left\{a^n, x, x^{-1}\right\}$ for some x. If $x \in \langle a \rangle$, then $b \notin \langle S \rangle$, and if $x = a^i b$ for some i, then $x^2 = a^n$ and $\langle S \rangle = \langle x \rangle$ is cyclic.

If n is odd, then every subgroup of $\langle a \rangle$ is normal in B_{4n} and there is no other nontrivial normal subgroup. If n is even, then besides the subgroups of $\langle a \rangle$, there are two other nontrivial normal subgroups, namely $N_1 = \left\langle a^2, b \right\rangle$ and $N_2 = \left\langle a^2, ab \right\rangle$. When n is odd, the only index 2 normal subgroup is $\langle a \rangle$ and for n even, $\langle a \rangle$, N_1 and N_2 are the only three normal subgroups of index 2 in B_{4n}. It is not hard to see that for $n > 2$ even, N_1 and N_2 are both dicyclic of order $4.\frac{n}{2} = 2n$. For $n = 2$, they are both cyclic of order 4. So in general, for every even or odd natural number $n \ge 2$, any index 2 subgroup of the group B_{4n} is itself a B-group.

Now suppose M is a nontrivial normal subgroup of B_{4n} excluding $\langle a \rangle$, N_1 and N_2. Then $M = \left\langle a^i \right\rangle$ for some natural number $i \ne 1, 2n$ which divides $2n$, and $|M| = \frac{2n}{i}$. If $i = 2$, then $\frac{B_{4n}}{M} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ according to whether n is even or odd, respectively. For $i \ge 3$, if i does not divide n, then i is even and $\frac{B_{4n}}{M} \simeq B_{4(i/2)}$, and if i divides n, then $\frac{B_{4n}}{M} \simeq D_{2i}$.

In the rest of this paper, when talking about the dicyclic group, B_{4n} with $n \ge 2$, we always assume it is generated by a and b, that is $B_{4n} = \langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$. Moreover, we will use the conventions $\rho := \rho_a$ and $\tau := \rho_b$; So $R(B_{4n}) = \langle \rho, \tau \rangle$, $\tau^{-1}\rho\tau = \rho^{-1}$ and $o(\rho) = 2n$.

4 The Core-Free Case

In this section, our goal is to prove the following important result:

Theorem 4.1 *Let $n \ge 3$, $G = B_{4n}$, and $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \le X \le Aut(\Gamma)$. Assume further, that $Core_X(R(G)) = 1$. Then Γ is one of the following graphs:*

(a) K_{4n}.
(b) $K_{2n,2n}$.
(c) $K_{2n,2n} - (2n)K_2$.
(d) *The incidence or non-incidence graph of a projective space, i.e. a $2-(\frac{q^{m+1}-1}{q-1}, \frac{q^m-1}{q-1}, \frac{q^{m-1}-1}{q-1})$ design, where q is an odd prime power and $m > 1$ is odd, with $2n = \frac{q^{m+1}-1}{q-1}$.*

(e) $X(2, r, \varphi(x))$ *where* $\varphi(x)$ *is some nonlinear binary irreducible polynomial of degree dividing* r*. This case is possible only for* $n = 2^{r+1} \geq 8$.

We are going to make the required preparations in order to be able to prove this theorem. Let $G = B_{4n}$ and $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \leq X \leq Aut(\Gamma)$. Because $R(G) \simeq G$ is a B-group and $(R(G)|V(\Gamma))$ is regular, $(X|V(\Gamma))$ must be either imprimitive or doubly transitive. If it is doubly transitive, then $\Gamma \simeq K_{4n}$ is complete, as Γ has at least one edge. So hereafter in the discussion of 2-arc transitive Cayley graphs of B_{4n}, we will assume that $(X|V(\Gamma))$ is imprimitive.

Proposition 4.2 *Let* $n \geq 2$, $G = B_{4n}$ *and* $\Gamma = Cay(G, S)$ *be a connected Cayley graph of* G*. Let* $(X|V(\Gamma))$ *be imprimitive with* $\mathcal{B}$ *as an imprimitivity block system, where* $R(G) \leq X \leq Aut(\Gamma)$*. There exists a positive integer* m *which divides* $2n$ *such that one of the following two cases occurs:*

(i) *For each block* $B \in \mathcal{B}$*, there exists a vertex* v *such that* $B = \langle \rho^m \rangle v$.
(ii) *For each block* $B \in \mathcal{B}$*, there exist two vertices* u *and* v *which are not in the same orbit of* $(\langle \rho \rangle |V(\Gamma))$*, such that* $B = \langle \rho^m \rangle u \bigcup \langle \rho^m \rangle v$.

Proof The action of $\langle \rho \rangle$ on $V(\Gamma)$ is semiregular and has exactly two orbits, say $V_1 = \langle \rho \rangle v_1$ and $V_2 = \langle \rho \rangle v_2$, each of size $| \langle \rho \rangle | = 2n$. There are two possibilities for each block. Call $B \in \mathcal{B}$ of type 1, if it is contained in one of V_1 or V_2, and of type 2, if the intersection of B with both V_1 and V_2 is nonempty.

Let B be of type 1, say, $B \subset V_1$. Because $|B| \geq 2$, there exists a 2-element subset of B of the form $\{u, \rho^j(u)\}$ for some $1 \leq j < 2n$. Take m to be the smallest such j. Now $B \cap \rho^m B \neq \emptyset$ implies $B = \rho^m B$ and hence $\langle \rho^m \rangle u \subset B$. As $V_1 = \langle \rho \rangle v_1 = \langle \rho \rangle u$, for an arbitrary $x \in B$ we have $x = \rho^j(u)$ for some j. Let $j = qm + r$ where $0 \leq r < m$. If $r \neq 0$, then $\rho^m(u) \in B$ contradicting the choice of m. Hence $r = 0$ and $x \in \langle \rho^m \rangle u$. So $B = \langle \rho^m \rangle u$. Let $\alpha = gcd(m, 2n) = mx + 2ny$ for integers x and y. Then $\rho^\alpha(u) = \rho^{mx}(u) \in B$ and so $\alpha \geq m$ which implies $\alpha = m$. I.e. m divides $2n$ and $|B| = | \langle \rho^m \rangle | = \frac{2n}{m}$. If $B' \in \mathcal{B}$ is another block of type 1, similarly $B' = \left\langle \rho^k \right\rangle v$ for some k which divides $2n$ and $|B'| = \frac{2n}{k}$, which implies $k = m$.

Now let B be of type 2. If $B = \{u, v\}$ has only two elements, then B is of the form $\langle \rho^m \rangle u \bigcup \langle \rho^m \rangle v$ for $m = 2n$. If $|B| \geq 3$, at least one of the intersections has more than one element, say $|B \cap V_1| \geq 2$. As for type 1 blocks, there is the least integer m with $1 \leq m < 2n$ for which $B \cap V_1$ has a 2-element subset of the form $\{u, \rho^m(u)\}$. Again $B = \rho^m B$. If $v \in B \cap V_2$ is arbitrary, then $V_1 = \langle \rho \rangle u$ and $V_2 = \langle \rho \rangle v$. We have $\langle \rho^m \rangle u \bigcup \langle \rho^m \rangle v \subset B$. For proving equality, assume $x \in B$ is arbitrary; Either $x = \rho^j(u)$ or $x = \rho^j(v)$ for some j. Conclude that $j = qm$ for some q, and then $B = \langle \rho^m \rangle u \bigcup \langle \rho^m \rangle v$. As above, the technique of gcd shows that m divides $2n$. If $B' = \left\langle \rho^k \right\rangle u' \bigcup \left\langle \rho^k \right\rangle v'$ is another type 2 block, then $|B| = \frac{4n}{m}$ and $|B'| = \frac{4n}{k}$ implies $k = m$.

It remains to show that either all blocks in $\mathcal{B}$ are of type 1, or all are of type 2. To this end, it suffices to show that if some $B \in \mathcal{B}$ is of type 1, then all blocks in $\mathcal{B}$ will be of type 1. Let $B = \langle \rho^m \rangle u \subset V_1$. For each $1 \leq i \leq m - 1$, $\rho^i B \subset V_1$ is also in $\mathcal{B}$ and $B, \rho B, \ldots, \rho^{m-1} B$ are mutually disjoint. So $V_1 = B \cup \rho B \cup \cdots \cup \rho^{m-1} B$. Now

$B' = \tau(B)$ is also a block in $\mathcal{B}$, and $v = \tau(u)$ must be in V_2, implying $V_2 = \langle\rho\rangle\, v$. From $\tau^{-1}\rho\tau = \rho^{-1}$ we obtain $\tau\,\langle\rho^m\rangle = \langle\rho^m\rangle\,\tau$ or $B' = \langle\rho^m\rangle\, v$. Now $B, \rho B, \ldots, \rho^{m-1}B$, $B', \rho B', \ldots, \rho^{m-1}B'$ are mutually disjoint blocks whose union equals $V(\Gamma)$. An arbitrary block $C \in \mathcal{B}$ is of the form gB for some $g \in X$ and must intersect one of the above blocks. If for example $C \cap \rho^i B' \neq \emptyset$, then $B \cap g^{-1}\rho^i\tau B \neq \emptyset$ or $B = g^{-1}\rho^i\tau B$. So $C = \rho^i B'$. That is, every block in $\mathcal{B}$ is of type 1. □

Proposition 4.3 *Let $n \geq 2$, $G = B_{4n}$ and $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \leq X \leq Aut(\Gamma)$. Let $(X|V(\Gamma))$ be imprimitive with $\mathcal{B} = \{\langle\rho^m\rangle\, u | u \in V(\Gamma)\}$ as an imprimitivity block system satisfying case (i) of Proposition 4.2. Then $Core_X(R(G)) \neq 1$ provided that $m \geq 2$.*

Proof Consider the action of X on $\mathcal{B}$ defined by $g.B = g(B)$ for every $g \in X$ and every $B \in \mathcal{B}$, and let N be its kernel. Clearly $\langle\rho^m\rangle \leq N$ and consequently $\langle\rho^m\rangle\, u \subset Nu$ for each vertex u. Each $f \in N$ fixes every block setwise; hence $Nu \subset \langle\rho^m\rangle\, u$ and so $Nu = \langle\rho^m\rangle\, u$ for each vertex u. If $m \geq 2$, then $|Nu| = |\langle\rho^m\rangle\, u| = \frac{2n}{m} \leq n$ and the number of orbits of $(N|V(\Gamma))$ is at least 4. So we can use Theorem 2.1 to conclude that $(N|V(\Gamma))$ is semiregular and hence $|N| = |Nu| = |\langle\rho^m\rangle\, u| = |\langle\rho^m\rangle|$, which results in $N = \langle\rho^m\rangle$. As blocks in $\mathcal{B}$ have at least two elements, $m \neq 2n$, and hence $Core_X(R(G)) \neq 1$, because $\langle\rho^m\rangle$ is a non-identity subgroup of $R(G)$ normal in X. □

Let Ω be a finite set and P_1 and P_2 two partitions for Ω. The partition P_1 is called a *refinement* for P_2 if each element of P_2 is a union of some elements from P_1, and P_1 is called a *genuine* refinement for P_2 if it is a refinement and moreover $P_1 \neq P_2$ and there is at least one element in P_1 with cardinality greater than 1. Let $\mathcal{B}$ be a G-invariant partition for the set of vertices of a graph Γ; $\mathcal{B}$ is said to be minimal if there is no G-invariant partition for $V(\Gamma)$ which is a genuine refinement for $\mathcal{B}$. Let Γ be a graph and $X \leq Aut(\Gamma)$ such that $(X|V(\Gamma))$ is imprimitive with $\mathcal{B}$ as an imprimitivity block system. Then we can define an action of X on $\mathcal{B}$ by $g.B = g(B)$ for every $g \in X$ and every $B \in \mathcal{B}$. Associated to this action, are the following groups, defined for each $B \in \mathcal{B}$:

$$X_B = \{g \in X | gB = B\}$$
$$X_{(B)} = \{g \in X_B | gb = b; \forall b \in B\}$$

Clearly X_B acts on B. The following lemma is a restatement of Lemma 2.2 of [21]. Note the X-symmetricity of Γ is not really needed in the proof.

Lemma 4.4 [21] *Let Γ be a connected graph, such that $X \leq Aut(\Gamma)$ is transitive on $V(\Gamma)$ and $(X|V(\Gamma))$ is imprimitive with $\mathcal{B}$ as an imprimitivity block system. For an arbitrary block $B \in \mathcal{B}$, let $N \trianglelefteq X_B$. Then given any fixed $b \in B$, $\mathcal{B}_N = \{g(Nb)|g \in X\}$ is an X-invariant partition for $V(\Gamma)$, which is a refinement for $\mathcal{B}$ and*

(i) $\mathcal{B}_N = \{\{\alpha\} : \alpha \in V(\Gamma)\}$ *if and only if* $N \leq X_{(B)}$.
(ii) $\mathcal{B}_N = \mathcal{B}$ *if and only if* $(N|B)$ *is transitive.*

Lemma 4.5 *Let G be a finite abelian group, $1 \neq a \in G$ with $o(a) \neq 2$ and $\frac{G}{\langle a\rangle} \simeq \mathbb{Z}_2$. Then G has a nontrivial characteristic subgroup N where $N \leq \langle a\rangle$.*

Proof Assume $n = o(a)$. There exists some $x \in G$ with $x \notin \langle a \rangle$ and $x^2 \in \langle a \rangle$. Suppose $x^2 = a^j$ for some j. We distinguish the following cases:

(i) n is odd or n and j are both even. The congruence $2i = -j \ (mod\ n)$ has a solution for i and if $y = xa^i$, then $o(y) = 2$ and $G = \langle a \rangle \langle y \rangle$ is an internal direct product. If n is not a power of 2, there exists an odd prime p which divides n. If S_p is a Sylow p-subgroup of $\langle a \rangle$, then S_p is also a Sylow p-subgroup of G and is a characteristic subgroup of G. If $n = 2^r$ and $r \geq 2$, then $N = \langle a^2 \rangle$ is a nontrivial characteristic subgroup of G.

(ii) n is even and $j = 2r + 1$ is odd. If $y = xa^{-r}$, then $o(y) = 2n$, $G = \langle y \rangle$ is cyclic and $N = \langle a \rangle$ is a characteristic subgroup of G. □

Lemma 4.6 *Let $q = p^e$, p prime. Then none of the quotients of subgroups of $\frac{P\Gamma L_2(q)}{PSL_2(q)}$ could be isomorphic to S_3.*

Proof Let $F = \frac{P\Gamma L_2(q)}{PSL_2(q)}$ and $A = \frac{PGL_2(q)}{PSL_2(q)} \simeq \mathbb{Z}_2$. Then $\frac{F}{A} \simeq \mathbb{Z}_e$. If $K \trianglelefteq H \leq F$ and $\frac{H}{K} \simeq S_3$, then $\frac{H'}{H' \cap K} \simeq \mathbb{Z}_3$ and H' has an element of order 3. But $H' \leq F' \leq A$ and so the order of elements of H' is at most 2. □

Proposition 4.7 *Let $n \geq 2$, $G = B_{4n}$ and $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \leq X \leq Aut(\Gamma)$. Also assume $(X|V(\Gamma))$ is imprimitive with $\mathcal{B}$ as an imprimitivity block system whose block size is minimum. If $\mathcal{B}$ satisfies case (ii) of Proposition 4.2, then Γ is bipartite and either $m = 2$ or $m = n = 2^{r+1} \geq 8$. The latter case could happen only if $\Gamma \cong X(2, r, \varphi(x))$ where $\varphi(x)$ is some nonlinear binary irreducible polynomial of degree dividing r.*

Proof According to Proposition 4.2, m divides $2n$. Clearly $m > 1$. If $m = 2n$, then consider a typical block $B = \{u, v\} \in \mathcal{B}$, where u and v are in different orbits of $\langle \rho \rangle$. Because $R(G)$ is transitive on $V(\Gamma)$, there is some $f \in R(G)$ with $v = f(u)$ so that $B = \{u, f(u)\}$. Now $f^{-1}B \cap B$ contains u and we must have $f^{-1}B = B$. Hence $f(B) = B$ and so $f^2(u) = u$. The action of $R(G)$ on the vertices is regular and so f^2 is the identity map. If $f = \rho_g$, then $o(g) = 2$ and $g = a^n$, as the only element of order 2 in G is a^n. But then $v = \rho_g(u) = ua^n = \rho^n(u)$, and u and v are in the same orbit of $\langle \rho \rangle$, a contradiction.

Now assume $m \leq n$. If $n = 2$, then we must have m =2 and we are done. So suppose $3 \leq m \leq n$. Consider the action of X on $\mathcal{B}$ and let N be its kernel. Let $B = \langle \rho^m \rangle u \bigcup \langle \rho^m \rangle v$ be an arbitrary block in $\mathcal{B}$. Elements of N fix B setwise, so $Nu \subset B$ and $Nv \subset B$. On the other hand, $\langle \rho^m \rangle \leq N$ implies $B \subset Nu \bigcup Nv$. Hence $B = Nu \bigcup Nv$. Now $(N|V(\Gamma))$ is not transitive and its orbits form a nontrivial imprimitivity block system for $(X|V(\Gamma))$. Hence the minimality of the block size of $\mathcal{B}$ forces $Nu = Nv = B$, and $\mathcal{B}$ is the set of orbits of $N \trianglelefteq X$. Observe that $|B| = \frac{4n}{m} \leq \frac{|V(\Gamma)|}{3}$ and the number of orbits of $(N|V(\Gamma))$ is at least 3. So according to Theorem 2.1, $(N|V(\Gamma))$ is semiregular and $|N| = |Nu| = 2|\langle \rho^m \rangle|$ which implies $[N : \langle \rho^m \rangle] = 2$. Clearly N is transitive on each $B \in \mathcal{B}$.

In the following, we distinguish two cases.

Case 1: $3 \leq m = n$.

If this case happens, then $|N| = |B| = 4$. Let $B \in \mathcal{B}$ and consider the action of X_B on B. This is transitive because $N \leq X_B$, but we claim it cannot be imprimitive.

If imprimitive, it would have a block $\Delta = \{\delta_1, \delta_2\}$ of cardinality 2 and assuming $\Delta' = B - \Delta$, $\left\{\Delta, \Delta'\right\}$ is an X_B-invariant partition of B and the setwise stabilizer, H, of Δ in X_B, is a normal subgroup of X_B. Now according to Proposition 4.4, $\mathcal{B}_H$ is an X-invariant partition for $V(\Gamma)$, and is a refinement of $\mathcal{B}$. But $\mathcal{B}$ is an X-invariant partition for $V(\Gamma)$ whose block size is minimum. This means that either $\mathcal{B}_H = \mathcal{B}$, or $\mathcal{B}_H = \{\{\alpha\} : \alpha \in V(\Gamma)\}$. According to Proposition 4.4, the latter implies $H \le X_{(B)}$, which is not the case, because there is some $g \in N$ with $g\delta_1 = \delta_2$, and this g must lie in H. If $\mathcal{B}_H = \mathcal{B}$, again Proposition 4.4 implies that H is transitive on B, which is again impossible since $H\Delta = \Delta$. So the action of X_B on B should be primitive. There are only 2 primitive permutation groups of degree 4: the symmetric group S_4 and the alternating group A_4 (see e.g. [2]). So $\dfrac{X_B}{X_{(B)}} \simeq S_4$ or A_4. For a subgroup $H \le X_B$ we denote by $\overline{H}$ the subgroup $\dfrac{HX_{(B)}}{X_{(B)}}$ of $\dfrac{X_B}{X_{(B)}}$. Let $B = \langle\rho^n\rangle\, u \bigcup \langle\rho^n\rangle\, v$ where u and v are in different orbits of $\langle\rho\rangle$ and hence $v = \rho^i\tau(u)$ for some i. It's easy to verify that $R(G) \cap X_B = \{1, \rho^n, \rho^i\tau, \rho^{n+i}\tau\} = \left\langle\rho^i\tau\right\rangle \simeq \mathbb{Z}_4$. Now $\overline{R(G) \cap X_B} \simeq R(G) \cap X_B$ is a subgroup of $\overline{X_B}$ and A_4 doesn't have an element of order 4; so we may only have $\overline{X_B} \simeq S_4$. Now $N \trianglelefteq X_B$ implies that $\overline{N} \simeq N$ is a normal subgroup of S_4; hence $N \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Aut(N) \simeq S_3$. So $\frac{X}{C_X(N)}$ is isomorphic to a subgroup of S_3. If $|\frac{X}{C_X(N)}| = d \le 3$, then $X^{(d)} \subset C_X(N)$ and hence $(\overline{X_B})^{(d)} \subset C_{\overline{X_B}}(\overline{N}) = \overline{N}$. But $|(S_4)^{(d)}| > 4$ for $d = 1, 2, 3$. So $\frac{X}{C_X(N)} \simeq S_3$.

According to Theorem 2.1 $Aut(\Gamma_N)$ has a subgroup isomorphic to $\frac{X}{N}$ and Γ is a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of Γ_N which is $(\frac{X}{N}, 2)$-arc transitive. The order of ρN in $\frac{X}{N}$ is n because $\rho^n \in N$ and if $(\rho N)^i = N$, then $\rho^{2i} = 1$ which implies $i \ge n$. Noting that $B = \{u, \rho^n u, \rho^i\tau u, \rho^{n+i}\tau u\}$, we find that $B, \rho B, \ldots, \rho^{n-1}B$ are mutually disjoint and hence form all the elements of $\mathcal{B}$. The action of $\langle\rho N\rangle$ on $\mathcal{B}$ is regular and Γ_N is a Cayley graph of $\mathbb{Z}_n$. According to Theorem 2.5, Γ_N is either K_n, $K_{\frac{n}{2},\frac{n}{2}}$, $K_{\frac{n}{2},\frac{n}{2}} - (\frac{n}{2})K_2$ or a cycle. If $\Gamma_N \simeq K_{\frac{n}{2},\frac{n}{2}} - (\frac{n}{2})K_2$, then according to Theorem 2.7, $\frac{n}{2} = 4$ and hence the degree of Γ must be 3 which is impossible since the degree of Γ is $|S| \ge 4$. The degree argument also rules cycles out. So the only possibilities for Γ_N are K_n and $K_{\frac{n}{2},\frac{n}{2}}$.

(a) Assume $\Gamma_N \cong K_n$; Γ is a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of Γ_N. Fibres are the blocks in $\mathcal{B}$ and so X is a subgroup of the fibre-preserving group which acts 2-arc transitively on Γ. We can apply Theorem 2.6 to conclude that $q = n - 1 \ge 5$ is a prime power and $q \equiv 1(mod 4)$. As $\frac{X}{N}$ acts transitively on the set of 2-arcs from $\Gamma_N \simeq K_n$, $(\frac{X}{N}|V(\Gamma_N))$ must be a 3-transitive permutation group of degree $q + 1$. According to Theorem 2.9, there are 5 possibilities. Since $q + 1$ is not a power of 2, cases (iii), (iv) and (v) of Theorem 2.9 do not happen because in all these cases the degree of the 3-transitive permutation group is a power of 2. If case (ii) of Theorem 2.9 happens, then $PSL_2(q) \le \frac{X}{N} \le P\Gamma L_2(q)$. Let $G \trianglelefteq X$ such that $\frac{G}{N} \simeq PSL_2(q)$ is the socle of $\frac{X}{N}$. A straightforward discussion tells us that $G \le C_X(N)$. Now we have

$$\frac{X}{G} \simeq \frac{\frac{X}{N}}{\frac{G}{N}} \le \frac{P\Gamma L_2(q)}{PSL_2(q)}$$

and

$$\frac{\frac{X}{G}}{\frac{C_X(N)}{G}} \simeq \frac{X}{C_X(N)} \simeq S_3$$

which do not hold together according to Lemma 4.6. If case (i) of Theorem 2.9 happens, then $\frac{X}{N}$ is an almost simple group. Looking at the degree column of table 7.4 of [2], there are only two rows where the degree of a possibly 3-transitive permutation group can be of the form $q+1$ with $q \geq 5$ a prime power and $q \equiv 1 \ (mod\ 4)$. In the second row the socle is $PSL_2(q)$ which can be 3-transitive only for q even. In the first row the socle is A_{q+1}. If the socle is A_{q+1}, then according to the same table, the index of the socle is at most two and so $\frac{X}{N} = A_{q+1}$ or S_{q+1}. Now if $C_X(N) = N$, then $\frac{X}{N} = \frac{X}{C_X(N)} \simeq S_3$ which is not 3-transitive of degree $q+1 \geq 6$, and if $C_X(N) = X$, then $\overline{N} \leq Z(X)$ and hence $\overline{N} \leq Z(\overline{X_B}) = 1$ which is impossible. So assume $C_X(N) \neq N, X$. Thus $\frac{C_X(N)}{N}$ is a nontrivial normal subgroup of $\frac{X}{N}$. This rules out A_{q+1} and leaves us only with $\frac{X}{N} = S_{q+1}$ and hence $\frac{C_X(N)}{N} = A_{q+1}$. So $[X : C_X(N)] = 2$ which contradicts what we obtained earlier. This shows that Γ is never a cover of K_n.

(b) Suppose $\Gamma_N \cong K_{\frac{n}{2},\frac{n}{2}}$; According to Theorem 2.8, this implies that either $\Gamma_N \cong K_{3,3}$ which is impossible since the valency of Γ is at least 4, or $\frac{n}{2} = 2^r$, $r \geq 2$, and $\Gamma \cong X(2, r, \varphi(x))$ where $\varphi(x)$ is some nonlinear irreducible polynomial over $GF(2)$ of degree dividing r.

Case 2: $3 \leq m < n$.

We have $\left(\frac{N}{\langle \rho^m \rangle}\right)' = 1$ and so $N' \leq \langle \rho^m \rangle$. Consider the following two subcases.

(i) If $N' \neq 1$, then $N' \trianglelefteq^c N \trianglelefteq X$ implies $N' \trianglelefteq X$.
(ii) If $N' = 1$, then N is abelian and contains $\rho^m \neq 1$ of order $o(\rho^m) \neq 2$, whereby according to Proposition 4.5, N will have a nontrivial characteristic subgroup, M, contained in $\langle \rho^m \rangle$. Now $M \trianglelefteq^c N \trianglelefteq X$ implies $M \trianglelefteq X$.

In case (i), let $H = N'$ and in case (ii), take $H = M$. In both cases, $1 \neq H \trianglelefteq X$ and $H \leq \langle \rho^m \rangle$. Evidently, $H \neq N$. For a block $B \in \mathcal{B}$, $H \trianglelefteq X_B$; So according to Proposition 4.4, $\mathcal{B}_H$ is an X-invariant partition for $V(\Gamma)$, and is a refinement of $\mathcal{B}$. The minimality of the block size of $\mathcal{B}$ leads to either $\mathcal{B}_H = \mathcal{B}$, or $\mathcal{B}_H = \{\{\alpha\} : \alpha \in V(\Gamma)\}$. According to Proposition 4.4, the latter implies $H \leq X_{(B)}$, which is not the case, because $H \neq 1$ is included in N and is semiregular on B. If $\mathcal{B}_H = \mathcal{B}$, Proposition 4.4 implies that H is transitive on B and hence $|H| \geq |B| = |N|$. This leads to the impossible equality $H = N$ as we already have $H \leq N$. □

Let $\Gamma = Cay(G, S)$ be a connected bipartite Cayley graph of a group G. Then Γ has a unique bipartition $\{\Delta_1, \Delta_2\}$. In fact if Δ_1 is the partite containing $1 \in G$, then $S \subset \Delta_2$. Now every element of G which is a product of 2 elements of S, is again in Δ_1. Continuing, we see that Δ_1 is the set of elements of G which can be written as a product of an even number of elements from S, and Δ_2 is the set of elements of G which can be written as a product of an odd number of elements from S. Clearly these two subsets of G are unique, and hence the bipartition is unique. Now assume

$X \leq Aut(\Gamma)$ is transitive on $V(\Gamma)$ and define

$$X^+ := \{g \in X | g(\Delta_1) = \Delta_1\} = \{g \in X | g(\Delta_2) = \Delta_2\}$$

Then $\left[X : X^+\right] = 2$. Also let X_1^+ be obtained by restricting the domain of all elements of X^+ to Δ_1. Clearly $X_1^+ \simeq \frac{X^+}{K}$ where K is the kernel of the action of X^+ on Δ_1. As the bipartition $\{\Delta_1, \Delta_2\}$ is unique, we may unambiguously refer to X^+ and X_1^+, given X.

Lemma 4.8 *Let $n \geq 2$, $G = B_{4n}$ and $\Gamma = Cay(G, S)$ be a connected bipartite Cayley graph such that $X \leq Aut(\Gamma)$ is transitive on $V(\Gamma)$. Then for $i = 1$ or 2, the action of X^+ on Δ_i is either imprimitive or doubly transitive. Moreover X_1^+ has a subgroup of order $2n$ and an element of order n.*

Proof We can easily verify that $\left[R(G) : R^+\right] = 2$ where $R^+ = R(G) \cap X^+$. So R^+ is a subgroup of X^+ of order $2n$. Also R^+ is isomorphic to an index 2 subgroup of G, and hence isomorphic to $\langle a \rangle$, N_1 or N_2 (the last two cases only for even n). Each of these three groups has an element of order n. It also follows that R^+ is a B-group which acts regularly on Δ_i for i =1 and 2. Assume K_i is the kernel of the action of X^+ on Δ_i. We have the permutation group $(\frac{X^+}{K_i} | \Delta_i)$ whose action is essentially the same as X^+ on Δ_i. Now $K_i \cap R^+ = 1$ implies that $R^+ \simeq \frac{R^+ K_i}{K_i} \leq \frac{X^+}{K_i}$ and hence $(\frac{X^+}{K_i} | \Delta_i)$ is either imprimitive or doubly transitive. So the action of X^+ on Δ_i is either imprimitive or doubly transitive. Moreover $X_1^+ \simeq \frac{X^+}{K_1}$ has a subgroup isomorphic to R^+ which is of order $2n$ and has an element of order n. □

Clearly $K_{2n,2n}$ and $K_{2n,2n} - (2n)K_2$ are 2-arc transitive Cayley graphs of B_{4n} and are excluded in the next lemma.

Lemma 4.9 *Let $n \geq 3$, $G = B_{4n}$, and $\Gamma \neq K_{2n,2n}, K_{2n,2n} - (2n)K_2$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \leq X \leq Aut(\Gamma)$. Assume further, that $(X|V(\Gamma))$ is imprimitive with minimum block size equal to $2n$. Then Γ is the incidence graph of a symmetric 2-design D, with $X_1^+ \leq Aut(D)$ acting doubly transitively on the point set.*

Proof Choose $\mathcal{B}$ to be an imprimitivity block system for $(X|V(\Gamma))$ whose block size is minimum possible. So the block size of $\mathcal{B}$ is $2n$ and Γ is bipartite. Now the action of X^+ on Δ_1 is either 2-transitive or imprimitive, according to Lemma 4.8. If it is imprimitive with $\mathcal{B}_1$ as an imprimitivity system of blocks, and if $X = X^+ \cup X^+ g$, then applying g on the blocks in $\mathcal{B}_1$, we obtain an imprimitivity system of blocks $\mathcal{B}_2$ for the action of X^+ on Δ_2, and $\mathcal{B}_1 \cup \mathcal{B}_2$ would be an imprimitivity system of blocks for $(X|V(\Gamma))$. We have $|\mathcal{B}_1 \cup \mathcal{B}_2| \geq 4$, and so the block size of $\mathcal{B}_1 \cup \mathcal{B}_2$ is at most $\frac{4n}{4} = n$, contradicting the assumption.

So X^+ acts 2-transitively on Δ_i for $i = 1$ and 2. It follows that every pair of vertices from Δ_1 have the same number of neighbors in Δ_2. Call this number λ and take s to be the valency of Γ. As Γ is connected and $\Gamma \neq K_{2n,2n}, K_{2n,2n} - (2n)K_2$, we have $4 \leq s \leq 2n - 2$ and we may define a symmetric 2-design D in such a way that Γ

becomes its incidence graph. Take the set of points of D to be Δ_1 and for each vertex in Δ_2, put all its neighbors in one block. So blocks are subsets of Δ_1 of cardinality s so that they are neighbors of a common vertex in Δ_2. So D is a symmetric 2-$(2n, s, \lambda)$ design. Moreover, it is easily verified that $Aut(D)$ has a subgroup isomorphic to X_1^+, and because X_1^+ acts 2-transitively on the point set of D, it is a 2-transitive design. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1 As noted earlier, because B_{4n} is a B-group, $(X|V(\Gamma))$ is imprimitive or 2-transitive, and in the latter case, $\Gamma \simeq K_{4n}$. So assume $(X|V(\Gamma))$ is imprimitive and $\Gamma \neq K_{2n,2n}, K_{2n,2n} - (2n)K_2$. Choose $\mathcal{B}$ to be an imprimitivity block system for $(X|V(\Gamma))$ whose block size is minimum. If blocks in $\mathcal{B}$ satisfy case (i) of Proposition 4.2, then according to Proposition 4.3, the block size is $2n$ since $Core_X(R(G)) = 1$. If blocks in $\mathcal{B}$ satisfy case (ii) of Proposition 4.2, then according to Proposition 4.7, either $n = 2^{r+1} \geq 8$ and $\Gamma \cong X(2, r, \varphi(x))$ for some suitable $\varphi(x)$, or the block size is $2n$. In both cases, when the block size is $2n$, according to Lemma 4.9, Γ is the incidence graph of a symmetric $2-(2n, s, \lambda)$ design D, where X_1^+ is a 2-transitive permutation group on its point set, and where $s = |S|$. Also D', the complement of D, is a symmetric design and X_1^+ is a 2-transitive permutation group on its point set. The relation $\lambda(2n-1) = s(s-1)$ holds for D, from which it follows that $s \neq n$. If $s < n$, then D is one of the designs listed in Theorem 2.3, and if $s > n$, then $2 \leq 2n - s < n$ and D' is one of the designs listed in Theorem 2.3. So Γ is either the incidence graph of a design from Theorem 2.3, or the non-incidence graph of one of those designs. We show that D is none of the designs listed in (ii), (iii) and (iv) of Theorem 2.3 nor their complements. Note that X_1^+ plays the role of G in Theorem 2.3 and take N to be the unique minimal normal subgroup of X_1^+. Case (ii) is clearly not possible, since the number of points of D is even.

Case (iii): If this case happens for D or its complement, then the degree of the 2-transitive permutation group is 176 and N is nonabelian simple. Looking at the degree column of Table 7.4 of [2], one can easily verify that the only 2-transitive permutation groups of degree 176 correspond to $N = A_{176}$ or $N = HS$. If $N = A_{176}$, then X_1^+ is in fact 50-transitive and there is an element $f \in X_1^+$ which takes a 50-element block to a 50-element non-block subset of points, contradicting the fact that $X_1^+ \leq Aut(D)$. If $N = HS$, then according to the fourth column of the same table, $X_1^+ = N = HS$. According to Lemma 4.8, X_1^+ has a subgroup of order 176, but we discuss that HS doesn't have any such subgroup. In fact if $A \leq HS$ is of order 176, then it is contained in a maximal subgroup of HS. According to $\mathbb{ATLAS}$ [3], HS has only two maximal subgroups whose orders are divisible by 176, namely M_{11} and M_{22}. So A is a subgroup of M_{11} or M_{22}, and again A is included in a maximal subgroup of one of these two groups. But again if we look at the maximal subgroups of M_{11} and M_{22}, listed in $\mathbb{ATLAS}$, none has order divisible by 176.

Case (iv): If this case happens for the design D associated to Γ, then it follows from the detailed proof of Theorem 2.3, given in [11], that for each $m \geq 2$, D is isomorphic to a unique design constructed in [10], whose full automorphism group is a semidirect product of the translation group of the affine space, $AG(2m, 2)$, and the symplectic group $Sp(2m, 2)$. It follows that up to isomorphism, X_1^+ is a subgroup of $AGL(2m, 2)$, the affine general linear group. According to Lemma 4.8, X_1^+ would have an element of order 2^{2m-1}; On the other hand, $AGL(2m, 2)$ has no element of order 2^{2m-1}. Otherwise it requires the general linear group, $GL(2m, 2)$, to have an element of order 2^{2m-1}. By Theorem 1 of [4], if the order of $A \in GL(k, 2)$ is even for $k \geq 4$, then it is strictly less than 2^{k-1}, which is a contradiction. Evidently the same argument rules out D' and Γ is neither the incidence nor the non-incidence graph of this design. $\square$

5 Toward Classification of 2-Arc Transitive Dicirculants

In this section, we prove that if we know core-free connected 2-arc transitive dihedrants, then we will have a sort of classification theorem for connected 2-arc transitive dicirculants in terms of regular cyclic covers.

Lemma 5.1 *Let $n \geq 2$, $G = B_{4n}$ and $\Gamma = Cay(G, S)$ be a connected $(X, 2)$-arc transitive Cayley graph of G, where $R(G) \leq X \leq Aut(\Gamma)$. If $H = Core_X(R(G))$, then $[R(G) : H] \geq 5$ and H is cyclic.*

Proof If $H = R(G)$, then $K = \langle \rho^2 \rangle \trianglelefteq X$ since $K \trianglelefteq^c R(G)$. According to Theorem 2.1, Γ is a cover of the quotient graph Γ_K and hence their valencies are the same. But the degree of Γ is $|S| \geq 4$ whereas the degree of Γ_K is at most 3. If $[R(G) : H] = 2$, then $H \simeq \langle a \rangle$, N_1 or N_2. In any case, $K = \langle \rho^2 \rangle \trianglelefteq^c H$ implies $K \trianglelefteq X$ which would again lead to a contradiction as above.

Now that the index of H is at least 3, it must be contained in $\langle \rho \rangle$ and is cyclic. Finally, if $[R(G) : H] = 3$ or 4, then Γ would be a cover of Γ_H and again the degree argument leads to a contradiction. $\square$

We first resolve the cases $n = 1, 2$. For $n = 1$, $B_{4n} \simeq \mathbb{Z}_4$ and for $n = 2$, $B_{4n} = Q_8$ is the well known quaternion group of order 8. There are only two connected Cayley graphs on $\mathbb{Z}_4$, namely K_4, the complete graph, and C_4, the cycle on 4 vertices. Both of these graphs are 2-arc transitive.

For Q_8, let $\Gamma = Cay(Q_8, S)$ be a connected 2-arc transitive Cayley graph. We claim Γ is isomorphic to either K_8 or $K_{4,4}$. Let $H = Core_{Aut(\Gamma)}(R(Q_8))$, then according to Lemma 5.1, $[R(Q_8) : H] \geq 5$ and hence $|H| \leq \frac{8}{5}$, or $|H| = 1$. So Γ is core-free. Now Q_8 is a B-group, and again the action of $Aut(\Gamma)$ on the vertices is either imprimitive or doubly transitive. In the latter case, Γ must be K_8. So assume the action of $X = Aut(\Gamma)$ on the vertices is imprimitive and take $\mathcal{B}$ to be an imprimitivity system of blocks. According to Proposition 4.2, for some $m \in \{1, 2, 4\}$, $\mathcal{B}$ satisfies either case (i) or case (ii) of that proposition. If case (i) happens, then it follows from Proposition 4.3 that m =1, as we just showed that Γ is core-free. So in this case m =1 and Γ is bipartite. On the other hand, if case (ii) happens, then according to Proposition 4.7, Γ is bipartite and each partite has 4 vertices. Now it follows from the connectivity of Γ that its valency is at least 4 and hence $\Gamma \simeq K_{4,4}$.

Let C_1 be the class containing exactly the following graphs: K_{4n} for all $n \geq 2$, $K_{2n,2n}$ for all $n \geq 2$, $K_{2n,2n} - (2n)K_2$ for all $n \geq 3$, the incidence and non-incidence graphs of a projective space with $2n = \frac{q^{m+1}-1}{q-1}$ points, where q is an odd prime power, $m > 1$ is odd and $n \geq 3$, and finally, graphs $X(2, r, \varphi(x))$ for $r \geq 2$ where $\varphi(x)$ is any nonlinear binary irreducible polynomial of degree dividing r. Also let C_2 be the class containing all core-free 2-arc transitive dihedrants of valency at least 4. Then we can say the following about 2-arc transitive dicirculants:

Proposition 5.2 *Let $n \geq 3$ and $\Gamma = Cay(B_{4n}, S)$ be a connected 2-arc transitive dicirculant. Then Γ belongs to one of the following families:*

(a) C_1.
(b) *A regular $\mathbb{Z}_d$-cover of one of the graphs in C_1, with $2 \leq d \leq \frac{2n}{3}$ a divisor of $2n$.*
(c) *A regular $\mathbb{Z}_d$-cover of one of the graphs in C_2, with $2 \leq d \leq \frac{2n}{3}$ a divisor of $2n$.*

Proof Let $H = Core_{Aut(\Gamma)}(R(B_{4n}))$; If $H = 1$, then $\Gamma \in C_1$. If $H \neq 1$, according to Lemma 5.1, $[R(B_{4n}) : H] \geq 5$ and $H \simeq \langle a^i \rangle$ is cyclic, where i divides $2n$. It follows that $2i \geq 5$, or $i \geq 3$, which yields $d = |H| \leq \frac{2n}{3}$. Now according to Theorem 2.2, Γ is an H-cover of Γ_H which itself is a core-free 2-arc transitive Cayley graph of the group $\frac{B_{4n}}{\langle a^i \rangle}$. As we noted in Sect. 3, this quotient is either dihedral or dicyclic, for $i \geq 3$, and hence Γ_H lies in C_1 or C_2. $\square$

As an application of the results we have obtained so far, here we give a full classification for 2-arc transitive dicirculants of order $4p$, $p > 2$ prime. Note that the case $p = 2$ was resolved above.

Proposition 5.3 *For any odd prime p, a connected graph Γ of order $4p$ is a 2-arc transitive dicirculant if and only if Γ is isomorphic to one of the followings:*

1. K_{4p}
2. $K_{2p,2p}$
3. $K_{2p,2p} - (2p)K_2$

Proof Suppose $p > 2$ is a prime and let $G = B_{4p}$. Clearly K_{4p}, $K_{2p,2p}$ and $K_{2p,2p} - (2p)K_2$ are Cayley graphs of the group G and they are also 2-arc transitive. Conversely assume $\Gamma = Cay(G, S)$ is 2-arc transitive and connected. Let $H = Core_{Aut(\Gamma)}(R(G))$. It follows from Lemma 5.1, that $|H| \leq \frac{4p}{5} < p$. Also $|H|$ is a divisor of $4p$ and hence $H = 1$. So Γ is core-free and therefore it is isomorphic to one of the graphs listed in Theorem 4.1. Cases (d) and (e) of that Theorem can occur only under some conditions of the parameters. Since p is odd, Γ cannot be isomorphic to graphs in case (e). Also if Γ is isomorphic to a graph in case (d), then we must have $2p = \frac{q^{m+1}-1}{q-1}$ for some odd $m > 1$ and some odd prime power q. However with these constraints, this equation does not have any solution. In fact if $2p = \frac{q^{m+1}-1}{q-1}$ where m and q satisfy the aforementioned constraints, then $2p = q^m + q^{m-1} + \cdots + q + 1 = (q+1)t$ where $t = q^{m-1} - q^{m-2} + q^{m-3} - \cdots + q^2 - q + 1 > 1$. So $q + 1 > 2$ is an even divisor of $2p$ and hence $q + 1 = 2p$ which implies $t = 1$, a contradiction. $\square$

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