



Two Orthogonal C_4 -Free One-Factorizations of Complete Graphs

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Abstract

A pair of orthogonal one-factorizations \mathcal{F} and \mathcal{G} of the complete graph K_n is C_4 -free if for any two factors $F \in \mathcal{F}$ and $G \in \mathcal{G}$ the union $F \cup G$ does not include a cycle of length four. Such a concept was introduced by Blokhuis et al. (J Combin Theory B 82: 1–18, 2001), who used it to improve the upper bound for two-round rainbow colorings of K_n . In this paper, we focus on constructions for a pair of orthogonal C_4 -free one-factorizations of the complete graph K_n . Some infinite classes of such orthogonal decompositions are obtained.

Keywords One-factorization · Room square · C_4 -free

1 Introduction

A one-factor of a graph G is a regular spanning subgraph of degree one. A *one-factorization* of a graph G is a set $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ of edge-disjoint one-factors such that $E(G) = \bigcup_{i=1}^n E(F_i)$. Two one-factorizations are *orthogonal*, if any pair of one-factors one from the first factorization and one from the second factorization have at most one edge in common. The existence of a pair of orthogonal one-factorizations of complete graph K_{n+1} is equivalent to the existence of a Room square of side n .

Let S be a set of $n + 1$ elements called symbols. A *Room square* of size n on S is an $n \times n$ array, F , that satisfies the following properties:

1. Every cell of F is either empty or contains an unordered pair of symbols from S .
2. Each symbol of S occurs once in each row and column of F .
3. Every unordered pair of symbols occurs in precisely one cell of F .

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Room squares were named after T. G. Room who published a paper in 1955. It is well known that a Room square of size n exists if and only if n is odd and $n \neq 3$ or 5 [17].

One-factors and one-factorizations of complete graphs (and other graphs) arise naturally in the construction of round-robin tournaments and other competition schedules, and certainly they have been considered for a long time. One-factorizations have proved useful in the construction of designs such as Room squares and Steiner systems. For more details on one-factorizations of complete graphs, see the excellent surveys [1,5,9,16].

A one-factorization $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ of a graph G is said to be k -cycle free if the union of any two one-factors does not include a cycle of length k (denoted by C_k). The existence of a 4-cycle-free one-factorization of complete graph K_n for even $n \geq 6$ has already been observed in [12], where they were used to give the existence of simple quadruple systems with index three. Generally, for each even n and each even $k \geq 4$ with $k \neq \frac{n}{2}$, the complete graph K_n has a k -cycle-free one-factorization [10].

Theorem 1 [12] *A C_4 -free one-factorization of complete graphs K_n exists if and only if n is even and $n \geq 6$.*

A pair of orthogonal one-factorizations \mathcal{F} and \mathcal{G} of complete graph K_n is C_4 -free if for any two factors $F \in \mathcal{F}$ and $G \in \mathcal{G}$ the union $F \cup G$ does not include a cycle of length four. Such a concept was introduced by Blokhuis et al., who used it to improve the upper bound for two-round rainbow colorings of K_n [2].

A subgraph H of K_n is *total multicolored* (colored rainbow) if all its edges have different colors. A t -round χ -coloring is defined as a sequence ψ_1, \dots, ψ_t of t (not necessarily distinct) proper edge colorings of a complete graph, using at most χ colors in each of the colorings. For positive integers $k \leq n$ and t , let $\chi^t(k, n)$ denote the minimum number χ of colors for which there exists a t -round χ -coloring of K_n such that each subgraph K_k of K_n is total multicolored in at least one round.

Blokhuis et al. [2] proved that

$$\frac{\sqrt{(n-1)n}}{\sqrt{2}} \leq \chi^2(4, n) \leq 2n$$

for $n \geq 4$. When there is a pair of orthogonal C_4 -free one-factorizations of a complete graph, this upper bound can be improved greatly.

Theorem 2 [2] *If K_n has a pair of orthogonal C_4 -free one-factorizations, then $\chi^2(4, n) \leq n - 1$.*

In this paper, we shall focus on constructions of a pair of orthogonal C_4 -free one-factorizations of complete graphs. Here we are also interested in pairs of orthogonal C_4 -free one-factorizations of complete graphs with a strengthened property. A pair of orthogonal one-factorizations \mathcal{F} and \mathcal{G} of complete graph K_n is *totally C_4 -free* if for any two distinct factors $F, G \in \mathcal{F} \cup \mathcal{G}$ the union $F \cup G$ does not include a cycle of length four.

The rest of this paper is arranged as follows. In Sect. 2, we use starters to construct a pair of orthogonal totally C_4 -free one-factorizations of complete graphs. In Sect. 3,

a recursive construction is stated. In Sect. 4, we modify Horton’s Room squares [8] so that the two orthogonal one-factorizations are C_4 -free. Some infinite classes of two orthogonal C_4 -free one-factorizations of complete graphs are also given.

2 Starter Construction

The most useful technique for the direct construction of Room squares has been the technique of orthogonal starters. This technique was introduced in the mathematical literature in 1968 by Stanton and Mullin in [14].

Let G be an abelian group of odd order g . A starter in G is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g - 1)/2\}$ which satisfies the following two properties:

- (i) $\{s_i : 1 \leq i \leq (g - 1)/2\} \cup \{t_i : 1 \leq i \leq (g - 1)/2\} = G \setminus \{0\}$;
- (ii) $\{\pm(s_i - t_i) : 1 \leq i \leq (g - 1)/2\} = G \setminus \{0\}$.

It is well known that $F_a = \{\{\infty, a\}\} \cup \{\{s_i + a, t_i + a\} : 1 \leq i \leq (g - 1)/2\}$, $a \in G$, forms a one-factorization of the complete graph on $G \cup \{\infty\}$.

Let $S = \{\{s_i, t_i\} : 1 \leq i \leq (g - 1)/2\}$ and $T = \{\{u_i, v_i\} : 1 \leq i \leq (g - 1)/2\}$ be two starters in G . Without loss of generality, we may assume that $s_i - t_i = u_i - v_i$ for all i . Then S and T are said to be *orthogonal starters* if $u_i - s_i = u_j - s_j$ implies $i = j$, and if $u_i \neq s_i$ for all i .

Theorem 3 [14] *If there are two orthogonal starters in an abelian group of odd order g , then there is a Room square of side g , i.e., a pair of orthogonal one-factorizations of complete graph K_{g+1} .*

If $S = \{\{s_i, t_i\} : 1 \leq i \leq (g - 1)/2\}$ is a starter, then $-S = \{\{-s_i, -t_i\} : 1 \leq i \leq (g - 1)/2\}$ is also a starter. In any abelian group G of odd order, the set of pairs $P = \{\{x, -x\} : x \in G\}$ is a starter, called the *patterned starter*. A starter $S = \{\{s_i, t_i\} : 1 \leq i \leq (g - 1)/2\}$ is called *strong* if the sums $s_i + t_i$ are distinct and nonzero.

Theorem 4 [7] *If there is a strong starter S in an abelian group of odd order g , then S , $-S$ and P are pairwise orthogonal.*

Lemma 1 [6] *For odd $g \geq 5$, the one-factorization of K_{g+1} generated by patterned starter is C_4 -free if and only if $g \not\equiv 0 \pmod{3}$.*

Let q be a prime power with $q = ef + 1$ and $GF(q)$ be the finite field of q elements. Given a primitive element α of $GF(q)$, define $C_0^{(e,q)} = \{\alpha^{je} : 0 \leq j \leq f - 1\}$, the multiplicative group generated by α^e , and

$$C_i^{(e,q)} = \alpha^i C_0^{(e,q)}$$

for $1 \leq i \leq e - 1$. Then $C_0^{(e,q)}, C_1^{(e,q)}, \dots, C_{e-1}^{(e,q)}$ partition $GF(q)^* = GF(q) \setminus \{0\}$. The $C_i^{(e,q)}$ are known as *cyclotomic classes* of order e (with respect to $GF(q)$).

Theorem 5 [7] *Let $q \equiv 3 \pmod{4}$ be a prime power. For $w \in C_1^{(2,q)} \setminus \{-1\}$, define $T_w = \{\{x, wx\} : x \in C_0^{(2,q)}\}$. Then T_w is a strong starter. Further, T_w and T_u are orthogonal if $w, u \in C_1^{(2,q)} \setminus \{-1\}$ and $w \neq u$.*

Lemma 2 *Let $q \equiv 3 \pmod{4}$ be a prime power and T_w as above. The one-factorization generated by the starter T_w is C_4 -free if $\{w, \frac{1+w^2}{2w}\} \subseteq C_1^{(2,q)}$ and $w^3 \neq -1$.*

Proof Let $F_i = \{\{\infty, i\}\} \cup \{\{x + i, wx + i\} : x \in C_0^{(2,q)}\}$ for $i \in GF(q)$. Then $\mathcal{F} = \{F_i : i \in GF(q)\}$ is a one-factorization of the complete graph on $GF(q) \cup \{\infty\}$. Assume that this one-factorization is not C_4 -free. Since $F_i = F_0 + i$, without loss of generality, let $F_0 \cup F_i$ ($i \neq 0$) contain a cycle $\{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}\}$ of length 4 where $\{a, b\}, \{c, d\} \in F_0$ and $\{a, c\}, \{b, d\} \in F_i$. Then $\{a-i, c-i\}, \{b-i, d-i\} \in F_0$.

Case 1 $\infty \in \{a, b, c, d\}$.

Without loss of generality, let $a = \infty$. Then $b = 0$ and $c = i$. If $i \in C_0^{(2,q)}$, then by the definition of T_w and $-1 \in C_1^{(2,q)}$ we have that $d - i \in C_0^{(2,q)}$, $d = wi$ and $-i = w(d - i)$. It follows that $w^2 - w + 1 = 0$, which contradicts the assumption that $w^3 \neq -1$. If $i \in C_1^{(2,q)}$, then $d - i \in C_1^{(2,q)}$, $i = wd$ and $d - i = -wi$. It follows that $w^2 - w + 1 = 0$, which also contradicts the assumption that $w^3 \neq -1$.

Case 2 $\infty \notin \{a, b, c, d\}$, i.e., $\{a, b, c, d\} \subset GF(q)^*$.

Let $\{a, b\} = \{x, wx\}$ where $x \in C_0^{(2,q)}$. Since $x^{-1} \cdot F_0 = F_0$, we have that $\{a/x, b/x\}, \{c/x, d/x\} \in F_0$ and $\{a/x, c/x\}, \{b/x, d/x\} \in F_{i/x}$, i.e., $\{\{a/x, b/x\}, \{c/x, d/x\}, \{a/x, c/x\}, \{b/x, d/x\}\}$ is a cycle of length 4 in $F_0 \cup F_{i/x}$. For convenience, let $a = 1$ and $b = w$. Since $F_i = F_0 + i, \{1 - i, c - i\}, \{w - i, d - i\} \in F_0$. Clearly, $1 - i, c - i, w - i, d - i \in GF(q)^*$.

Subcase 1 $1 - i, w - i, c \in C_0^{(2,q)}$.

By the definition of T_w , we have that $c - i = w(1 - i), d - i = w(w - i)$ and $d = wc$. Simple computation shows that $w = 1$, a contradiction.

Subcase 2 $1 - i, w - i \in C_0^{(2,q)}$ and $c \in C_1^{(2,q)}$.

By the definition of T_w , we have $c - i = w(1 - i), d - i = w(w - i)$ and $c = wd$. Simple computation shows $i = \frac{w^2+w}{w-1}$. Since $1 - i, w - i \in C_0^{(2,q)}$, we have that $1 - i = \frac{-1-w^2}{w-1}, w - i = \frac{-2w}{w-1} \in C_0^{(2,q)}$. This is impossible because $\frac{1+w^2}{2w} \in C_1^{(2,q)}$ by assumption.

Subcase 3 $1 - i, c \in C_0^{(2,q)}$ and $w - i \in C_1^{(2,q)}$.

By the definition of T_w , we have $c - i = w(1 - i), w(d - i) = w - i$ and $d = wc$. Simple computation shows that $i = \frac{w^2+w}{w^2+1}$. Then $c = i + w - wi = \frac{2w}{w^2+1} \in C_0^{(2,q)}$. This is impossible because $\frac{1+w^2}{2w} \in C_1^{(2,q)}$ by assumption.

Subcase 4 $1 - i \in C_0^{(2,q)}$ and $w - i, c \in C_1^{(2,q)}$.

By the definition of T_w , we have $c - i = w(1 - i), w(d - i) = w - i$ and $c = wd$. Simple computation shows that $w = 1$, a contradiction.

Subcase 5 $1 - i \in C_1^{(2,q)}$ and $w - i, c \in C_0^{(2,q)}$.

By the definition of T_w , we have $1 - i = w(c - i)$, $d - i = w(w - i)$ and $d = wc$. Simple computation shows $i = \frac{w+1}{2}$. Then $c = i + \frac{1-i}{w} = \frac{w^2+1}{2w} \in C_0^{(2,q)}$, which also contradicts the assumption that $\frac{1+w^2}{2w} \in C_1^{(2,q)}$.

Subcase 6 $1 - i, c \in C_1^{(2,q)}$ and $w - i \in C_0^{(2,q)}$.

By the definition of T_w , we have $1 - i = w(c - i)$, $d - i = w(w - i)$ and $c = wd$. Then $i = 1 + w$ and $d = i + w^2 - wi = 1 = a$, a contradiction.

Subcase 7 $1 - i, w - i \in C_1^{(2,q)}$ and $c \in C_0^{(2,q)}$.

By the definition of T_w , we have $1 - i = w(c - i)$, $w - i = w(d - i)$ and $d = wc$. Simple computation shows that $w = 1$, a contradiction.

Subcase 8 $1 - i, w - i, c \in C_1^{(2,q)}$.

By the definition of T_w , we have $1 - i = w(c - i)$, $w - i = w(d - i)$ and $c = wd$. Simple computation shows that $i = \frac{1+w}{1-w}$. Then since $1 - i \in C_1^{(2,q)}$ and $w - i \in C_1^{(2,q)}$, we have $1 - i = \frac{2w}{w-1} \in C_1^{(2,q)}$ and $w - i = \frac{w^2+1}{w-1} \in C_1^{(2,q)}$, which also contradicts the assumption that $\frac{1+w^2}{2w} \in C_1^{(2,q)}$. □

Example 1 For $q = 19$, take $w = 13$. Then $\frac{1+w^2}{2w} = 8 \in C_1^{(2,19)}$. Set

$$F_i = \left\{ \{\infty, i\} \cup \{x + i, 13x + i\} : x \in C_0^{(2,19)} \right\} \quad \text{for } i \in GF(19).$$

It is easy to check that one-factorization $\mathcal{F} = \{F_i : i \in GF(19)\}$ is C_4 -free.

Lemma 3 *Let $q \equiv 3 \pmod{4}$ be a prime power and T_w be as above. The pair of orthogonal one-factorizations generated by the starter T_w and $-T_w$ is C_4 -free if $w, \frac{1+w^2}{2w} \in C_1^{(2,q)}$, $w \notin \{2, \frac{1}{2}\}$ and $w^3 \neq -1$.*

Proof Let $F_i = \{\{\infty, i\}\} \cup \{x + i, wx + i\} : x \in C_0^{(2,q)}\}$ and $G_i = \{\{\infty, i\}\} \cup \{-x + i, -wx + i\} : x \in C_0^{(2,q)}\}$ for $i \in GF(q)$. Then $\mathcal{F} = \{F_i : i \in GF(q)\}$ and $\mathcal{G} = \{G_i : i \in GF(q)\}$ are orthogonal one-factorizations of the complete graph on $GF(q) \cup \{\infty\}$ by Theorems 3, 4 and 5. Assume that this pair of one-factorizations is not C_4 -free. Since $F_i = F_0 + i$ and $G_i = G_0 + i$, without loss of generality, assume $F_0 \cup G_i$ ($i \in GF(q)$) contains a cycle $\{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}\}$ of length 4 where $\{a, b\}, \{c, d\} \in F_0$ and $\{a, c\}, \{b, d\} \in G_i$. Then $\{a - i, c - i\}, \{b - i, d - i\} \in G_0$.

Case 1 $\infty \in \{a, b, c, d\}$.

Without loss of generality, let $a = \infty$. Then $b = 0$ and $c = i$. If $i = 0$, then $c = b = 0$, a contradiction. If $i \in C_0^{(2,q)}$, then by the definition of T_w , G_0 and $-1 \in C_1^{(2,q)}$ we have $d = wi$ and $d - i = -wi$. It follows that $w = 1/2$, which contradicts the assumption that $w \neq 1/2$. If $i \in C_1^{(2,q)}$, then $i = wd$ and $w(d - i) = -i$. It follows that $w = 2$, which also contradicts the assumption that $w \neq 2$.

Case 2 $\infty \notin \{a, b, c, d\}$, i.e., $\{a, b, c, d\} \subset GF(q)^*$.

Let $\{a, b\} = \{x, wx\}$ where $x \in C_0^{(2,q)}$. Since $x^{-1} \cdot F_0 = F_0$, we have $\{a/x, b/x\}, \{c/x, d/x\} \in F_0$ and $\{a/x, c/x\}, \{b/x, d/x\} \in G_{i/x}$, i.e., $\{\{a/x, b/x\}, \{c/x, d/x\}, \{a/x, c/x\}, \{b/x, d/x\}\}$ must be a cycle of length 4 in $F_0 \cup G_{i/x}$. For convenience, let $a = 1$ and $b = w$. Since $G_i = G_0 + i$, $\{1 - i, c - i\}, \{w - i, d - i\} \in G_0$, i.e., $\{i - 1, i - c\}, \{i - w, i - d\} \in F_0$.

Since $q \equiv 3 \pmod{4}$, we have $-1 \in C_1^{(2,q)}$. According to which cyclotomic classes $i - 1, i - w, c$ belong to, there are eight possible subcases. No matter which case occurs, it must be equivalent to some subcase of Lemma 2. For example, the case when $i - 1, i - w, c \in C_0^{(2,q)}$ is equivalent to Subcase 7 of Lemma 2. The same arguments arrive at a contradiction to the assumptions. \square

In the theory of cyclotomy, the number of solutions of

$$x + 1 = y, \quad x \in C_i^{(e,q)}, \quad y \in C_j^{(e,q)}$$

is called *cyclotomic number* of order e respect to $GF(q)$ and denoted $(i, j)_e$. The following basic properties of the cyclotomic numbers were established by Dickson [4].

Theorem 6 [4] *Let q be a prime power with $q = ef + 1$. Then*

- (1) $(i, j)_e = (e - i, j - i)$,
- (2) $(i, j) = \begin{cases} (j, i)_e, & \text{if } 2 \mid f, \\ (j + \frac{e}{2}, i + \frac{e}{2}), & \text{if } 2 \nmid f; \end{cases}$
- (3) $\sum_{j=0}^{e-1} (i, j)_e = \begin{cases} f - 1, & \text{if } 2 \mid f \text{ and } i = 0, \\ f - 1, & \text{if } 2 \nmid f \text{ and } i = \frac{e}{2}, \\ f, & \text{otherwise;} \end{cases}$
- (4) $\sum_{i=0}^{e-1} (i, j)_e = \begin{cases} f - 1, & \text{if } j = 0, \\ f, & \text{if } j \neq 0. \end{cases}$

Some cyclotomic numbers have been determined. The reader is referred to [15].

Theorem 7 *For any prime power $q \equiv 3 \pmod{4}$ with $q \geq 11$, there is a pair of orthogonal totally C_4 -free one-factorizations of K_{q+1} .*

Proof Let $M = \left\{ w : w \in C_1^{(2,q)}, w \notin \left\{ 2, \frac{1}{2} \right\}, w^3 \neq -1 \text{ and } \frac{w^2+1}{2w} \in C_1^{(2,q)} \right\}$. By Lemmas 2 and 3, we need only show that M is not the empty set.

Let the characteristic of the finite field $GF(q)$ be p . Then $q = p^n$. Since $q \equiv 3 \pmod{4}$, it holds that $p \equiv 3 \pmod{4}$ and n is odd. Let α be a primitive element of $GF(q)$ and $m = \frac{p^n-1}{p-1}$. Then m is odd and $\{\alpha^{mi} : 0 \leq i \leq p - 2\} = GF(p)^*$. Let $2 = \alpha^{mj}$. Clearly, if 2 also belongs to $C_0^{(2,p)}$, then 2 belongs to $C_0^{(2,q)}$. Conversely, if 2 belongs to $C_0^{(2,q)}$, then j is even because m is odd. So, $2 = (\alpha^{mj/2})^2 \in C_0^{(2,p)}$. These show that 2 is a quadratic residue modulo p if and only if 2 is a quadratic residue in $GF(q)$. Also, by Theorem 6 it is easy to see that $(0, 0)_2 = (1, 0)_2 = (1, 1)_2 = \frac{f-1}{2}$ and $(0, 1)_2 = \frac{f+1}{2}$.

For $q \equiv 7, 23 \pmod{24}$ with $q \geq 11$, it is easy to see that $\text{char}(GF(q)) = p \equiv 7 \pmod{8}$. Since 2 is a quadratic residue in \mathbb{Z}_p by elementary number theory, 2 is a quadratic residue in $GF(q)$. Since $(0, 0)_2 = \frac{q-3}{4} \geq 4$, there are at least four elements $y \in C_0^{(2,q)}$ such that $1 + y \in C_0^{(2,q)}$. Since $-1 \in C_1^{(2,q)}$, there is an $x \in C_1^{(2,q)}$ such that $x^2 = y$. On the other hand, there are at most three elements z in $C_1^{(2,q)}$ such

that $z^3 = -1$. So, there is at least one element $w \in C_1^{(2,q)}$ such that $w^3 \neq -1$ and $1 + w^2 \in C_0^{(2,q)}$. Such an element w belongs to M , thus $M \neq \emptyset$.

For $q \equiv 11, 19 \pmod{24}$ with $q \geq 23$, it is easy to see that $\text{char}(GF(q)) = p \equiv 3 \pmod{8}$. Since 2 is a quadratic non-residue in \mathbb{Z}_p by elementary number theory, 2 is also a quadratic non-residue in $GF(q)$. Since $(0, 1)_2 = \frac{q+1}{4} \geq 6$, there are at least six elements $y \in C_0^{(2,q)}$ such that $1 + y \in C_1^{(2,q)}$. Since $-1 \in C_1^{(2,q)}$, there is an $x \in C_1^{(2,q)}$ such that $x^2 = y$. On the other hand, there are at most three elements z in $C_1^{(2,q)}$ such that $z^3 = -1$. So, there is at least one element $w \in C_1^{(2,q)}$ such that $w^3 \neq -1$, $w \neq 2, \frac{1}{2}$ and $1 + w^2 \in C_1^{(2,q)}$. Such an element w belongs to M , i.e., $M \neq \emptyset$. For $q = 11$, take $w = 8$ and for $q = 19$, take $w = 13$. It is routine check that $w \in M$. □

Example 2 For $q = 19$, take $w = 13$. Then $\frac{1+w^2}{2w} = 8 \in C_1^{(2,19)}$. Set

$$F_i = \left\{ \{\infty, i\} \cup \{x + i, 13x + i\} : x \in C_0^{(2,19)} \right\} \quad \text{and}$$

$$G_i = \left\{ \{\infty, i\} \cup \{18x + i, 6x + i\} : x \in C_0^{(2,19)} \right\} \quad \text{for } i \in GF(19).$$

It is easy to check that the pair of one-factorizations $\mathcal{F} = \{F_i : i \in GF(19)\}$ and $\mathcal{G} = \{G_i : i \in GF(19)\}$ is totally C_4 -free.

Let $q = 2^k t + 1$ be an odd prime power, where $t \geq 3$ is odd. For $w \in C_{2^{k-1}}^{(2^k,q)}$, let $A = \bigcup_{i=0}^{2^k-1} C_i^{(2^k,q)}$ and $S_w = \{x, wx\} : x \in A$. Then each S_w is a starter, where S_{-1} is the patterned starter. It is easy to see that each S_w is a strong starter for $w \in C_{2^{k-1}}^{(2^k,q)} \setminus \{-1\}$.

Lemma 4 *Let $q = 2^k t + 1 \geq 9$ be an odd prime power, where t is odd and $k \geq 1$. The pair of orthogonal one-factorizations of K_{q+1} generated by the starters S_w and S_{-1} is C_4 -free, where $w \in C_{2^{k-1}}^{(2^k,q)} \setminus \{-1, 2, \frac{1}{2}\}$.*

Proof Denote $F_i = \{\{\infty, i\} \cup \{x + i, -x + i\} : x \in A\}$ and $G_i = \{\{\infty, i\} \cup \{x + i, wx + i\} : x \in A\}$ for $i \in GF(q)$. Since S_w is a strong starter for each $w \in C_{2^{k-1}}^{(2^k,q)} \setminus \{-1\}$, by Theorems 3 and 4 we have $\mathcal{F} = \{F_i : i \in GF(q)\}$ and $\mathcal{G} = \{G_i : i \in GF(q)\}$ are orthogonal one-factorizations of the complete graph on $GF(q) \cup \{\infty\}$. Assume that this pair of one-factorizations is not C_4 -free. Since $F_i = F_0 + i$ and $G_i = G_0 + i$, $F_i \cup G_j$ contains a C_4 if and only if $F_{i-j} \cup G_0$ contains a C_4 . Let $F_i \cup G_0$ ($i \in GF(q)$) contains a cycle $\{a, b, c, d\}$ of length 4 where $\{a, b\}, \{c, d\} \in G_0$ and $\{a, c\}, \{b, d\} \in F_i$. Then $\{a-i, c-i\}, \{b-i, d-i\} \in F_0$.

Case 1 $\infty \in \{a, b, c, d\}$.

Without loss of generality, let $a = \infty$. Then $b = 0$ and $c = i$. By the definition of F_i we have $b + d = d = 2i = 2c$. Hence $\frac{d}{c} = 2$. On the other hand, it holds that $\frac{d}{c} = w$ or $\frac{1}{w}$ by the definition of G_0 . So, we arrive at a contradiction because $w \notin \{2, \frac{1}{2}\}$ by assumption.

Case 2 $\infty \notin \{a, b, c, d\}$, i.e., $\{a, b, c, d\} \subset GF(q)^*$.

Let $\{a, b\} = \{x, wx\}$ and $\{c, d\} = \{y, wy\}$ where $x, y \in A$ and $x \neq y$. By the definition of F_i , it holds that $a + c = 2i = b + d$. So, $x + y = wx + wy$ or $x + wy = y + wx$. It follows that $w = 1$, or $x + y = 0$, or $x = y$. Since $x, y \in A = \cup_{i=0}^{2^{k-1}-1} C_i^{(2^k, q)}$ and $t \geq 3$ is odd, we have $-1 \in C_{2^{k-1}}^{(2^k, q)}$ and $x + y \neq 0$. Also, $w \neq 1$ and $x \neq y$ by definition. Thus, we arrive at a contradiction. \square

Corollary 1 *Let $q = p^s = 2^k t + 1 \geq 9$ be an odd prime power where $p \geq 5$ is a prime and $t \geq 3$ is odd. Then there is a pair of orthogonal C_4 -free one-factorizations of K_{q+1} .*

Proof By Lemma 4, we need only to show that there is an element $w \in C_{2^{k-1}}^{(2^k, q)} \setminus \{-1, 2, \frac{1}{2}\}$. If $t > 3$, clearly such an element exists. If $t = 3$, it holds that $2 \notin C_{2^{k-1}}^{(2^k, q)}$. Assume that it does not hold. Then we have $2 = \alpha^{2^k s + 2^{k-1}}$, where α is a primitive element of $GF(q)$. It follows that $8 = 2^t = (\alpha^{2^k s + 2^{k-1}})^t = \alpha^{2^{k-1} t} = -1$. It follows that q is divisible by 3, a contradiction. Also, $\frac{1}{2} \notin C_{2^{k-1}}^{(2^k, q)}$. So, there is an element $w \in C_{2^{k-1}}^{(2^k, q)} \setminus \{-1, 2, \frac{1}{2}\}$. \square

Lemma 5 *Let $q = 2^k t + 1 \geq 9$ be an odd prime power where $t \geq 3$ is odd. If the one-factorization of K_{q+1} generated by S_w , $w \in C_{2^{k-1}}^{(2^k, q)}$, is not C_4 -free, then $w^2 - w + 1 = 0$, or there is an element $x \in A = \cup_{i=0}^{2^{k-1}-1} C_i^{(2^k, q)}$ such that both $\frac{2w}{1-w}x$ and $\frac{1+w}{1-w}x$ are contained in A , or both $\frac{w^2+1}{2w}x$ and $\frac{w-1}{2}x$ are contained in A .*

Proof Denote $G_i = \{\{\infty, i\}\} \cup \{\{x + i, wx + i\} : x \in A\}$ for $i \in GF(q)$. Then $\mathcal{G} = \{G_i : i \in GF(q)\}$ is a one-factorization of the complete graph on $GF(q) \cup \{\infty\}$. Assume that this one-factorization is not C_4 -free. Since $G_i = G_0 + i$, $G_i \cup G_j$ contains a C_4 if and only if $G_{i-j} \cup G_0$ contains a C_4 . Suppose $G_i \cup G_0$ ($i \in GF(q)^*$) contains a cycle $\{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}\}$ of length 4 where $\{a, b\}, \{c, d\} \in G_0$ and $\{a, c\}, \{b, d\} \in G_i$.

If $\infty \in \{a, b, c, d\}$, without loss of generality, let $a = \infty$. Then $b = 0$ and $c = i$. By the definition of G_0 , if $c \in A$, then $d = wc$ and $-c \notin A$. It follows that $d - c \in A$ and $-c = w(d - c)$. Thus, $w^2 - w + 1 = 0$, a contradiction. If $c \notin A$, then $c = wd$ and $-c \in A$. It follows that $d - c \notin A$ and $d - c = -wc$. Thus, $w^2 - w + 1 = 0$, a contradiction.

Suppose that $\infty \notin \{a, b, c, d\}$ and let $\{a, b\} = \{y, wy\}$ and $\{c, d\} = \{z, wz\}$ where $y, z \in A$ and $y \neq z$. We consider the following two cases.

Case 1 $\{y, z\}, \{wy, wz\} \in G_i$, i.e., $\{y - i, z - i\}, \{wy - i, wz - i\} \in G_0$.

If $z - i, wz - i \in A$, then $y - i = w(z - i)$ and $wy - i = w(wz - i)$. It follows that $(w - 1)y = w(w - 1)z$. Since $w \neq 1$, we have $y = wz$. This is impossible because $y, z \in A$ and $w \in C_{2^{k-1}}^{(2^k, q)}$. Similar arguments show that it is impossible for $y - i, wy - i \in A$.

If $z - i, wy - i \in A$, then $y - i = w(z - i)$ and $wz - i = w(wy - i)$. After subtracting the second equation from the first, we obtain $z = \frac{1+w}{2w}y$. It follows that $i = \frac{w+1}{2}y$. Since $z - i, wy - i \in A$, we deduce $z - i = \frac{1-w}{2w}y \in A$ and $wy - i = \frac{w-1}{2}y \in A$.

Clearly, $\frac{-1}{w} \in C_0^{(2^k, q)}$. Therefore, $\frac{1-w}{2w}y \in A$ if and only if $\frac{w-1}{2}y \in A$. So, if there is a cycle of length 4, there is an element $x \in A$ such that $\frac{1+w^2}{2w}x \in A$ and $\frac{w-1}{2}x \in A$. When $y - i, wz - i \in A$, similar arguments also show that there is such an element $x \in A$.

Case 2 $\{y, wz\}, \{z, wy\} \in G_i$, i.e., $\{y - i, wz - i\}, \{z - i, wy - i\} \in G_0$.

If $z - i, wz - i \in A$, then $y - i = w(wz - i)$ and $wy - i = w(z - i)$. It follows that $y = -wz$. By substitution, we get $i = \frac{w^2+w}{w-1}z$. Since $z - i, wz - i \in A$, both $\frac{2w}{1-w}z$ and $\frac{1+w^2}{1-w}z$ belong to A . When $y - i, wy - i \in A$, same arguments also show that there is an element $x \in A$ such that both $\frac{2w}{1-w}x$ and $\frac{1+w^2}{1-w}x$ belong to A .

If $z - i, y - i \in A$, then $wy - i = w(z - i)$ and $wz - i = w(y - i)$. It follows that $w(y - z) = w(z - y)$. Then $w = 0$, a contradiction.

If $wy - i, wz - i \in A$, then $y - i = w(wz - i)$ and $z - i = w(wy - i)$. It follows that $y - z = w^2(z - y)$. Then $w^2 = -1$, contradicting $w^2 \in C_0^{(2^k, q)}$. □

Lemma 6 *Let q be a prime power with $q \not\equiv 0 \pmod{3}$. If $q \equiv 5 \pmod{8}$ or $q \equiv 9 \pmod{16}$, then there is a pair of orthogonal totally C_4 -free one-factorizations of K_{q+1} for $q \geq 13$.*

Proof Let the characteristic of the finite field $GF(q)$ be p and $q = p^n$. We shall show that there is an element $w \in C_{2^{k-1}}^{(2^k, q)} \setminus \{-1, 2, \frac{1}{2}\}$ such that $\frac{w-1}{2} \in C_{2^{k-1}}^{(2^k, q)}$ and $w^2 - w + 1 \neq 0$. Then by Lemmas 4 and 5, the pair of orthogonal one-factorizations of K_{q+1} generated by the starters S_w and S_{-1} is totally C_4 -free.

If $q \equiv 5 \pmod{8}$, then $p \equiv 5 \pmod{8}$ and n is odd. Similar to the proof Theorem 7, 2 is a quadratic non-residue in $GF(q)$. Write $q = s^2 + 4t^2$. By [15, Lemma 19 in Part 1], we have $(3, 2)_4 = \frac{q+1+2s-8t}{16}$ and $(1, 2)_4 = \frac{q+1+2s+8t}{16}$. Clearly, $(s \pm 4t)^2 = s^2 + 16t^2 \pm 8st \leq 5(s^2 + 4t^2) = 5q$. When $q \geq 113$, $(3, 2)_4 = \frac{q+1+2s-8t}{16} \geq \frac{q+1-2\sqrt{5q}}{16} \geq 4$. Also, $(1, 2)_4 \geq 4$. If $2 \in C_1^{(4, q)}$, there are at least four elements $w \in C_2^{(4, q)}$ such that $w - 1 \in C_3^{(4, q)}$, which yields $\frac{w-1}{2} \in C_2^{(4, q)}$. So, there is at least one element $w \in C_2^{(4, q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_2^{(4, q)}$. If $2 \in C_3^{(4, q)}$, there are at least four elements $w \in C_2^{(4, q)}$ such that $w - 1 \in C_1^{(4, q)}$, which yields $\frac{w-1}{2} \in C_2^{(4, q)}$. So, there is at least one element $w \in C_2^{(4, q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_2^{(4, q)}$.

For $q \equiv 9 \pmod{16}$, write $q = s^2 + 4t^2$ where $s \equiv 1 \pmod{4}$. By [15, Lemma 30 in Part 1], if $2 \in C_0^{(4, q)}$, then $(0, 4)_8 = \frac{q+1-18s}{64}$ and $(4, 4)_8 = \frac{q-15-2s}{64}$; if $2 \notin C_0^{(4, q)}$, then $(2, 4)_8 = \frac{q+1-2s-16t}{64}$ and $(6, 4)_8 = \frac{q+1-2s+16t}{64}$. Clearly, $(-s \pm 8t)^2 = s^2 + 64t^2 \pm 16st \leq 17(s^2 + 4t^2) = 17q$. When $q \geq 666$ and $2 \in C_2^{(4, q)}$, we have $(2, 4)_8 = \frac{q+1-2s-16t}{64} \geq \frac{q+1-2\sqrt{17q}}{64} \geq 4$. Similarly, $(6, 4)_8 \geq 4$ when $q \geq 666$ and $2 \in C_2^{(4, q)}$. Also, when $q \geq 936$ and $2 \in C_0^{(4, q)}$, we have $(0, 4)_8 = \frac{q+1-18s}{64} \geq \frac{q+1-18\sqrt{q}}{64} \geq 6$. Similarly, $(2, 4)_8 \geq 6, (4, 4)_8 \geq 6$ and $(6, 4)_8 \geq 6$ when $q \geq 936$ and $2 \in C_0^{(4, q)}$.

For $q \equiv 9 \pmod{16}$, we have $p \equiv 3, 5 \pmod{8}$ and $n \equiv 2 \pmod{4}$, or $p \equiv 9 \pmod{16}$ and n is odd. Similar to the proof Theorem 7, 2 is a quadratic residue in $GF(q)$. Moreover, $2 \in C_2^{(4, q)}$ if $p \equiv 5 \pmod{8}$, and $2 \in C_4^{(8, q)}$ if $p \equiv 3 \pmod{8}$.

For $q = p^n \equiv 9 \pmod{16}$ with $p \equiv 5 \pmod{8}$ and $q \geq 666$, if $2 \in C_2^{(8,q)}$, there are at least four elements $w \in C_4^{(8,q)}$ such that $w - 1 \in C_6^{(8,q)}$. This is because $(6, 4)_8 \geq 4$. So, there is at least one element $w \in C_4^{(8,q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_4^{(8,q)}$. If $2 \in C_6^{(8,q)}$, since $(2, 4)_8 \geq 4$, there is at least one element $w \in C_4^{(8,q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_4^{(8,q)}$.

For $q = p^n \equiv 9 \pmod{16}$ with $p \equiv 3 \pmod{8}$ and $q \geq 936$. Since $(0, 4)_8 \geq 6$, $(2, 4)_8 \geq 6$, $(4, 4)_8 \geq 6$ and $(6, 4)_8 \geq 6$, there is at least one element $w \in C_4^{(8,q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_4^{(8,q)}$.

For $q = p^n \equiv 9 \pmod{16}$ with $p \equiv 9 \pmod{16}$ and $q \geq 936$, if $2 \in C_2^{(8,q)}$, there are at least six elements $w \in C_4^{(8,q)}$ such that $w - 1 \in C_6^{(8,q)}$. This is because $(6, 4)_8 \geq 6$. So, there is at least one element $w \in C_4^{(8,q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_4^{(8,q)}$. If $2 \in C_4^{(8,q)}$, $2 \in C_6^{(8,q)}$ or $2 \in C_0^{(8,q)}$, a similar argument shows that there is at least one element $w \in C_4^{(8,q)}$ such that $w \notin \{-1, 2, \frac{1}{2}\}$, $w^2 - w + 1 \neq 0$ and $\frac{w-1}{2} \in C_4^{(8,q)}$.

We next consider small prime powers q . For any prime $q \in \{29, 53, 61, 101, 109, 137, 233, 281, 313, 409, 457, 521, 569, 601, 617, 761, 809, 857\}$, we list the required element w which satisfies the conditions in Lemma 5. For $q \in \{37, 41, 73, 89\}$, a strong

q	w	q	w	q	w
29	9	53	6	61	36
233	170	281	126	313	108
569	444	601	599	617	171
q	w	q	w	q	w
101	64	109	4	137	18
409	144	457	254	521	404
761	18	809	373	857	102

starter S is constructed in \mathbb{Z}_q with multiplier group $\{\xi^i : 1 \leq i \leq N\}$. The shortened list of pairs is as follows:

- $q = 37, \xi = 7, N = 9 : \{1, 2\}, \{3, 17\}.$
- $q = 41, \xi = 10, N = 5 : \{1, 2\}, \{3, 15\}, \{4, 39\}, \{6, 11\}.$
- $q = 73, \xi = 2, N = 9 : \{1, 3\}, \{5, 15\}, \{9, 33\}, \{13, 35\}.$
- $q = 89, \xi = 2, N = 11 : \{1, 3\}, \{5, 18\}, \{11, 31\}, \{19, 43\}.$

It is readily checked that the two orthogonal one-factorizations of K_{q+1} generated by S and $-S$ are totally C_4 -free.

For $q \in \{25, 121, 169, 361\}$, let the finite field $GF(q)$ be generated by the primitive polynomial $f(x)$ and let θ be a primitive element of $GF(q)$. Denote the field elements $0, \theta^1, \theta^2, \dots, \theta^{q-1}$ by $0, 1, 2, \dots, q-1$. A strong starter S is constructed over $GF(q)$ with multiplier group $\{\xi^i : 1 \leq i \leq N\}$. The shortened list of pairs is as follows. It

q	f(x)	ξ	N	Shortened list of pairs
25	$x^2 + 3x + 3$	θ^8	3	{1,2}, {3,7}, {4,13}, {6,8}.
121	$x^2 + 10x + 8$	θ^8	15	{1,2}, {3,4}, {5,64}, {6,39}.
169	$x^2 + 9x + 11$	θ^8	21	{1,2}, {3,4}, {5,63}, {6,136}.
361	$x^2 + 15x + 15$	θ^{40}	9	{1,2}, {3,4}, {5,6}, {7,8}, {9,10}, {11,12}, {13,14},{15,16}, {17,18}, {19,20},{21,35}, {22,193}, {23,105}, {24,271}, {26,38},{27,109}, {28,199},{30,276},{32,280}, {34,37}.

is easy to check that the two orthogonal one-factorizations of K_{q+1} by S and $-S$ are totally C_4 -free.

For $q = 13$, we construct two orthogonal starters in \mathbb{Z}_q as follows:

$$S = \{\{1, 2\}, \{3, 6\}, \{4, 11\}, \{5, 9\}, \{7, 12\}, \{8, 10\}\},$$

$$T = \{\{1, 6\}, \{2, 9\}, \{3, 5\}, \{4, 7\}, \{8, 12\}, \{10, 11\}\}.$$

It is easy to check that the two orthogonal one-factorizations of K_{q+1} generated by S and T are totally C_4 -free. □

3 Product Construction

A Latin square of order n is an $n \times n$ matrix with entries from an n -set V , where every row and every column is a permutation of V . A one-factorization of complete bipartite graph $K_{n,n}$ is equivalent to a Latin square of order n . A C_4 -free one-factorization of $K_{n,n}$ is equivalent a Latin square without a subsquare of order 2. In 1991, Heinrich [3] proved that Latin squares without a subsquare of order 2 exist for all orders $n \notin \{2, 4\}$.

In order to state our product construction for a pair of orthogonal one-factorizations of K_n , we need a pair of orthogonal one-factorizations of the complete bipartite graph such that the union of any two factors does not include a cycle of length 4, which we succinctly call totally C_4 -free.

Let $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ and $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ be a pair of orthogonal one-factorizations of $K_{n,n}$ with bipartition X and Y where $|X| = |Y| = n$ and $X \cap Y = \emptyset$. By definition, $|F_i \cap G_j| = 1$ for $i, j \in \{1, 2, \dots, n\}$. Define an $n \times n$ array such that its (i, j) -th entry is $F_i \cap G_j$. By definition, the union of the i -th row is F_i , the union of the j -th column is G_j . Define two $n \times n$ arrays L_1 and L_2 with rows indexed by X and columns indexed by Y . The (x, y) -th entry of L_1 is i if $\{x, y\}$ in F_i and (x, y) -th entry of L_2 is j if $\{x, y\}$ in G_j . It is well known that such two arrays are Latin squares and orthogonal. Two Latin squares $A = (a_{i,j})$ and $B = (b_{i,j})$ of order n from n -sets V and S , respectively, are said to be *orthogonal* if $\{(a_{i,j}, b_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq n\} = V \times S$. Conversely, a pair of orthogonal one-factorizations of $K_{n,n}$ can be obtained from two orthogonal Latin squares.

Suppose that the two orthogonal one-factorizations of $K_{n,n}$ are totally C_4 -free. Since there is no C_4 in any union of $F_i \cup F_j$, there is no subsquare of order 2 in L_1 (Otherwise, there exist $x_1 \neq x_2$ and $y_1 \neq y_2$ such that $L_1(x_1, y_1) = L_1(x_2, y_2)$)

and $L_1(x_1, y_2) = L_1(x_2, y_1)$, then pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are in $F_{L_1(x_1, y_1)}$, and pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are in $F_{L_1(x_1, y_2)}$. These pairs form a C_4 in $F_{L_1(x_1, y_2)} \cup F_{L_1(x_1, y_2)}$, which contradicts that no $F_i \cup F_j$ include a C_4). Likewise, L_2 contains no subsquare of order 2. Further, there does not exist $x_1 \neq x_2$ and $y_1 \neq y_2$ such that $L_1(x_1, y_1) = L_1(x_2, y_2)$ and $L_2(x_1, y_2) = L_2(x_2, y_1)$ (Otherwise, pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are in $F_{L_1(x_1, y_1)}$, and pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are in $G_{L_2(x_1, y_2)}$. These pairs form a C_4 in $F_{L_1(x_1, y_2)} \cup G_{L_2(x_1, y_2)}$, which contradicts that no $F_i \cup G_j$ include a C_4). Conversely, we obtain two orthogonal totally C_4 -free one-factorizations of $K_{n,n}$ from two orthogonal Latin squares $L_1 = (L_1(x, y))$ and $L_2 = (L_2(x, y))$ of order n each without a subsquare of order 2 and there do not exist $x_1 \neq x_2, y_1 \neq y_2$ such that $L_1(x_1, y_1) = L_1(x_2, y_2)$ and $L_2(x_1, y_2) = L_2(x_2, y_1)$. This fact is stated below.

Lemma 7 *There are two orthogonal totally C_4 -free one-factorizations of the complete bipartite graph $K_{n,n}$ if and only if there are two orthogonal Latin squares $L_1 = (L_1(x, y))$ and $L_2 = (L_2(x, y))$ of order n each without a subsquare of order 2 and there do not exist $x_1 \neq x_2, y_1 \neq y_2$ such that $L_1(x_1, y_1) = L_1(x_2, y_2)$ and $L_2(x_1, y_2) = L_2(x_2, y_1)$.*

Lemma 8 *For any integer $n \equiv 1, 5 \pmod{6}$ where $n \geq 5$, there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$.*

Proof Define two one-factorizations of $K_{n,n}$ on $\mathbb{Z}_n \times \{1, 2\}$ with bipartition $\mathbb{Z}_n \times \{i\}, i \in \{1, 2\}$ as follows:

$$\mathcal{F} = \{F_a = \{(x, 1), (a - x, 2)\} : x \in \mathbb{Z}_n\} : a \in \mathbb{Z}_n\},$$

$$\mathcal{G} = \{G_a = \{(x, 1), (a - 2x, 2)\} : x \in \mathbb{Z}_n\} : a \in \mathbb{Z}_n\}.$$

It is easy to check that \mathcal{F} and \mathcal{G} are orthogonal.

We prove that \mathcal{F} is C_4 -free. Assume that $F_a \cup F_b$ ($a \neq b$) includes a C_4 . Suppose that two edges $\{(x, 1), (a - x, 2)\}$ and $\{(y, 1), (a - y, 2)\}$ ($x \neq y$) of F_a are contained in a C_4 . Then the other two edges $\{(x, 1), (a - y, 2)\}$ and $\{(y, 1), (a - x, 2)\}$ are contained in F_b . It follows that $x + a - y = y + a - x = b$. So, $2x = 2y$. Since n is odd, we have $x = y$, which is a contradiction. Similarly, we can show that \mathcal{G} is C_4 -free.

We show that for any $F_a \in \mathcal{F}$ and $G_b \in \mathcal{G}$, $F_a \cup G_b$ does not include a C_4 . If it is not true, then we can assume that two edges $\{(x, 1), (a - x, 2)\}$ and $\{(y, 1), (a - y, 2)\}$ ($x \neq y$) of F_a are contained in a C_4 . Then the other two edges $\{(x, 1), (a - y, 2)\}$ and $\{(y, 1), (a - x, 2)\}$ are contained in G_b . It follows that $2x + a - y = 2y + a - x = b$. So, $3x = 3y$. Since $n \equiv 1, 5 \pmod{6}$, we have $x = y$, which is a contradiction. This completes the proof. □

Lemma 9 *For any $q = 3^k$ with $k \geq 2$, there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{q,q}$.*

Proof Let α be a primitive element of the finite field $GF(q)$. Define two one-factorizations of $K_{q,q}$ on $GF(q) \times \{1, 2\}$ with bipartition $GF(q) \times \{i\}, i \in \{1, 2\}$ as

follows:

$$\mathcal{H} = \{H_a = \{(x, 1), (\alpha(a - x), 2)\} : x \in GF(q)\} : a \in GF(q)\},$$

$$\mathcal{G} = \{G_a = \{(x, 1), ((a - x)(1 + \alpha), 2)\} : x \in GF(q)\} : a \in GF(q)\}.$$

It is easy to check that \mathcal{F} and \mathcal{G} are orthogonal.

We prove that \mathcal{H} is C_4 -free. If it does not hold, assume that $H_a \cup H_b$ ($a \neq b$) includes a C_4 . Suppose that two edges $\{(x, 1), (\alpha(a - x), 2)\}$ and $\{(y, 1), (\alpha(a - y), 2)\}$ ($x \neq y$) of H_a are contained in a C_4 . Then the other two edges $\{(x, 1), (\alpha(a - y), 2)\}$ and $\{(y, 1), (\alpha(a - x), 2)\}$ are contained in H_b . It follows that $\alpha(a - y) = \alpha(b - x)$ and $\alpha(a - x) = \alpha(b - y)$. So, $\alpha(x - y) = \alpha(y - x)$. Then $x = y$, a contradiction. Similarly, we can show that \mathcal{G} is C_4 -free.

We show that for any $H_a \in \mathcal{H}$ and $G_b \in \mathcal{G}$, $H_a \cup G_b$ does not include a C_4 . If it is not true, then we can assume that two edges $\{(x, 1), (\alpha(a - x), 2)\}$ and $\{(y, 1), (\alpha(a - y), 2)\}$ ($x \neq y$) of H_a are contained in a C_4 . Then the other two edges $\{(x, 1), (\alpha(a - y), 2)\}$ and $\{(y, 1), (\alpha(a - x), 2)\}$ are contained in G_b . It follows that $\alpha(a - y) = (1 + \alpha)(b - x)$ and $\alpha(a - x) = (1 + \alpha)(b - y)$. So, $\alpha(x - y) = (1 + \alpha)(y - x)$. Then $\alpha = -\frac{1}{2} = 1$, which is a contradiction since α is a primitive element. This completes the proof. □

Lemma 10 *If there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{m,m}$ and a pair of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$, then there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{mn,mn}$*

Proof Let $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ and $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$ be two orthogonal totally C_4 -free one-factorizations of $K_{m,m}$ on $V \times \{1, 2\}$ with bipartition $V \times \{1\}$ and $V \times \{2\}$. Let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_m\}$ be two orthogonal totally C_4 -free one-factorizations of $K_{n,n}$ on $S \times \{1, 2\}$ with bipartition $S \times \{1\}$ and $S \times \{2\}$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, define

$$A(F_i, H_j) = \{(x_1, y_1, 1), (x_2, y_2, 2)\} : \{(x_1, 1), (x_2, 2)\} \in F_i, \{(y_1, 1), (y_2, 2)\} \in H_j\},$$

$$B(G_i, Q_j) = \{(x_1, y_1, 1), (x_2, y_2, 2)\} : \{(x_1, 1), (x_2, 2)\} \in G_i, \{(y_1, 1), (y_2, 2)\} \in Q_j\}.$$

It is routine to check that $\{A(F_i, H_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{B(G_i, Q_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ form a pair of orthogonal totally C_4 -free one-factorizations of $K_{mn,mn}$. □

Applying Lemma 10 with the known pairs of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$ in Lemmas 8 and 9, we obtain the following theorem.

Theorem 8 *For any odd positive integer n with $\gcd(n, 9) \neq 3$, there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$.*

With the aid of a pair of orthogonal totally C_4 -free one-factorizations of $K_{m,m}$, we are now in a position to state a recursive construction of pairs of orthogonal totally C_4 -free one-factorizations of K_n . Such a construction is obtained by modifying the construction for Room squares.

Theorem 9 *Suppose that there is a pair of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$, two orthogonal totally C_4 -free one-factorizations of K_{n+1} and K_{m+1} , then there are two orthogonal totally C_4 -free one-factorizations of K_{nm+1} .*

Proof We prove it in term of Room squares. By the equivalence between a Room square and a pair of orthogonal one-factorizations of complete graph, let M and N be the standardized Room squares of orders m and n , respectively, which correspond to pairs of orthogonal totally C_4 -free one-factorizations of complete graph of orders m and n . The symbol sets are $\{\infty, 1, 2, \dots, m\}$ and $\{\infty, 1, 2, \dots, n\}$ respectively. Let A and B be two orthogonal Latin squares of order n on $\{1, 2, \dots, n\}$, which corresponds to a pair of orthogonal totally C_4 -free one-factorizations of $K_{n,n}$, i.e., A and B have no subsquare of order 2 and there do not exist four elements $x, y, z, w \in \{1, 2, \dots, n\}, x \neq z, y \neq w$ such that $A(x, y) = A(z, w)$ and $B(x, w) = B(z, y)$. We shall construct a Room square of order mn on $S = \{\infty, 1_1, 1_2, \dots, 1_m, 2_1, \dots, 2_m, \dots, n_1, n_2, \dots, n_m\}$.

For $1 \leq i \leq m$, we write A_i to mean A with every entry given a subscript i : If A has $(1,1)$ entry x , then A_i has $(1,1)$ entry x_i . B_i is defined similarly. For $1 \leq i, j \leq m$, Q_{ij} is an $n \times n$ array of unordered pairs whose (k, l) entry consists of the (k, l) entries of A_i and B_j . We also subscript the arrays N , the pair $\{x, y\}$ from N would be replaced by $\{x_i, y_i\}$, where $x_i = x$ when $x = \infty$, the empty cells of N would correspond to empty cells of N_i .

We first convert M into an $mn \times mn$ array U by replacing each of its cells by an $n \times n$ array. An empty cell is replaced by an empty $n \times n$ array; the entry $\{\infty, i\}$ is replaced by N_i ; and the entry $\{i, j\}, i \neq j$, is replaced by Q_{ij} .

By the Multiplication Theorem [13], U is a Room square of order mn . It is left to check that the two orthogonal one-factorizations of K_{mn+1} on S from rows and columns are totally C_4 -free.

Assume that there is a cycle of length 4 in the union of any two distinct one-factors from rows and columns of U . Let $\{a_s, b_t\}, \{c_i, d_j\}, \{a_s, c_i\}, \{b_t, d_j\}$ be such a cycle where $\{a_s, b_t\}, \{c_i, d_j\}$ are from one-factor and $\{a_s, c_i\}, \{b_t, d_j\}$ are from another one-factor. From the above construction of U , it is easy to see that $\{s, t\}, \{i, j\}$ must be from a one-factor of M and $\{s, i\}, \{t, j\}$ must be also from a one-factor of M (they may be from the same one-factor). If $s = t$, then $i = s = j$. These four unordered pairs are from the same Room square N_s , i.e., this cycle is contained in the union of two one-factors from rows and columns of N_s . This gives a contradiction because N_s is totally C_4 -free. A similar contradiction occurs for $i = j$. Let $s \neq t$ and $i \neq j$. If $\{s, t\} = \{i, j\}$, then these four unordered pairs are from the same array Q_{ij} . This is impossible because there is no C_4 in any union of two rows, or two columns, or one row and one column of Q_{ij} . If $\{s, t\} \neq \{i, j\}$, by projection we obtain that $\{s, t\}, \{i, j\}$ are in the same row or column of Room square M and $\{s, i\}$ and $\{t, j\}$ are in the same row or column of Room square M . We also arrive a contradiction since M is totally C_4 -free. This completes the proof. \square

Theorem 10 *Let n be an positive integer of the form $n = 3^{a_1}5^{a_2}7^{a_3}p_1^{\alpha_1} \dots p_t^{\alpha_t}$ where each $p_i \geq 11$ is a prime, $p_i \not\equiv 1 \pmod{16}$, $a_1, a_2, a_3, \alpha_1, \dots, \alpha_t$ are non-negative integers and $a_1 \notin \{1, 2\}, a_2, a_3 \neq 1$. Then there is a pair of orthogonal totally C_4 -free one-factorizations of K_{n+1} .*

Proof By Theorem 7, there exists a pair of orthogonal totally C_4 -free one-factorizations of $K_{3^{2k+1}+1}$ with $k > 0$. We construct a pair of orthogonal totally C_4 -free one-factorizations of K_{3^4+1} as follows. For $n = 81$, let the finite field $GF(81)$ be generated by the primitive polynomial $f(x) = x^4 + 5x + 5$ and let θ be a primitive element

of $GF(81)$. Denote the field elements $0, \theta^1, \theta^2, \dots, \theta^{80}$ by $0, 1, 2, \dots, 80$. A strong starter S is constructed in $GF(81)$ with multiplier group $\{\theta^{16}, \theta^{32}, \theta^{48}, \theta^{64}, \theta^{80}\}$. The shortened list of pairs is below.

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 58\}, \{11, 60\}, \{13, 62\}, \{15, 64\}.$$

It is easy to check that the two orthogonal one-factorizations of K_{82} over $GF(81) \cup \{\infty\}$ generated by S and $-S$ are totally C_4 -free. So, by applying Theorems 7 and 9 with the known two orthogonal totally C_4 -free one-factorizations of $K_{3i, 3i}$ in Theorem 8, we obtain a pair of orthogonal totally C_4 -free one-factorizations of K_{3a_1+1} for each $a_1 \geq 3$.

By Lemma 6, there exists a pair of orthogonal totally C_4 -free one-factorizations of K_{n+1} for $n \in \{25, 125\}$. Applying Theorem 9 with the known two orthogonal totally C_4 -free one-factorizations of $K_{5i, 5i}$ in Theorem 8, we obtain a pair of orthogonal totally C_4 -free one-factorizations of K_{5a_2+1} for each $a_2 \neq 1$.

By Theorem 7, there exists a pair of orthogonal totally C_4 -free one-factorizations of $K_{7^{2k+1}+1}$ with $k > 0$. For $n = 49$, let the finite field $GF(49)$ be generated by the primitive polynomial $f(x) = x^2 + 5x + 5$ and let θ be a primitive element of $GF(49)$. Denote the field elements $0, \theta^1, \theta^2, \dots, \theta^{48}$ by $0, 1, 2, \dots, 48$. A strong starter S is constructed in $GF(49)$ with multiplier group $\{\theta^{16}, \theta^{32}, \theta^{48}\}$. The shortened list of pairs is below.

$$\{1, 2\}, \{3, 4\}, \{5, 8\}, \{6, 7\}, \{9, 26\}, \{11, 44\}, \{13, 32\}, \{14, 47\}.$$

It is easy to check that the two orthogonal one-factorizations of K_{50} over $GF(49) \cup \{\infty\}$ generated by S and $-S$ are totally C_4 -free. Applying Theorem 9 with the known two orthogonal totally C_4 -free one-factorizations of $K_{7i, 7i}$ in Theorem 8, we obtain a pair of orthogonal totally C_4 -free one-factorizations of K_{7a_3+1} for each $a_3 \neq 1$.

By Theorem 7 and Lemma 6, there exists a pair of orthogonal totally C_4 -free one-factorizations of K_{n+1} for $n \in \{p_1, p_2, \dots, p_t\}$. Further, applying Theorem 9 with the known two orthogonal totally C_4 -free one-factorizations of $K_{m,m}$ in Theorem 8 yields the result. □

4 Two Orthogonal C_4 -free One-factorizations

In this section, we only consider two orthogonal C_4 -free one-factorizations of K_n , i.e., any union of two one-factors from two distinct one-factorizations does not include a C_4 .

We modify Horton’s Room squares [8] so that the corresponding two orthogonal one-factorizations are C_4 -free.

Theorem 11 *Let G be an abelian group of odd order n where $n \not\equiv 0 \pmod{3}$ and $n \geq 7$. If there is a strong starter $X = \{\{x_i, y_i\} : 1 \leq i \leq \frac{n-1}{2}\}$ in G such that $\frac{x_i}{y_i} \notin \{\frac{1}{2}, 2\}$ for $1 \leq i \leq \frac{n-1}{2}$, $x_i + x_j \neq y_i + y_j$ and $x_i + y_j \neq x_j + y_i$ for any*

$1 \leq i < j \leq \frac{n-1}{2}$, then there is a pair of orthogonal C_4 -free one-factorizations of K_{5n+1} .

Proof We first construct two orthogonal starters on $\mathbb{Z}_5 \times G$. For $(i, j) \in \mathbb{Z}_5 \times G$, let

$$F(i, j) = \{\{\infty, (i, j)\}\} \cup \{(s + i, t + j), (i - s, j - t) : (s, t) \in \mathbb{Z}_5 \times G \setminus \{(0, 0)\}\}.$$

Then all $F(i, j)$ form a one-factorization of K_{5n+1} on $\{\infty\} \cup (\mathbb{Z}_5 \times G)$ and $F(0, 0) \setminus \{\{\infty, (0, 0)\}\}$ is the patterned starter on $\mathbb{Z}_5 \times G$.

For the starter X , we shall find two distinct non-zero elements a and b in G such that $a + b \neq 0$ and $5a \neq 3b$ and there is no pair $\{x, y\}$ in X whose sum is a or b . This can be done as follows. For $a, b \neq 0$, if $5a = 3b$ and $5b = 3a$, then $25a = 9a$. Since n is odd, it follows from $16a = 0$ that $a = 0$, a contradiction. Hence, it holds that $5a \neq 3b$, or $5b \neq 3a$ for any $a \neq 0$ and $b \neq 0$. Without loss of generality, let $5a \neq 3b$. There are only $(n - 1)/2$ elements in the set of sums of X , while there are $n - 1$ non-zero elements in G . Since $n \geq 7$, we have $n - 1 - \frac{n-1}{2} \geq 3$. Thus, we conclude that there exist two distinct non-zero elements a, b in G satisfying our conditions. Let $h = \frac{b-a}{4}$ and $g = \frac{a}{2}$. Now partition the nonzero elements of G into two sets P and N such that x is in P if and only if $-x$ is in N . We add the restrictions: If $-6x = a$, then let x belong to P ; since $5a \neq 3b$, we have $-6(-h) \neq a$ and we can let h belong to P ; since $a + b \neq 0$, we have $-6 \times \frac{h}{3} \neq a$ and we can let $-\frac{h}{3} \in P$. Now we consider the sets of pairs:

$$\begin{aligned} A &= \{(0, x), (0, y) : \{x, y\} \in X\}, \\ B &= \{(1, x + g), (2, 2x + g) : x \in P, x \neq h\}, \\ C &= \{(4, x + g), (3, 2x + g) : x \in P, x \neq h\}, \\ D &= \{(1, x + g), (3, 2x + g) : x \in N\}, \\ E &= \{(4, x + g), (2, 2x + g) : x \in N\}, \\ F &= \{(1, h + g), (2, g), \{(4, h + g), (3, g)\}, \\ &\quad \{(1, g), (4, g)\}, \{(2, 2h + g), (3, 2h + g)\}\}. \end{aligned}$$

From the proof of the main theorem in [8], the union of these 6 sets forms a strong starter M in $\mathbb{Z}_5 \times G$. By Theorem 4, this strong starter M and the patterned starter $F(0, 0) \setminus \{\{\infty, (0, 0)\}\}$ are orthogonal. Let $M(0, 0) = M \cup \{\{\infty, (0, 0)\}\}$ and $M(i, j) = M(0, 0) + (i, j)$ for $(i, j) \in \mathbb{Z}_5 \times G$. By Theorem 3, the two one-factorizations $\{M(i, j) : (i, j) \in \mathbb{Z}_5 \times G\}$ and $\{F(i, j) : (i, j) \in \mathbb{Z}_5 \times G\}$ are orthogonal. It is left to show that any union of $M(i, j)$ and $F(i_1, j_1)$ does not include a cycle of length four. Since $M(i, j) = M(0, 0) + (i, j)$ and $F(i, j) = F(0, 0) + (i, j)$, we need only to show that any union of $M(0, 0)$ and $F(i, j)$ does not include a cycle of length four.

Assume that the union of $M(0, 0)$ and $F(i, j)$ contains a C_4 . Let $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\} \in M(0, 0)$ and $\{(a_1, b_1), (a_4, b_4)\}, \{(a_3, b_3), (a_2, b_2)\} \in F(i, j)$, where $(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)$ are distinct. According to the occurrence of ∞ in this C_4 , we divided the proof into two cases.

Case 1 $\infty \in \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$.

Without loss of generality, let $(a_1, b_1) = \infty$. Then $(a_2, b_2) = (0, 0)$ and $(a_4, b_4) = (i, j)$. By the definition of $F(i, j)$, $(0, 0) + (a_3, b_3) = (2i, 2j)$ since $\{(a_3, b_3), (a_2, b_2)\} \in F(i, j)$. It follows that $\{(i, j), (2i, 2j)\} \in M$. If $i = 0$, then

$\{j, 2j\} \in X$ from the definition of M . This contradicts the assumption on the strong starter X that $\frac{y}{z} \notin \{\frac{1}{2}, 2\}$ for any $\{y, z\} \in X$. If $i = 1$, then $\{(1, j), (2, 2j)\} \in B$ or $\{(1, j), (2, 2j)\} = \{(1, h + g), (2, g)\} \in F$. When the former case occurs, by the definition of B there is some $x \in P$ such that $x + g = j$ and $2j = 2x + g$. Then $g = 0$, this is a contradiction because $g = \frac{a}{2} \neq 0$. When $\{(1, j), (2, 2j)\} = \{(1, h + g), (2, g)\}$, we have $2h + g = 0$. This is also a contradiction because $2h + g = 2 \times \frac{b-a}{4} + \frac{a}{2} = \frac{b}{2} \neq 0$. Similar arguments also show that it is impossible for $i = 4$. If $i = 2$, then $\{(2, j), (4, 2j)\} \in E$. By the definition of E , there is some $x \in N$ such that $2x + g = j$ and $2j = x + g$. Then $3x + g = 0$, i.e., $-6x = a$. By the restriction of P and N , there is no element $x \in N$ such that $-6x = a$. We then get a contradiction. Similar arguments also show that it is impossible for $i = 3$.

Case $2 \infty \notin \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$.

By the definition of $F(i, j)$, it holds that $(2i, 2j) = (a_1, b_1) + (a_4, b_4) = (a_2, b_2) + (a_3, b_3)$.

If $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\} \in A$, then $\{b_1, b_2\}, \{b_3, b_4\} \in X$ and $b_1 + b_4 = b_2 + b_3$. It is a contradiction because $b_1 + b_4 \neq b_2 + b_3$ by assumption. Clearly, this is impossible when $\{(a_1, b_1), (a_2, b_2)\} \in A$ and $\{(a_3, b_3), (a_4, b_4)\} \notin A$.

If $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\} \in B$, then there are two distinct elements x, y from $P \setminus \{h\}$ such that $x + g + 2y + g = 2x + g + y + g$. It follows that $x = y$, a contradiction. If $\{(a_1, b_1), (a_2, b_2)\} \in B$ and $\{(a_3, b_3), (a_4, b_4)\} \notin B$, then $\{(a_3, b_3), (a_4, b_4)\}$ must be from C or F . When $\{(a_3, b_3), (a_4, b_4)\} \in C$, there are two elements $x, y \in P \setminus \{h\}$ such that $\{(a_1, b_1), (a_2, b_2)\} = \{(1, x + g), (2, 2x + g)\}$ and $\{(a_3, b_3), (a_4, b_4)\} = \{(4, y + g), (3, 2y + g)\}$. It follows that $x + g + y + g = 2x + g + 2y + g$. Then $y = -x$, which contradicts the restriction on P and N that $x \in P$ if and only if $-x \in N$. Consider $\{(a_3, b_3), (a_4, b_4)\} \in F$. When $\{(a_3, b_3), (a_4, b_4)\} = \{(1, h + g), (2, g)\}$, there is an element $x \in P \setminus \{h\}$ such that $x + g + g = 2x + g + h + g$. Then $x = -h$. This is a contradiction since $-h \in N$. When $\{(a_3, b_3), (a_4, b_4)\} = \{(4, h + g), (3, g)\}$, there is an element $x \in P \setminus \{h\}$ such that $x + g + h + g = 2x + g + g$. Then $x = h$, a contradiction. When $\{(a_3, b_3), (a_4, b_4)\} = \{(2, 2h + g), (3, 2h + g)\}$, then there is an element $x \in P \setminus \{h\}$ such that $x + g + 2h + g = 2x + g + 2h + g$. Then $x = 0$, a contradiction.

If $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\} \in C$, then there are two distinct elements $x, y \in P \setminus \{h\}$ such that $x + g + 2y + g = 2x + g + y + g$. It follows that $x = y$, a contradiction. For $\{(a_1, b_1), (a_2, b_2)\} \in C$ and $\{(a_3, b_3), (a_4, b_4)\} \notin C$, we need only consider $\{(a_3, b_3), (a_4, b_4)\}$ from F . When $\{(a_3, b_3), (a_4, b_4)\} = \{(1, h + g), (2, g)\}$, there is an element $x \in P \setminus \{h\}$ such that $x + g + h + g = 2x + g + g$. Then $x = h$, a contradiction. When $\{(a_3, b_3), (a_4, b_4)\} = \{(4, h + g), (3, g)\}$, there is an element $x \in P \setminus \{h\}$ such that $x + g + g = 2x + g + h + g$. Then $x = -h$. This is a contradiction since $-h \in N$. When $\{(a_3, b_3), (a_4, b_4)\} = \{(2, 2h + g), (3, 2h + g)\}$, there is an element $x \in P \setminus \{h\}$ such that $x + g + 2h + g = 2x + g + 2h + g$. Then $x = 0$, a contradiction.

If $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\} \in D$, then there are two distinct elements x, y from N such that $x + g + 2y + g = 2x + g + y + g$. It follows that $x = y$, a contradiction. If $\{(a_1, b_1), (a_2, b_2)\} \in D$ and $\{(a_3, b_3), (a_4, b_4)\} \notin D$, then $\{(a_3, b_3), (a_4, b_4)\}$ must be from E or $\{(a_3, b_3), (a_4, b_4)\} = \{(1, g), (4, g)\} \in F$. If $\{(a_3, b_3), (a_4, b_4)\} \in$

E , then there are two elements $x, y \in N$ such that $\{(a_1, b_1), (a_2, b_2)\} = \{(1, y + g), (3, 2y + g)\}$ and $\{(a_3, b_3), (a_4, b_4)\} = \{(4, x + g), (2, 2x + g)\}$. It follows that $x + g + y + g = 2x + g + 2y + g$. Then $y = -x$, which contradicts the restriction on P and N that $x \in P$ if and only if $-x \in N$. If $\{(a_3, b_3), (a_4, b_4)\} = \{(1, g), (4, g)\}$, then there is an element $x \in N$ such that $x + g + g = 2x + g + g$. Then $x = 0$, a contradiction. Similar arguments show that this is impossible for $|E| \geq 1$. If two edges are from F , then $h = 0$, a contradiction. So, there does not exist a C_4 in the union of $M(i, j)$ and $F(i_1, j_1)$. This completes the proof. \square

Example 3 For $n = 11$, let

$$X = \{\{1, 7\}, \{4, 6\}, \{5, 2\}, \{9, 8\}, \{3, 10\}\}.$$

It is easy to check that $\frac{x_i}{y_i} \notin \{\frac{1}{2}, 2\}$ for $\{x_i, y_i\} \in X$, $x_i + x_j \neq y_i + y_j$ and $x_i + y_j \neq x_j + y_i$ for any $\{x_i, y_i\} \in X$ and $i \neq j$. Let $a = 3$ and $b = 1$. Then we have $a + b \neq 0$, $5a \neq 3b$ and $5b \neq 3a$. Set $h = \frac{b-a}{4} = 5$, $g = \frac{a}{2} = 7$, $P = \{1, 2, 3, 4, 5\}$ and $N = \{6, 7, 8, 9, 10\}$. Now we consider the sets of pairs:

$$\begin{aligned} A &= \{(0, 1), (0, 7)\}, \{(0, 4), (0, 6)\}, \{(0, 5), (0, 2)\}, \{(0, 9), (0, 8)\}, \{(0, 3), (0, 10)\}, \\ B &= \{(1, 8), (2, 9)\}, \{(1, 9), (2, 0)\}, \{(1, 10), (2, 2)\}, \{(1, 0), (2, 4)\}, \\ C &= \{(4, 8), (3, 9)\}, \{(4, 9), (3, 0)\}, \{(4, 10), (3, 2)\}, \{(4, 0), (3, 4)\}, \\ D &= \{(1, 2), (3, 8)\}, \{(1, 3), (3, 10)\}, \{(1, 4), (3, 1)\}, \{(1, 5), (3, 3)\}, \{(1, 6), (3, 5)\}, \\ E &= \{(4, 2), (2, 8)\}, \{(4, 3), (2, 10)\}, \{(4, 4), (2, 1)\}, \{(4, 5), (2, 3)\}, \{(4, 6), (2, 5)\}, \\ F &= \{(1, 1), (2, 7)\}, \{(4, 1), (3, 7)\}, \{(1, 7), (4, 7)\}, \{(2, 6), (3, 6)\}, \\ M(0, 0) &= \{\{\infty, (0, 0)\}\} \cup A \cup B \cup C \cup D \cup E \cup F, \\ M(i, j) &= M(0, 0) + (i, j) \text{ for } (i, j) \in \mathbb{Z}_5 \times G, \text{ and} \\ F(i, j) &= \{\{\infty, (i, j)\}\} \cup \{(s, t), (2i - s, 2j - t)\} : (s, t) \in \mathbb{Z}_5 \times G \setminus \{(i, j)\} \\ &\text{for } (i, j) \in \mathbb{Z}_5 \times G. \end{aligned}$$

It is readily checked that any union of $M(i, j)$ and $F(i_1, j_1)$ does not include a cycle of length four.

Similar to the proof of Theorem 9, we have the following theorem.

Theorem 12 *Suppose there is a pair of orthogonal C_4 -free one-factorizations of $K_{n,n}$, two orthogonal C_4 -free one-factorizations of K_{n+1} and K_{m+1} , then there is a pair of orthogonal C_4 -free one-factorizations of K_{nm+1} .*

Theorem 13 *Let $n = 3^a 5^b p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_i > 5$ are distinct primes, $\alpha_1, \dots, \alpha_k$ are non-negative integers and $a \notin \{1, 2\}$. If each p_i is a non-Fermat prime and $5^b p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \geq 7$, then there is a pair of orthogonal C_4 -free one-factorizations of K_{n+1} .*

Proof For $n \in \{7, 35\}$, it is easy to check that the two one-factorizations generated by the following starter A and the patterned starter in \mathbb{Z}_n are orthogonal and C_4 -free.

$$\begin{aligned} n = 7 &: \{\{1, 3\}, \{2, 6\}, \{4, 5\}\} \\ n = 35 &: \{\{1, 3\}, \{2, 5\}, \{4, 9\}, \{6, 7\}, \{8, 12\}, \{10, 19\}, \{11, 24\}, \{13, 25\}, \{14, 29\}, \\ &\{15, 31\}, \{16, 33\}, \{17, 28\}, \{18, 26\}, \{20, 34\}, \{21, 27\}, \{22, 32\}, \{23, 30\}\} \end{aligned}$$

By applying Theorem 12 repeatedly with the known pair of orthogonal C_4 -free one-factorizations of $K_{7,7}$ in Theorem 8, there is a pair of orthogonal C_4 -free one-factorizations of $K_{7^{k+1}}$ for any positive integer k . Similarly, by Theorems 12 and 8, there is a pair of orthogonal C_4 -free one-factorizations of $K_{5 \times 7^{k+1}}$ for any positive integer k .

If n is not divisible by 5, we have $n = 3^a p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where all $p_i > 5$ are distinct primes and $a \notin \{1, 2\}$. By Corollary 1 and Theorem 10, there is a pair of orthogonal C_4 -free one-factorizations of K_{3^a+1} and $K_{p_i^{\alpha_i+1}}$ for $a > 2$ and each $p_i > 7$ is a non-Fermat prime. Since there is a pair of orthogonal C_4 -free one-factorizations of $K_{p_i^{\alpha_i+1}}$ for prime $p_i \geq 7$ and positive integer α_i , by applying Theorems 12 and 8, there is a pair of orthogonal C_4 -free one-factorizations of K_{n+1} .

If n is divisible by 25, we have $n = 3^a 5^b p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where all $p_i > 5$ are distinct primes, $b \geq 2$ and $a \notin \{1, 2\}$. By Theorem 10, there is a pair of orthogonal C_4 -free one-factorizations of $K_{5^{b+1}}$. By applying Theorem 12 with the known pair of orthogonal C_4 -free one-factorizations of $K_{5^a, 5^a}$ in Theorem 8 and a pair of orthogonal C_4 -free one-factorizations of $K_{3^a p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} + 1}$, there is a pair of orthogonal C_4 -free one-factorizations of K_{n+1} .

If n is divisible by 35 and n is not divisible by 25, we have $n = 3^a \cdot 5 \cdot 7^s p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i \geq 11$, $s \geq 1$ and $a \notin \{1, 2\}$. From above, there is a pair of orthogonal C_4 -free one-factorizations of $K_{5 \times 7^{b+1}}$. By applying Theorem 12 with the known pair of orthogonal C_4 -free one-factorizations of $K_{5 \times 7^b, 5 \times 7^b}$ in Theorem 8 and a pair of orthogonal C_4 -free one-factorizations of $K_{3^a p_2^{\alpha_2} \cdots p_k^{\alpha_k} + 1}$, there is a pair of orthogonal C_4 -free one-factorizations of K_{n+1} .

If n is divisible by 5 and n is not divisible by 175, we have $n = 3^a \cdot 5 \cdot p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_i \geq 11$ and $a \notin \{1, 2\}$. Since $5 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \geq 7$, we can let $\alpha_1 \geq 1$. Since each p_i is a non-Fermat prime, from the proof of Corollary 1 there is a strong starter S_w in $GF(p_i)$ satisfying the conditions of Theorem 11. Thus, there is a pair of orthogonal C_4 -free one-factorizations of $K_{5 p_1+1}$. By applying Theorem 12 with the known pair of orthogonal C_4 -free one-factorizations of $K_{5 p_1, 5 p_1}$ in Theorem 8 and a pair of orthogonal C_4 -free one-factorizations of $K_{3^a p_1^{\alpha_1-1} \cdots p_k^{\alpha_k} + 1}$, there is a pair of orthogonal C_4 -free one-factorizations of K_{n+1} . This completes the proof. \square

5 Conclusion

In this paper, we consider the constructions of two orthogonal C_4 -free one-factorizations of K_n , which can be used to improve the upper bound for two-round rainbow colorings of K_n . Although some constructions of two orthogonal one-factorizations of K_n have been modified successfully, the existence of two orthogonal C_4 -free one-factorizations of K_n is not completely resolved. A further interesting problem is the existence of two orthogonal *totally* C_4 -free one-factorizations of K_n .

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