Graphs and Combinatorics (2018) 34:1295–1314
https://doi.org/10.1007/s00373-018-1950-4

ORIGINAL PAPER

Graphs with Almost All Edges in Long Cycles

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Received: 23 May 2017 / Revised: 24 August 2018 / Published online: 25 September 2018

Abstract
For an edge e of a given graph G, define $c_e(G)$ be the length of a longest cycle of G containing e. Wang and Lv (2008) gave a tight function $f_0(n,k)$ (for integers $n \geq 3$ and $k \geq 4$), such that for any 2-connected graph G on n vertices with more than $f_0(n,k)$ edges, every edge belongs to a cycle of length at least k, i.e., $c_e(G) \geq k$ for every edge $e \in E(G)$. In this work we give a tight function $f(n,k)$ (for integers $n \geq k \geq 6$), such that for any 2-connected graph G on n vertices with more than $f(n,k)$ edges, we have that $c_e(G) \geq k$ for all but at most one edge of G.

Keywords Cycles · 2-Connected graphs · Extremal graphs

1 Introduction

The graphs considered here are finite, undirected and simple (no loops or parallel edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order of a graph G is the number of its vertices. Define $e(G) = |E(G)|$. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of m disjoint copies of the same graph G is denoted by mG. The *join* of two disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from their union by joining each vertex of G_1 to each vertex of G_2.

A classical result of Erdös and Gallai [2] is that for an integer $k \geq 2$, if G is a graph on n vertices with more than $\frac{k}{2}(n-1)$ edges, then G contains a cycle of length more than k. The result is best possible when $n-1$ is divisible by $k-1$, in view of the graph consisting of copies of K_k all having exactly one vertex in common. Woodall

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[6] improved the result by giving best possible bounds for the remaining cases when $n-1$ is not divisible by $k-1$. Caccetta and Vijayan [1] gave an alternative proof of the same result, and in addition, characterized the structure of the extremal graphs. For 2-connected graphs, Woodall [6] obtained the bound for the case when $2 \leq k \leq \frac{2n+2}{3}$, and Fan et al. [4] completed all the rest cases when $\frac{2}{3}n+1 \leq k \leq n-1$ by using an edge-switching technique.

Let $c_e(G)$ be the length of a longest cycle which contains e in G. In [5], Wang and Lv gave the maximum number of edges a 2-connected graph can have with at least one edge e of G such that $c_e(G) \leq k-1$, as the following theorem states. For integers $n \geq 3$ and $k \geq 4$, define $f_0(n,k) = q\binom{k-3}{2} + \binom{r}{2} + 2(n-2) + 1$, where $n-2 = q(k-3)+r, 0 \leq r < k-3$.

Theorem 1.1 [5] *For integers $n \geq 3$ and $k \geq 4$, let G be a 2-connected graph on n vertices. If there exists an edge uv of G such that $c_{uv}(G) \leq k-1$, then*

$$e(G) \leq f_0(n,k),$$

with equality if and only if (i) $G \cong uv \vee (qK_{k-3} \cup K_r)$; *or* (ii) $G \cong (uv \vee q'K_{k-3}) \cup (uv \vee K_{t-2} \vee \overline{K}_{n'-t})$, *with $k = 2t$ and $r = \frac{k-2}{2}$ or $\frac{k-4}{2}$, where $t \geq 3$, $0 \leq q' < q$ and $n' = n - q'(k-3)$.*

Let $F_G = \{e | e \in G \text{ and } c_e(G) \leq k-1\}$. In Theorem 1.1, it means that if $e(G) > f_0(n,k)$, then $c_e(G) \geq k$ for every $e \in E(G)$, i.e., $|F_G| = 0$. As a generalization of Theorem 1.1, we give a tight function $f(n,k)$, such that for any 2-connected graph G on n vertices with $e(G) > f(n,k)$, then $c_e(G) \geq k$ for all but at most one edge of G, i.e., $|F_G| \leq 1$.

For integers $n \geq k \geq 6$, define $f_1(n,k) = q_1\binom{k-4}{2} + \binom{r_1}{2} + 2(n-2) + 1$, where $n-3 = q_1(k-4) + r_1, q_1 \geq 0, 0 \leq r_1 < k-4$; $f_2(n,k) = \binom{\frac{k}{2}}{2} + \frac{k}{2}\left(n - \frac{k}{2}\right)$, if k is even, otherwise $f_2(n,k) = \binom{\frac{k-1}{2}}{2} + \frac{k-1}{2}\left(n - \frac{k-1}{2}\right) + 1$; $f_3(n,k) = \binom{k-2}{2} + 3(n-k+2)$. We get the following result.

Theorem 1.2 *For integers $n \geq k \geq 6$, let G be a 2-connected graph on n vertices. If*

$$e(G) > f(n,k),$$

then $|F_G| \leq 1$, where $f(n,k) = \max\{f_1(n,k), f_2(n,k), f_3(n,k)\}$.

We shall show that the function $f(n,k)$ is tight. For integers $n \geq k \geq 6$, let

$$G_1 = K_2 \vee (K_1 \cup q_1 K_{k-4} \cup K_{r_1}), \text{ where } n-3 = q_1(k-4) + r_1, q_1 \geq 0 \quad \text{and} \quad 0 \leq r_1 < k-4,$$

$$G_2 = \begin{cases} K_{\frac{k}{2}} \vee \left(n - \frac{k}{2}\right) K_1, & \text{if } k \text{ is even,} \\ K_{\frac{k-1}{2}} \vee \left(K_2 \cup \left(n - \frac{k+3}{2}\right) K_1\right), & \text{otherwise,} \end{cases}$$

$$G_3 = K_3 \vee \left(K_{k-5} \cup (n-k+2) K_1\right).$$

It's easy to see that $|F_{G_i}| \geq 2$ and $e(G_i) = f_i(n,k)$ for $i = 1, 2, 3$. In this sense, Theorem 1.2 is best possible.

Let H be a subgraph of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$. When no confusion can occur, we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. For subgraphs F and H, $E(F, H)$ denotes the set, and $e(F, H)$ the number, of edges with one end in F and the other end in H. For simplicity, we write $E(F)$ and $e(F)$ for $E(F, F)$ and $e(F, F)$, respectively. In particular, $e(G) = |E(G)|$. Note $G - H$ denotes the graph obtained from G by deleting all vertices of H together with all the edges with at least one end in H. For $E' \subseteq E(G)$, $G - E'$ denotes the graph obtained from G by deleting all the edges of E'. Let $S \subseteq V(G)$. A subgraph H is *induced* by S if $V(H) = S$ and $xy \in E(H)$ if and only if $xy \in E(G)$, we denote H by $G[S]$. We say S is an *independent* set if $E(S) = \emptyset$. Let $P = a_1a_2 \ldots a_n$ be a path. We can assume that P has an orientation which is consistent with the increasing order of the indices of a_i, $1 \leq i \leq n$. For $a \in V(P)$, define a^- and a^+ to be the vertices on P immediately before and after a, respectively, according to the orientation of P. Similar definition can be given for an oriented cycle C.

2 Some Lemmas

The concept of edge-switching is given by Fan in [3]. Let uv be an edge in a graph G and let $Z = N(v)\backslash(N(u) \cup \{u\})$. An *edge-switching* from v to u is to delete $\{vz|z \in Z\}$ and add $\{uz|z \in Z\}$. The resulting graph, denoted by $G[v \to u]$, is called an *edge-switching graph* of G (from v to u). Let $H = \{uz|z \in Z\}$. Then we have the following lemma.

Lemma 2.1 *If G is a connected graph and uv is an edge of G, let $G' = G[v \to u]$, then the following statements are true.*

(a) *For any edge $e = ux,\ x \in N_G(u)$, we have that $c_e(G') \leq c_e(G)$.*
(b) *For any edge $e = vy,\ y \in N_G(v)\backslash\{u\}$, we have that $c_{uy}(G') \leq c_{vy}(G)$.*
(c) *For any edge e which isn't incident with u and v in G, we have that $c_e(G') \leq c_e(G)$.*

Proof (a) Suppose, to the contrary, that there is an edge $e = ux,\ x \in N_G(u)$, such that $c_e(G') > c_e(G)$. That is, there is a cycle C' in G', which contains e and with $e(C') > c_e(G)$. In the following, we shall always find a cycle C in G, such that $e \in C$ and $e(C) \geq e(C') > c_e(G)$. That's a contradiction which completes the proof.

If $E(C') \cap H = \emptyset$, then we can choose $C = C'$. Thus, we can assume that $E(C') \cap H \neq \emptyset$. Since $|E(C') \cap H| \leq 1$, we can assume that $|E(C') \cap H| = 1$. Let $E(C') \cap H = \{uy\}$.

If $x = v$, then without loss of generality, we can assume that $C' = uvz \ldots yu$, where $uy \in H$ and $z \in N_G(u) \cap N_G(v)$. (See Fig. 1a). Then let $C = (C'\backslash\{uy, vz\}) \cup \{uz, vy\}$.

If $x \neq v$, then there are two subcases. If $v \notin C'$, then we can assume that $C' = ux \ldots yu$, where $uy \in H$. (See Fig. 1b). Then let $C = (C'\backslash\{uy\}) \cup \{uv, vy\}$. If $v \in C'$, then we can assume that $C' = ux \ldots z_1vz_2 \ldots yu$, where $uy \in H$ and $\{z_1, z_2\} \subseteq N_G(u) \cap N_G(v)$. (See Fig. 2). Then let $C = (C'\backslash\{uy, vz_2\}) \cup \{uz_2, vy\}$.

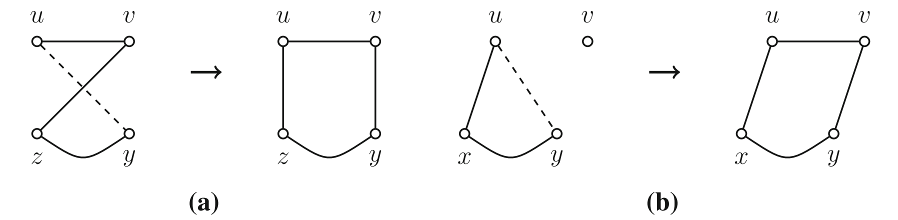

Fig. 1 The cases of $C' = uvz \dots yu$ and $C' = ux \dots yu$

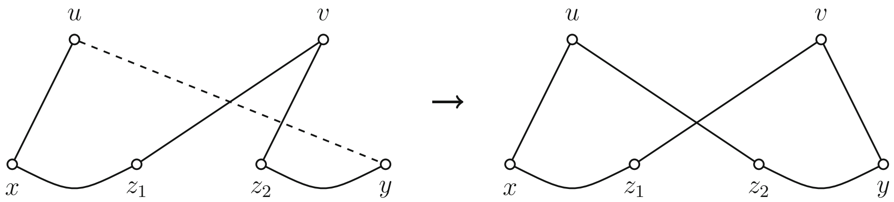

Fig. 2 The case of $C' = ux \dots z_1 v z_2 \dots yu$

(b) Note that for any $y \in N_G(v)\backslash\{u\}$, whenever $uy \in H$ or not, the discussions in the following are the same. Similar with the proof of (a), suppose, to the contrary, that for some $y \in N_G(v)\backslash\{u\}$, $c_{uy}(G') > c_{vy}(G)$. Assume that C' is a cycle in G' such that $uy \in C'$ and $e(C') = c_{uy}(G')$. We shall find a cycle C in G, such that $e = vy \in C$ and $e(C) \geq e(C') > c_{vy}(G)$. This produces a contradiction.

If $v \notin C'$, then we assume that $C' = uy \dots xu$. If $ux \notin H$, then let $C = (C'\backslash\{uy\}) \cup \{uv, vy\}$. If $ux \in H$, then let $C = (C'\backslash\{ux, uy\}) \cup \{vx, vy\}$.

If $v \in C'$, then there are two subcases. If $uv \in E(C')$, then without loss of generality, we can assume that $C' = uy \dots zvu$. Then let $C = (C'\backslash\{uy, vz\}) \cup \{uz, vy\}$. If $uv \notin E(C')$, then we assume that $C' = uy \dots z_1 v z_2 \dots wu$. If $uw \notin H$, then let $C = (C'\backslash\{uy, vz_1\}) \cup \{uz_1, vy\}$. If $uw \in H$, then let $C = (C'\backslash\{uw, uy, vz_1, vz_2\}) \cup \{uz_1, uz_2, vw, vy\}$.

(c) The proof is similar with the above discussion. We shall omit the details here. □

The following lemma is easy to prove, so we omit the details here. Let $e = xy$ be an edge of G. By G/e we denote the graph obtained from G by contracting the edge e into a new vertex w which becomes adjacent to all the former neighbors of x and of y.

Lemma 2.2 *Let G be a 2-connected graph and let uv be an edge of G.*

(i) *If G isn't isomorphic to K_3 and G/uv isn't 2-connected, then $\{u, v\}$ is a vertex cut of G.*
(ii) *If $N(u) \cap N(v) \neq \emptyset$, and the edge-switching graph $G[v \to u]$ isn't 2-connected, then $\{u, v\}$ is a vertex cut of G.*

Lemma 2.3 *For integers $n \geq 0$ and $m > 0$, define $l(n, m) = q\binom{m}{2} + \binom{r}{2}$, where $n = qm + r$, $q \geq 0$ and $0 \leq r < m$. Then*

$$l(n, m + 1) \geq l(n, m).$$

Proof Let $l(n, m+1) = q'\binom{m+1}{2} + \binom{r'}{2}$, where $n = q'(m+1) + r'$, $q' \geq 0$ and $0 \leq r' < m+1$. Clearly $q' \leq q$.

If $q' = q$, then $r' = r - q$. Thus

$$\begin{aligned} l(n, m+1) - l(n, m) &= \frac{1}{2}[q'm(m+1) + r'(r'-1) - qm(m-1) - r(r-1)] \quad (2.1)\\ &= \frac{1}{2}[q^2 + q(2m - 2r + 1)]. \end{aligned}$$

Since $r < m$, $l(n, m+1) \geq l(n, m)$.

If $q' = q - 1$, then $r' = m - (q - 1 - r)$. Using $q' = q - 1$ in (2.1),

$$l(n, m+1) - l(n, m) = \frac{1}{2}[2qm - m(m+1) + r'(r'-1) - r(r-1)]. \quad (2.2)$$

Using $m + 1 = r' - r + q$ in (2.2),

$$\begin{aligned} l(n, m+1) - l(n, m) &= \frac{1}{2}[2qm - m(r' - r + q) + r'(r'-1) - r(r-1)]\\ &= \frac{1}{2}[qm - r'(m - r' + 1) + r(m - r + 1)]. \quad (2.3) \end{aligned}$$

Since $m - r' + 1 = q - r \leq q$ and $r' \leq m$, $r'(m - r' + 1) \leq qm$. Note that $r < m$. By (2.3), $l(n, m+1) - l(n, m) \geq 0$. That is, $l(n, m+1) \geq l(n, m)$.

If $q' \leq q - 2$, note that $q = \frac{n-r}{m}$ and $q' = \frac{n-r'}{m+1}$, then we obtain $\frac{n-r'}{m+1} \leq \frac{n-r}{m} - 2$. That is, $n \geq m(m+1) + r(m+1) + m(m+1-r') \geq m(m+1)$. Using $qm = n - r$ and $q'(m+1) = n - r'$ in (2.1),

$$\begin{aligned} l(n, m+1) - l(n, m) &= \frac{1}{2}[(n - r')m + r'(r'-1) - (n-r)(m-1) - r(r-1)]\\ &= \frac{1}{2}[n - r'm + r'(r'-1) + r(m-r)]. \quad (2.4) \end{aligned}$$

Since $r' < m+1$, $r'm < (m+1)m \leq n$. Note that $r < m$. By (2.4), $l(n, m+1) - l(n, m) \geq 0$. That is, $l(n, m+1) \geq l(n, m)$.

Consequently, in each case we have that $l(n, m+1) \geq l(n, m)$. This completes the proof of Lemma 2.3. □

By Lemma 2.3, we can easily get the following result.

Corollary 2.4 *For integers $n \geq 0$ and $m > 0$, define $l(n, m) = q\binom{m}{2} + \binom{r}{2}$, where $n = qm + r$, $q \geq 0$ and $0 \leq r < m$. Then $l(n, m_1) \geq l(n, m_2)$, for integers $m_1 \geq m_2 > 0$.*

Lemma 2.5 *For integers $0 \leq r_1 \leq r_2 < k$, let $r_1 + r_2 = qk + r$, where $q \geq 0$ and $0 \leq r < k$, then we have that*

$$\binom{r_1}{2} + \binom{r_2}{2} \leq q\binom{k}{2} + \binom{r}{2},$$

the equality holds if and only if $r_1 = 0$ *or* $r_2 = 0$.

Proof Since $0 \le r_1 \le r_2 < k$, we have that $0 \le r_1 + r_2 < 2k$, which implies that $0 \le q \le 1$.

If $q = 0$, then $r = r_1 + r_2$. So

$$\binom{r_1}{2} + \binom{r_2}{2} = \binom{r_1 + r_2}{2} - r_1 r_2 \le \binom{r}{2},$$

the equality holds if and only if $r_1 = 0$ or $r_2 = 0$.

If $q = 1$, then $r_1 + r_2 = k + r$. Let $l = r_1 + r_2 = k + r$. Note that $r_1 \le r_2$ and $r < k$. So $r_1 \le \frac{l}{2}$ and $r < \frac{l}{2}$. Since $r_2 < k$, we have that $r_1 > r$.

$$\binom{r_1}{2} + \binom{r_2}{2} - \binom{k}{2} - \binom{r}{2} = \binom{r_1}{2} + \binom{l - r_1}{2} - \binom{l - r}{2} - \binom{r}{2}$$
$$= r(l - r) - r_1(l - r_1). \tag{2.5}$$

Let $f(x) = x(l - x)$. Since $0 \le r < r_1 \le \frac{l}{2}$ and $f(x)$ is a strictly increasing function on the interval $[0, \frac{l}{2}]$, $f(r) < f(r_1)$. That is, $r(l - r) < r_1(l - r_1)$. By (2.5), $\binom{r_1}{2} + \binom{r_2}{2} < \binom{k}{2} + \binom{r}{2}$.

In each case, we have that $\binom{r_1}{2} + \binom{r_2}{2} \le q\binom{k}{2} + \binom{r}{2}$, and the equality holds if and only if $r_1 = 0$ or $r_2 = 0$. □

Lemma 2.6 *For integers* $n \ge k \ge 6$, *define*

$$f(n, k) = \max\{f_1(n, k), f_2(n, k), f_3(n, k)\},$$

where $f_i(n, k)$ $(1 \le i \le 3)$ *is defined as in Theorem 1.2. For integers* $n \ge 2$ *and* $k \ge 6$, *define*

$$g(n, k) = q'\binom{k - 5}{2} + \binom{r'}{2} + 2(n - 2) + 1,$$

where $n - 2 = q'(k - 5) + r'$, $q' \ge 0$ *and* $0 \le r' < k - 5$. *Then we have that*

$$f(n_1, k) + g(n_2, k) - 1 \le f(n, k),$$

where n_1, n_2 *are integers,* $n \ge n_1 \ge k \ge 6$ *and* $n = n_1 + n_2 - 2$.

Proof Let

$$f_1(n_1, k) = q_1\binom{k - 4}{2} + \binom{r_1}{2} + 2(n_1 - 2) + 1,$$
$$g(n_2, k) = q_2\binom{k - 5}{2} + \binom{r_2}{2} + 2(n_2 - 2) + 1,$$
$$f_1(n, k) = q\binom{k - 4}{2} + \binom{r}{2} + 2(n - 2) + 1,$$

where

$$n_1 - 3 = q_1(k-4) + r_1, q_1 \geq 0 \quad \text{and} \quad 0 \leq r_1 < k-4;$$
$$n_2 - 2 = q_2(k-5) + r_2, q_2 \geq 0 \quad \text{and} \quad 0 \leq r_2 < k-5;$$
$$n - 3 = q(k-4) + r, q \geq 0 \text{ and } 0 \leq r < k-4.$$

Claim 1 $f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n, k)$.

Define $h(n_2, k) = q_3\binom{k-4}{2} + \binom{r_3}{2} + 2(n_2 - 2) + 1$, where $n_2 - 2 = q_3(k-4) + r_3$, $q_3 \geq 0$ and $0 \leq r_3 < k-4$. By Lemma 2.3, $q_2\binom{k-5}{2} + \binom{r_2}{2} \leq q_3\binom{k-4}{2} + \binom{r_3}{2}$. Thus

$$g(n_2, k) \leq h(n_2, k). \tag{2.6}$$

Since $n_1 - 3 = q_1(k-4) + r_1$, $n_2 - 2 = q_3(k-4) + r_3$ and $n = n_1 + n_2 - 2$, we have that $n - 3 = (q_1 + q_3)(k-4) + (r_1 + r_3)$. Note that $0 \leq r_1 < k-4$ and $0 \leq r_3 < k-4$. Let $r_1 + r_3 = q'(k-4) + r'$, where $q' \geq 0$ and $0 \leq r' < k-4$. Hence, by Lemma 2.5,

$$\binom{r_1}{2} + \binom{r_3}{2} \leq q'\binom{k-4}{2} + \binom{r'}{2}. \tag{2.7}$$

And $n - 3 = (q_1 + q_3)(k-4) + q'(k-4) + r' = (q_1 + q_3 + q')(k-4) + r'$, $0 \leq r' < k-4$. Since $n - 3 = q(k-4) + r$, $q \geq 0$ and $0 \leq r < k-4$, it follows that $q = q_1 + q_3 + q'$ and $r = r'$.

$$\begin{aligned} f_1(n_1, k) + h(n_2, k) - 1 &= q_1\binom{k-4}{2} + \binom{r_1}{2} + 2(n_1 - 2) \\ &\quad +1 + q_3\binom{k-4}{2} + \binom{r_3}{2} + 2(n_2 - 2) + 1 - 1 \\ &= (q_1 + q_3)\binom{k-4}{2} + \binom{r_1}{2} + \binom{r_3}{2} + 2(n_1 + n_2 - 4) + 1. \end{aligned} \tag{2.8}$$

Using $n = n_1 + n_2 - 2$ and (2.7) in (2.8),

$$\begin{aligned} f_1(n_1, k) + h(n_2, k) - 1 &\leq (q_1 + q_3)\binom{k-4}{2} + q'\binom{k-4}{2} + \binom{r'}{2} + 2(n-2) + 1 \\ &= q\binom{k-4}{2} + \binom{r}{2} + 2(n-2) + 1 \\ &= f_1(n, k). \end{aligned}$$

Then by (2.6), we have that $f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n_1, k) + h(n_2, k) - 1 \leq f_1(n, k)$.

Claim 2 $f_2(n_1, k) + g(n_2, k) - 1 \leq f_2(n, k)$.

If k is even, then $f_2(n,k)=\binom{\frac{k}{2}}{2}+\frac{k}{2}(n-\frac{k}{2})$. Note that $q_2(k-5)=n_2-2-r_2$.

$$\begin{aligned}
&f_2(n_1,k)+g(n_2,k)-1\\
&=\binom{\frac{k}{2}}{2}+\frac{k}{2}\left(n_1-\frac{k}{2}\right)+\frac{q_2(k-5)(k-6)}{2}+\binom{r_2}{2}+2(n_2-2)+1-1\\
&=\binom{\frac{k}{2}}{2}+\frac{k}{2}\left(n_1-\frac{k}{2}\right)+\frac{(n_2-2-r_2)(k-6)}{2}+\binom{r_2}{2}+2(n_2-2)\\
&=\binom{\frac{k}{2}}{2}+\frac{k}{2}\left(n_1+n_2-2-\frac{k}{2}\right)+\frac{1}{2}r_2[r_2-(k-5)]+(2-n_2). \qquad (2.9)
\end{aligned}$$

Using $n=n_1+n_2-2$ in (2.9),

$$f_2(n_1,k)+g(n_2,k)-1=f_2(n,k)+\frac{1}{2}r_2[r_2-(k-5)]+(2-n_2). \qquad (2.10)$$

Note that $r_2<k-5$ and $n_2\geq 2$. By (2.10), $f_2(n_1,k)+g(n_2,k)-1\leq f_2(n,k)$.

If k is odd, then $f_2(n,k)=\binom{\frac{k-1}{2}}{2}+\frac{k-1}{2}(n-\frac{k-1}{2})+1$. Note that $q_2(k-5)=n_2-2-r_2$ and $n=n_1+n_2-2$.

$$\begin{aligned}
&f_2(n_1,k)+g(n_2,k)-1\\
&=\binom{\frac{k-1}{2}}{2}+\frac{k-1}{2}\left(n_1-\frac{k-1}{2}\right)+1+\frac{q_2(k-5)(k-6)}{2}+\binom{r_2}{2}\\
&\quad+2(n_2-2)+1-1\\
&=\binom{\frac{k-1}{2}}{2}+\frac{k-1}{2}\left(n_1-\frac{k-1}{2}\right)+1+\frac{(n_2-2-r_2)(k-6)}{2}+\binom{r_2}{2}\\
&\quad+2(n_2-2)\\
&=f_2(n,k)+\frac{1}{2}[(2-n_2)+r_2(r_2-(k-5))].
\end{aligned}$$

Since $n_2\geq 2$ and $r_2<k-5$, it follows that $f_2(n_1,k)+g(n_2,k)-1\leq f_2(n,k)$.

Claim 3 $f_3(n_1,k)+g(n_2,k)-1\leq f(n,k)$.

If $k=6,7$, then $f_3(n_1,k)=f_2(n_1,k)$. If $k=8$, since $n_1\geq k\geq 8$, then $f_3(n_1,k)\leq f_2(n_1,k)$. That is, $f_3(n_1,k)\leq f_2(n_1,k)$ for $6\leq k\leq 8$. By Claim 2, $f_3(n_1,k)+g(n_2,k)-1\leq f_2(n_1,k)+g(n_2,k)-1\leq f_2(n,k)\leq f(n,k)$. Therefore, the result is true for $6\leq k\leq 8$.

If $n_2 \leq 5$, then

$$\begin{aligned} g(n_2, k) &= q_2\binom{k-5}{2} + \binom{r_2}{2} + 2(n_2-2) + 1 \\ &\leq \binom{q_2(k-5)+r_2}{2} + 2(n_2-2) + 1 \\ &= \binom{n_2-2}{2} + 2(n_2-2) + 1 \\ &= \frac{(n_2-2)(n_2-3)}{2} + 2(n_2-2) + 1 \\ &\leq 3(n_2-2) + 1. \end{aligned}$$

Hence,

$$\begin{aligned} f_3(n_1, k) + g(n_2, k) - 1 &\leq \binom{k-2}{2} + 3(n_1-k+2) + 3(n_2-2) + 1 - 1 \\ &= \binom{k-2}{2} + 3(n-k+2) \\ &= f_3(n, k) \\ &\leq f(n, k). \end{aligned}$$

Thus we may suppose that $k \geq 9$ and $n_2 \geq 6$. In the following, we shall compare $f_3(n_1, k)$ with $f_1(n_1, k)$ and use Claim 1 which has been proved to obtain our result. Note that $f_1(n_1, k) = q_1\binom{k-4}{2} + \binom{r_1}{2} + 2(n_1-2) + 1$, where $n_1 - 3 = q_1(k-4) + r_1$. Since $n_1 \geq k$, we have that $q_1 \geq 1$. We distinguish two cases according to q_1 and r_1.

Case 1 $q_1 \geq 2$ or $r_1 \geq 4$.

$$\begin{aligned} f_1(n_1, k) &= q_1\binom{k-4}{2} + \binom{r_1}{2} + 2(n_1-2) + 1 \\ &= \binom{k-2}{2} + (q_1-1)\binom{k-4}{2} + \binom{r_1}{2} + 2(n_1-k+2) \\ &= \binom{k-2}{2} + [(q_1-1)(k-4) + r_1 + 1] + 2(n_1-k+2) \\ &\quad + \frac{(q_1-1)(k-4)(k-7)}{2} + \frac{r_1(r_1-3)}{2} - 1 \\ &= \binom{k-2}{2} + 3(n_1-k+2) + \frac{(q_1-1)(k-4)(k-7)}{2} + \frac{r_1(r_1-3)}{2} - 1 \\ &= f_3(n_1, k) + \frac{(q_1-1)(k-4)(k-7)}{2} + \frac{r_1(r_1-3)}{2} - 1. \end{aligned}$$

Note that $k \geq 9$ and $r_1 \geq 0$. Clearly, if $q_1 \geq 2$ or $r_1 \geq 4$, then $\frac{(q_1-1)(k-4)(k-7)}{2} + \frac{r_1(r_1-3)}{2} - 1 \geq 0$. That is, $f_1(n_1, k) \geq f_3(n_1, k)$. By Claim 1, $f_3(n_1, k) + g(n_2, k) - 1 \leq f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n, k) \leq f(n, k)$.

Case 2 $q_1 = 1$ and $r_1 \leq 3$.
Since $n_1 - 3 = (k-4) + r_1$, we have that $r_1 = n_1 - k + 1 \geq 1$.

$$\begin{aligned} f_1(n_1, k) &= \binom{k-4}{2} + \binom{r_1}{2} + 2(n_1 - 2) + 1 \\ &= \binom{k-2}{2} + 3(n_1 - k + 2) + \binom{r_1}{2} - (n_1 - k + 1 + 1) \\ &= f_3(n_1, k) + \binom{r_1}{2} - (r_1 + 1) \\ &= f_3(n_1, k) + \frac{r_1(r_1 - 3)}{2} - 1. \end{aligned} \tag{2.11}$$

Since $1 \leq r_1 \leq 3$, $\frac{r_1(r_1-3)}{2} \geq -1$. By (2.11), $f_1(n_1, k) \geq f_3(n_1, k) - 2$. That is,

$$f_3(n_1, k) \leq f_1(n_1, k) + 2. \tag{2.12}$$

In the following, we shall prove that

$$f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n, k) - 2. \tag{2.13}$$

$$\begin{aligned} & f_1(n_1, k) + g(n_2, k) - 1 \\ &= \binom{k-4}{2} + \binom{r_1}{2} + 2(n_1 - 2) + 1 \\ &\quad + q_2\binom{k-5}{2} + \binom{r_2}{2} + 2(n_2 - 2) + 1 - 1. \end{aligned} \tag{2.14}$$

If $q_2 \geq 1$, then $q_2 - 1 \geq 0$. Note that $r_1 \geq 1$. By (2.14), we have that

$$\begin{aligned} & f_1(n_1, k) + g(n_2, k) - 1 \\ &= \binom{k-4}{2} + \left[\binom{r_1}{2} + \binom{k-5}{2}\right] \\ &\quad + 2(n_1 - 2) + (q_2 - 1)\binom{k-5}{2} + \binom{r_2}{2} + 2(n_2 - 2) + 1 \\ &= \binom{k-4}{2} + \left[\binom{r_1 - 1}{2} + \binom{k-4}{2} - (k - 4 - r_1)\right] \\ &\quad + 2(n_1 - 2) + (q_2 - 1)\binom{k-5}{2} \\ &\quad + \binom{r_2}{2} + 2(n_2 - 2) + 1. \end{aligned} \tag{2.15}$$

Let $n_1' = n_1 + (k-5)$ and $n_2' = n_2 - (k-5)$. Clearly, $n_1' \geq n_1 \geq k$ and $n = n_1' + n_2' - 2$. Then

$$f_1(n_1', k) = 2\binom{k-4}{2} + \binom{r_1 - 1}{2} + 2(n_1' - 2) + 1,$$
$$g(n_2', k) = (q_2 - 1)\binom{k-5}{2} + \binom{r_2}{2} + 2(n_2' - 2) + 1.$$

Since $k \geq 9$ and $1 \leq r_1 \leq 3$, we have that $k - 4 - r_1 \geq 2$. By (2.15),

$$\begin{aligned} f_1(n_1, k) + g(n_2, k) - 1 &\leq 2\binom{k-4}{2} + \binom{r_1 - 1}{2} - 2 + 2(n_1 + (k-5) - 2) \\ &\quad + (q_2 - 1)\binom{k-5}{2} + \binom{r_2}{2} + 2(n_2 - (k-5) - 2) + 1 \\ &= f_1(n_1', k) + g(n_2', k) - 3. \end{aligned} \tag{2.16}$$

By Claim 1, $f_1(n_1', k) + g(n_2', k) - 1 \leq f_1(n, k)$. Using this in (2.16), we have that $f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n, k) - 2$.

If $q_2 = 0$, then $r_2 \geq 4$ since $n_2 \geq 6$. Note that $n_1 - 3 = (k-4) + r_1$ and $n_2 - 2 = r_2$, where $0 \leq r_1 < k-4$ and $0 \leq r_2 < k-5$.

$$\begin{aligned} &f_1(n_1, k) + g(n_2, k) - 1 \\ &= \binom{k-4}{2} + \binom{r_1}{2} + 2(n_1 - 2) + 1 + \binom{r_2}{2} + 2(n_2 - 2) + 1 - 1 \\ &= \binom{k-4}{2} + \left[\binom{r_1 - 1}{2} + \binom{r_2 + 1}{2} - (r_2 - r_1 + 1)\right] + 2(n-2) + 1. \end{aligned} \tag{2.17}$$

Note that $r_1 - 1 < k - 4$ and $r_2 + 1 < k - 4$. Let $(r_1 - 1) + (r_2 + 1) = q'(k-4) + r'$, where $q' \geq 0$ and $0 \leq r' < k - 4$. Then by Lemma 2.5,

$$\binom{r_1 - 1}{2} + \binom{r_2 + 1}{2} \leq q'\binom{k-4}{2} + \binom{r'}{2}. \tag{2.18}$$

Since $r_1 \leq 3$ and $r_2 \geq 4$, we have that $r_2 - r_1 + 1 \geq 2$. Using (2.18) in (2.17), we obtain

$$\begin{aligned} f_1(n_1, k) + g(n_2, k) - 1 &\leq \binom{k-4}{2} + q'\binom{k-4}{2} + \binom{r'}{2} - 2 + 2(n-2) + 1 \\ &= (q' + 1)\binom{k-4}{2} + \binom{r'}{2} + 2(n-2) + 1 - 2. \end{aligned} \tag{2.19}$$

Note that $n - 3 = n_1 + n_2 - 5 = (k-4) + r_1 + r_2 = (q'+1)(k-4) + r' = q(k-4) + r$, where $q' + 1 \geq 0$ and $0 \leq r' < k - 4$. Hence, $f_1(n, k) = (q'+1)\binom{k-4}{2} + \binom{r'}{2} + 2(n-2) + 1$. By (2.19), we have that

$$f_1(n_1, k) + g(n_2, k) - 1 \leq f_1(n, k) - 2.$$

This completes the proof of (2.13).

Combining (2.12) with (2.13), we obtain

$$\begin{aligned} f_3(n_1,k)+g(n_2,k)-1 &\le f_1(n_1,k)+2+g(n_2,k)-1 \\ &\le f_1(n,k)-2+2=f_1(n,k)\le f(n,k). \end{aligned}$$

In either case, we have that $f_3(n_1,k)+g(n_2,k)-1\le f(n,k)$, and we complete the proof of Claim 3.

By Claims 1, 2 and 3, we can easily obtain that

$$f(n_1,k)+g(n_2,k)-1\le f(n,k).$$

This ends the proof of the lemma. □

3 Proof of Theorem 1.2

The proof needs the following theorems. The first one is a result of Fan et al. [4]. Define $t(n,k)=\max\{\binom{k-1}{2}+2(n-k+1),\binom{k+1-\lfloor\frac{k}{2}\rfloor}{2}+\lfloor\frac{k}{2}\rfloor(n-k-1+\lfloor\frac{k}{2}\rfloor)\}$.

Theorem 3.1 [4] *For integers $3\le k\le n$, let G be a 2-connected graph on n vertices. If the length of a longest cycle of G is not more than k, then $e(G)\le t(n,k)$.*

For 2-connected graph G, let $c_{(e,e')}(G)$ be the length of a longest cycle containing both e and e' in G.

Theorem 3.2 *Let G be a 2-connected graph of order $n\ge 5$.*

(i) *If $e(G)>\binom{n-1}{2}+3$, then any two edges of G lie on a common cycle of length n.*
(ii) *If $e(G)\ge\binom{n-1}{2}+3$, then any two edges of G lie on a common cycle of length more than $n-2$.*

Proof We begin with a claim.

Claim. If $e(G)\ge\binom{n-1}{2}+3$, then for any two edges e_1 and e_2 of G, there is a Hamilton path P of G containing both e_1 and e_2, and one endvertex of P is neither incident with e_1 nor incident with e_2 in P.

If $e(G)\ge\binom{n-1}{2}+3$, then by Theorem 1.1, $c_e(G)=n$ for any edge e of G. Let $C=u_1u_2\dots u_n$ be a Hamilton cycle containing e_1. If $e_2\in C$, note that $n\ge 5$, then there exists an edge $e'\in C$ ($e'\ne e_1,e_2$) such that one end of e' isn't incident with e_1 and e_2. Then $P=C-e'$ is a Hamilton path with the required properties. If $e_2\notin C$, then without loss of generality, we can assume that $e_1=u_iu_{i+1}$ and $e_2=u_ju_k$, where $1\le i<j<k-1\le n-1$. Clearly, we can choose $P=u_{j+1}\overrightarrow{C}u_ku_j\overleftarrow{C}u_{k+1}$ (note that $u_{n+1}=u_1$). It ends the proof of the claim.

Now we shall prove (i) and (ii) respectively.

(i) Suppose to the contrary that there are two edges e_1 and e_2 with $c_{(e_1,e_2)}(G)<n$. Since $e(G)>\binom{n-1}{2}+3$, by the claim, there is a Hamilton path $P=u_1u_2\dots u_n$

containing e_1 and e_2, and without loss of generality, we may assume that $e_1 = u_k u_{k+1}$ and $e_2 = u_l u_{l+1}$, where $2 \le k < l \le n-1$. Clearly, $u_1 u_n \notin G$.

If $u_n u_i \in G$, where $2 \le i \le n-1$, $i \ne k, l$, then $u_1 u_{i+1} \notin G$, for otherwise, $C = u_1 u_{i+1} \overrightarrow{P} u_n u_i \overleftarrow{P} u_1$ is a Hamilton cycle of G containing e_1 and e_2, a contradiction. Hence, for each vertex u_i of $N(u_n) \backslash \{u_k, u_l\}$, there is a vertex u_{i+1} of $V(G) \backslash \{u_1\}$ not adjacent to u_1. Thus, $d(u_1) \le (n-1) - (d(u_n) - 2)$, that is, $d(u_1) + d(u_n) \le n+1$. Note that $u_1 u_n \notin G$. Then

$$e(G) = d(u_1) + d(u_n) + e(G - \{u_1, u_n\}) \le n + 1 + \binom{n-2}{2} = \binom{n-1}{2} + 3.$$

This contradiction completes the proof of (i).

(ii) Suppose to the contrary that there are two edges e_1 and e_2 such that $c_{(e_1,e_2)}(G) \le n-2$. Since $e(G) \ge \binom{n-1}{2} + 3$, by similar discussion as above, we have that there is a Hamilton path $P = u_1 u_2 \ldots u_n$ containing e_1 and e_2, where $e_1 = u_k u_{k+1}$ and $e_2 = u_l u_{l+1}$ $(2 \le k < l \le n-1)$, and $d(u_1) + d(u_n) \le n+1$. Clearly, $u_1 u_n \notin G$ and $u_2 u_n \notin G$ since $c_{(e_1,e_2)}(G) \le n-2$.

Note that $e(G) = e(G - u_n) + d(u_n)$ and $e(G) \ge \binom{n-1}{2} + 3$. We have that $d(u_n) \ge 3$. If $d(u_n) = 3$, then $G - u_n \cong K_{n-1}$. In this case, it's easy to see that any two edges lie on a common cycle of length more than $n-2$, a contradiction. Hence, we may assume that $d(u_n) \ge 4$.

If $u_n u_i \in G$, where $3 \le i \le n-1$, $i \ne k, l$, then $u_2 u_{i+1} \notin G$, for otherwise, $C = u_2 u_{i+1} \overrightarrow{P} u_n u_i \overleftarrow{P} u_2$ is a cycle containing e_1 and e_2 of order $n-1$, a contradiction. Hence, $N(u_2) \cap (N^+(u_n) \backslash \{u_{k+1}, u_{l+1}\}) = \emptyset$. Since $d(u_n) \ge 4$, $|N^+(u_n) \backslash \{u_{k+1}, u_{l+1}\}| \ge 2$. So

$$d(u_2) \le |V(G) \backslash \{u_2\}| - |N^+(u_n) \backslash \{u_{k+1}, u_{l+1}\}| \le (n-1) - 2 = n-3.$$

Thus,

$$\begin{aligned} e(G) &= e(G - \{u_1, u_2, u_n\}) + d(u_1) + d(u_2) + d(u_n) - e(G[\{u_1, u_2, u_n\}]) \\ &\le \binom{n-3}{2} + (n+1) + (n-3) - 1 \\ &< \binom{n-1}{2} + 3. \end{aligned}$$

This contradiction completes the proof of (ii), and of the theorem. □

The following theorem is a special case of Theorem 1.2 when $k = n$. We state it here in order to make the proof of Theorem 1.2 not too lengthy.

Theorem 3.3 *Let G be a 2-connected graph of order $n \ge 6$. Let $F^* = \{e | e \in G$ and $c_e(G) \le n-1\}$. If $|F^*| \ge 2$, then $e(G) \le f(n, n)$, where $f(n, n)$ is defined as in Theorem 1.2.*

Proof Without loss of generality, we can suppose that G is edge maximal with respect to the condition that $|F^*| \geq 2$. Then for any two nonadjacent vertices u and v of G, we have that $c_{e'}(G+uv) = n$ for some $e' \in F^*$. It means that there is a uv-path $P: u = u_1u_2\ldots u_n = v$ containing e', say $e' = u_ku_{k+1}$ $(1 \leq k \leq n-1)$ in G. Since $c_{e'}(G) \leq n-1$, we get that $N(u) \cap (N^+(v)\backslash\{u_{k+1}\}) = \emptyset$. Thus, $d(u) \leq (n-1)-(d(v)-1)$. That is, $d(u)+d(v) \leq n$ for any nonadjacent vertices u and v of G.

If G is isomorphic to the graph obtained from K_{n-1} by adding one vertex joined to t $(2 \leq t \leq n-1)$ vertices of K_{n-1}, then it's easy to see that there is at most one edge e such that $c_e(G) \leq n-1$, a contradiction. So there must exist four vertices, say $u_{i_1}, u_{i_2}, u_{i_3}$ and u_{i_4}, such that $u_{i_1}u_{i_2} \notin G$ and $u_{i_3}u_{i_4} \notin G$. Let $V' = \{u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}\}$. Then

$$\begin{aligned} e(G) &= e(G[V']) + e(V', V(G)\backslash V') + e(G-V') \\ &= \sum_{j=1}^{4} d(u_{i_j}) - e(G[V']) + e(G-V'). \end{aligned} \tag{3.1}$$

Note that $d(u_{i_1}) + d(u_{i_2}) \leq n$ and $d(u_{i_3}) + d(u_{i_4}) \leq n$. So $\sum_{j=1}^{4} d(u_{i_j}) \leq 2n$. If $\sum_{j=1}^{4} d(u_{i_j}) < 2n$, $e(G[V']) \geq 1$ or $e(G-V') < \binom{n-4}{2}$, then by (3.1), we have that $e(G) \leq 2n + \binom{n-4}{2} - 1 = f_3(n,n) \leq f(n,n)$. Thus, we can assume that $\sum_{j=1}^{4} d(u_{i_j}) = 2n$, V' is an independent set and $G-V' \cong K_{n-4}$. If there are two vertices of V', say u_{i_1} and u_{i_2}, such that $N(u_{i_1}) \neq N(u_{i_2})$, then there must exist a vertex w such that $w \in N(u_{i_1})\backslash N(u_{i_2})$ or $w \in N(u_{i_2})\backslash N(u_{i_1})$. Without loss of generality, we can assume that $w \in N(u_{i_1})\backslash N(u_{i_2})$. Then $V'' = \{u_{i_1}, u_{i_2}, w, u_{i_4}\}$ is a vertex set with $u_{i_1}u_{i_4} \notin G$, $u_{i_2}w \notin G$ and $e(G[V'']) \geq 1$. We may proceed as above to get that $e(G) \leq f(n,n)$. Hence, we can assume that all vertices of V' have the same neighborhood. Let $d(u_{i_j}) = t$, $j = 1, 2, 3, 4$.

Now we can see that G is isomorphic to the graph obtained from K_{n-4} by adding four isolated vertices each joined to the same t vertices of K_{n-4}. Since $\sum_{j=1}^{4} d(u_{i_j}) = 2n$, we get that $t = \frac{n}{2}$. It implies n is even, and $n \geq 8$ since $n-4 \geq t$. If $n \geq 12$, then $t \geq 6$. It is easy to check that $c_e(G) = n$ for any edge e of G, a contradiction. If $n = 8$ or 10, then $e(G) = \binom{n-4}{2} + 4t = \binom{n-4}{2} + 2n = f_2(n,n) \leq f(n,n)$. It completes the proof of Theorem 3.3. □

Proof of Theorem 1.2 Note that $F_G = \{e | e \in G \text{ and } c_e(G) \leq k-1\}$. Suppose that G is a 2-connected graph of order n $(n \geq 6)$ such that $|F_G| \geq 2$. We shall prove that $e(G) \leq f(n,k)$.

We apply induction on n $(n \geq k \geq 6)$. If $n = 6$, then $k = 6$ and $f(6,6) = 12$. By Theorem 1.1, if $e(G) > f_0(6,6) = 12$, then $F_G = \emptyset$. Since $|F_G| \geq 2$, we have that $e(G) \leq 12 = f(6,6)$. Assume that the result is true for those graphs of order less than n $(n > 6)$. Let G be a 2-connected graph of order n such that $|F_G| \geq 2$.

Claim 1 If G has a 2-vertex cut $\{u, v\}$ with $uv \in E(G)$, then $e(G) \leq f(n,k)$.

Assume that $G-\{u,v\}$ has s components, say H_i, $1\le i\le s$ ($s\ge 2$). Let $G_i=G[V(H_i)\cup\{u,v\}]$ and $n_i=|V(G_i)|$, $1\le i\le s$. We shall show that $|F_G\cap E(G_i)|\ge 2$ for some i ($1\le i\le s$).

If $c_{uv}(G)\le k-1$, then $uv\in F_G$. Since $|F_G|\ge 2$, there exists an edge $e'\in F_G$ and $e'\ne uv$. Without loss of generality, we can assume that $e'\in E(G_{i_0})$. Since $uv\in E(G_{i_0})$, $|F_G\cap E(G_{i_0})|\ge 2$. If $c_{uv}(G)\ge k$, then $c_{uv}(G_{j_0})\ge k$ for some j_0 ($1\le j_0\le s$). Let C be a longest cycle which contains uv in G_{j_0}. For any edge $e\notin E(G_{j_0})$, say $e\in E(G_l)$, $l\ne j_0$, since G_l is 2-connected and $e\ne uv$, e and uv must lie on a common cycle C' in G_l by Menger's Theorem. So $(C\cup C')-uv$ is a cycle containing e and with length more than k in G. Therefore, $F_G\subseteq E(G_{j_0})$. It means $|F_G\cap E(G_{j_0})|=|F_G|\ge 2$.

Without loss of generality, we can assume that $|F_G\cap E(G_1)|\ge 2$. Choose e_1 and e_2 from $F_G\cap E(G_1)$, such that $c_{(e_1,uv)}(G_1)=\max\{c_{(e,uv)}(G_1)|e\in F_G\cap E(G_1)\}$. Clearly, $e_1\ne uv$ and $c_{(e_1,uv)}(G_1)\ge 3$. Let $G_1'=G-H_1$ and $n_1'=|V(G_1')|$. We have that $n_1'=n-n_1+2$.

If $c_{(e_1,uv)}(G_1)=3$, without loss of generality, we may assume that $e_1=uw$, then we have that $d_{G_1}(v)=2$. Since $c_{e_1}(G)\ge c_{(e_1,uv)}(G_1)+c_{uv}(G_1')-2$ and $c_{e_1}(G)\le k-1$, we get that $c_{uv}(G_1')\le k-2$. Note that $n_1'\ge 3$, $k-1>3$ and G_1' is 2-connected. By Theorem 1.1,

$$e(G_1')\le f_0(n_1',k-1).$$

If $n_1=3$, then

$$e(G)=e(G_1)+e(G_1')-1\le \binom{3}{2}+f_0(n_1',k-1)-1=f_1(n,k)\le f(n,k).$$

If $n_1\ge 4$, note that $d_{G_1}(v)=2$, $|V(G_1-v)|\ge 3$ and G_1-v is 2-connected, then $c_{(e,uv)}(G_1)\ge 4$ for any edge $e\in E(G_1)\backslash\{e_1,uv\}$. By the choice of e_1, we have that $F_G\cap E(G_1)\subseteq\{e_1,uv\}$. Since $|F_G\cap E(G_1)|\ge 2$, $F_G\cap E(G_1)=\{e_1,uv\}$. That is, $c_{uv}(G)\le k-1$. And since $c_{uv}(G)\ge c_{uv}(G_1)=c_{e_1}(G_1-v)+1$, we get that $c_{e_1}(G_1-v)\le k-2$. Note that $|V(G_1-v)|=n_1-1\ge 3$ and $k-1>3$. By Theorem 1.1,

$$e(G_1-v)\le f_0(n_1-1,k-1).$$

Thus,

$$\begin{aligned}e(G)&=e(G_1)+e(G_1')-1\\&=e(G_1-v)+d_{G_1}(v)+e(G_1')-1\\&\le f_0(n_1-1,k-1)+2+f_0(n_1',k-1)-1\\&\le f_1(n,k)\le f(n,k).\end{aligned}$$

If $c_{(e_1,uv)}(G_1)\ge 4$, then $c_{uv}(G_1')\le c_{e_1}(G)-c_{(e_1,uv)}(G_1)+2\le k-3$. Note that $n_1\ge 4$, $n_1'\ge 3$ and $k-2>3$. By Theorem 1.1,

$$e(G_1')\le f_0(n_1',k-2)=g(n_1',k), \tag{3.2}$$

where $g(n_1', k) = q\binom{k-5}{2} + \binom{r}{2} + 2(n_1' - 2) + 1$, $n_1' - 2 = q(k-5) + r$, $q \geq 0$ and $0 \leq r < k - 5$.

We consider three cases.

Case 1 $n_1 \geq k$.

Since $c_{e_i}(G_1) \leq c_{e_i}(G) \leq k-1$ $(i = 1, 2)$, $|F_{G_1}| \geq 2$. Note that G_1 is 2-connected graph of order n_1, $k \leq n_1 < n$. By induction hypothesis, $e(G_1) \leq f(n_1, k)$. By (3.2) and Lemma 2.6,

$$e(G) = e(G_1) + e(G_1') - 1 \leq f(n_1, k) + g(n_1', k) - 1 \leq f(n, k).$$

Case 2 $n_1 = k - 1$.

If $e(G_1) \leq \binom{k-2}{2} + 2 = \binom{k-4}{2} + 2(n_1 - 2) + 1$, then by (3.2),

$$\begin{aligned} e(G) &= e(G_1) + e(G_1') - 1 \\ &\leq \binom{k-4}{2} + 2(n_1 - 2) + 1 + q\binom{k-5}{2} + \binom{r}{2} \\ &\quad +2(n_1' - 2) + 1 - 1, \end{aligned} \tag{3.3}$$

where $n_1' - 2 = q(k-5) + r$, $q \geq 0$ and $0 \leq r < k - 5$. By Lemma 2.3,

$$q\binom{k-5}{2} + \binom{r}{2} \leq q'\binom{k-4}{2} + \binom{r'}{2}, \tag{3.4}$$

where $n_1' - 2 = q'(k-4) + r'$, $q' \geq 0$ and $0 \leq r' < k - 4$. Using (3.4) in (3.3),

$$e(G) \leq (q' + 1)\binom{k-4}{2} + \binom{r'}{2} + 2(n-2) + 1 = f_1(n, k) \leq f(n, k).$$

Note that $n_1 = k-1 \geq 5$. If $e(G_1) = \binom{k-2}{2}+3$, then by Theorem 3.2, $c_{(e_1,uv)}(G_1) \geq n_1 - 1 = k-2$. Thus, $c_{uv}(G_1') \leq c_{e_1}(G) - c_{(e_1,uv)}(G_1) + 2 \leq (k-1)-(k-2)+2 = 3$. Note that $n_1' \geq 3$. By Theorem 1.1, $e(G_1') \leq 2(n_1' - 2) + 1$. Thus,

$$e(G) = e(G_1) + e(G_1') - 1 \leq \binom{k-2}{2} + 3 + 2(n_1' - 2) + 1 - 1. \tag{3.5}$$

Using $n_1' = n - n_1 + 2$, $n_1 = k - 1$ and $n \geq k$ in (3.5),

$$\begin{aligned} e(G) &\leq \binom{k-2}{2} + 2(n-k+1) + 3 \leq \binom{k-2}{2} + 3(n-k+2) \\ &= f_3(n, k) \leq f(n, k). \end{aligned}$$

If $e(G_1) > \binom{k-2}{2} + 3$, then by Theorem 3.2, $c_{(e_1,uv)}(G_1) = n_1 = k - 1$. Since $c_{uv}(G_1') \geq 3$, we have that $c_{e_1}(G) \geq c_{(e_1,uv)}(G_1) + c_{uv}(G_1') - 2 \geq k$. It's a contradiction.

Case 3 $4 \le n_1 < k-1$.
If $e(G_1) \le \binom{n_1-1}{2} + 3 = \binom{n_1-3}{2} + 2(n_1-2) + 2$, then

$$\begin{aligned} e(G) &= e(G_1) + e(G_1') - 1 \\ &\le \binom{n_1-3}{2} + 2(n_1-2) + 2 + q\binom{k-5}{2} + \binom{r}{2} + 2(n_1'-2) + 1 - 1. \end{aligned} \tag{3.6}$$

If $q \ge 1$, then by Corollary 2.4,

$$(q-1)\binom{k-5}{2} + \binom{r}{2} \le q_1\binom{k-4}{2} + \binom{r_1}{2}, \tag{3.7}$$

where $(q-1)(k-5) + r = q_1(k-4) + r_1$, $q_1 \ge 0$ and $0 \le r_1 < k-4$. Note that $0 < n_1 - 3 < k-4$. Then

$$\begin{aligned} \binom{n_1-3}{2} + \binom{k-5}{2} &= \binom{n_1-4}{2} + \binom{k-4}{2} - ((k-5) - (n_1-4)) \\ &\le \binom{n_1-4}{2} + \binom{k-4}{2} - 1. \end{aligned} \tag{3.8}$$

Since $0 \le n_1 - 4 < k-4$ and $0 \le r_1 < k-4$, by Lemma 2.5,

$$\binom{n_1-4}{2} + \binom{r_1}{2} \le q_2\binom{k-4}{2} + \binom{r_2}{2}, \tag{3.9}$$

where $(n_1-4) + r_1 = q_2(k-4) + r_2$, $q_2 \ge 0$ and $0 \le r_2 < k-4$. Using (3.7), (3.8) and (3.9) in (3.6),

$$\begin{aligned} e(G) &\le \binom{n_1-3}{2} + \binom{k-5}{2} + (q-1)\binom{k-5}{2} + \binom{r}{2} + 2(n-2) + 2 \\ &\le (q_1+q_2+1)\binom{k-4}{2} + \binom{r_2}{2} + 2(n-2) + 1. \end{aligned}$$

Clearly, $n-3 = (q_1+q_2+1)(k-4) + r_2$. So $f_1(n,k) = (q_1+q_2+1)\binom{k-4}{2} + \binom{r_2}{2} + 2(n-2) + 1$. Therefore, $e(G) \le f_1(n,k) \le f(n,k)$.

If $q = 0$, then $n_1' - 2 = r$. Note that $0 < r < k-4$ and $0 < n_1 - 3 < k-4$. By Lemma 2.5,

$$\binom{n_1-3}{2} + \binom{r}{2} < q_3\binom{k-4}{2} + \binom{r_3}{2}, \tag{3.10}$$

where $(n_1-3)+r=q_3(k-4)+r_3$, $q_3\geq 0$ and $0\leq r_3<k-4$. Note that $q=0$ and $n_1'=n-n_1+2$. Using (3.10) in (3.6),

$$\begin{aligned}
e(G) &\leq \binom{n_1-3}{2}+2(n_1-2)+\binom{r}{2}+2(n_1'-2)+2\\
&< q_3\binom{k-4}{2}+\binom{r_3}{2}+2(n-2)+2\\
&\leq q_3\binom{k-4}{2}+\binom{r_3}{2}+2(n-2)+1.
\end{aligned}$$

It is easy to see that $n-3=q_3(k-4)+r_3$. Hence, $f_1(n,k)=q_3\binom{k-4}{2}+\binom{r_3}{2}+2(n-2)+1$. So $e(G)\leq f_1(n,k)\leq f(n,k)$.

If $e(G_1)>\binom{n_1-1}{2}+3$, note that $n_1\geq 5$ since $\binom{4-1}{2}+3=\binom{4}{2}$, then by Theorem 3.2, $c_{(e_1,uv)}(G_1)=n_1$. So $c_{uv}(G_1')\leq c_{e_1}(G)-c_{(e_1,uv)}(G_1)+2\leq (k-1)-n_1+2=k-n_1+1$. Since $n_1<k-1$, $k-n_1+2>3$. By Theorem 1.1,

$$e(G_1')\leq f_0(n_1',k-n_1+2).$$

Let $f_0(n_1',k-n_1+2)=q_4\binom{k-n_1-1}{2}+\binom{r_4}{2}+2(n_1'-2)+1$, where $n_1'-2=q_4(k-n_1-1)+r_4$, $q_4\geq 0$ and $0\leq r_4<k-n_1-1$. Since $n_1'-2=n-n_1\geq k-n_1$, $q_4\geq 1$. Then

$$\begin{aligned}
e(G) &= e(G_1)+e(G_1')-1\\
&\leq \binom{n_1}{2}+\left[q_4\binom{k-n_1-1}{2}+\binom{r_4}{2}+2(n_1'-2)+1\right]-1\\
&= \left[\binom{n_1-2}{2}+2(n_1-2)+1\right]\\
&\quad+\left[\binom{k-n_1-1}{2}+(q_4-1)\binom{k-n_1-1}{2}+\binom{r_4}{2}+2(n_1'-2)\right] \qquad (3.11)
\end{aligned}$$

Since $4\leq n_1<k-1$, $0<k-n_1-1<k-4$. By Corollary 2.4,

$$(q_4-1)\binom{k-n_1-1}{2}+\binom{r_4}{2}\leq q_5\binom{k-4}{2}+\binom{r_5}{2}, \qquad (3.12)$$

where $(q_4-1)(k-n_1-1)+r_4=q_5(k-4)+r_5$, $q_5\geq 0$ and $0\leq r_5<k-4$. If $4\leq n_1<k-2$, then $0<n_1-2<k-4$ and $0<k-n_1-1<k-4$. By Lemma 2.5,

$$\binom{n_1-2}{2}+\binom{k-n_1-1}{2}\leq \binom{k-4}{2}+\binom{1}{2}=\binom{k-4}{2}. \qquad (3.13)$$

If $n_1=k-2$, then $\binom{n_1-2}{2}+\binom{k-n_1-1}{2}=\binom{k-4}{2}$. It means (3.13) holds for $4\leq n_1<k-1$. Using (3.12) and (3.13) in (3.11),

$$e(G)\leq (q_5+1)\binom{k-4}{2}+\binom{r_5}{2}+2(n-2)+1.$$

Clearly, $n-3=(q_5+1)(k-4)+r_5$. So $f_1(n,k)=(q_5+1)\binom{k-4}{2}+\binom{r_5}{2}+2(n-2)+1$. That is, $e(G)\leq f_1(n,k)\leq f(n,k)$. It completes the proof of Case 3, and so the proof of Claim 1.

Claim 2 Let $e_1,\ e_2\in F_G, uv\in E(G)$ and $uv\neq e_i\ (i=1,2)$. If $N_G(u)\cap N_G(v)=\emptyset$, then $e(G)\leq f(n,k)$.

If $k=n$, then by Theorem 3.3, $e(G)\leq f(n,n)$. So we can assume that $6\leq k\leq n-1$. Let $G'=G/uv$. We identify u and v with a new vertex w in G'. If $e_i\ (i=1,2)$ is not incident with u and v, then clearly $c_{e_i}(G')\leq c_{e_i}(G)\leq k-1$. If $e_i=ux$ (or vy), $i=1,2$, where $x\in N(u)-\{v\}$ ($y\in N(v)-\{u\}$), then it's easy to see that $c_{wx}(G')\leq c_{ux}(G)\leq k-1$ ($c_{wy}(G')\leq c_{vy}(G)\leq k-1$). So $|F_{G'}|\geq 2$. Since $|V(G')|=n-1>3$, G' isn't isomorphic to K_3. By Claim 1 and Lemma 2.2, G' is 2-connected. Note that $|V(G')|=n-1$, and $6\leq k\leq n-1$. Then by induction hypothesis, $e(G')\leq f(n-1,k)$. Thus, $e(G)=e(G')+1\leq f(n-1,k)+1\leq f(n,k)$. It completes the proof of Claim 2.

Let $\mathcal{G}=\{G|G$ is 2-connected graph of order n with $|F_G|\geq 2\}$, and $m^*=\max\{e(G)|G\in\mathcal{G}\}$. We only need to show that $m^*\leq f(n,k)$. For $G\in\mathcal{G}$, let $F_G=\{e_1,e_2,\ldots,e_l\}$, where $e_i=u_iv_i$ for $1\leq i\leq l$. We define $l_{(e,e')}(G)$ to be the minimum length of cycles containing e and e' in G. Let $l(G)=\min\{l_{(e_i,e_j)}(G)|e_i,e_j\in F_G,\ 1\leq i<j\leq l\}$. Now we choose $G_a\in\mathcal{G}$ and $e(G_a)=m^*$, and subject to this, let $l(G_a)$ be as small as possible. By Claim 1, we can assume that G_a has no vertex cut $\{u,v\}$ with $uv\in G_a$. We shall show that $l(G_a)=3$.

Since G_a is 2-connected, any two distinct edges must lie on a common cycle by Menger's theorem. So $l(G_a)\geq 3$. Without loss of generality, we may assume that $l_{(e_1,e_2)}(G_a)=l(G_a)$. Let C be a cycle containing e_1 and e_2 with $e(C)=l(G_a)$. If $l(G_a)\geq 4$, then let xy be an edge of C with $xy\neq e_i$ for $i=1,2$. If $N_{G_a}(x)\cap N_{G_a}(y)=\emptyset$, then by Claim 2, $e(G_a)\leq f(n,k)$; otherwise, we do edge-switching from y to x in G_a. Let $G'_a=G_a[y\to x]$. Then by Lemma 2.2 and our assumption, we have that G'_a is 2-connected. If y is not incident with e_1 and e_2, then by Lemma 2.1 (a) and (c), we get that $c_{e_i}(G'_a)\leq c_{e_i}(G_a)\leq k-1$, for $i=1,2$. Let x' be another neighbor of y in C. Note that $xx'\notin G_a$ by the choice of C. Then $C'=(C-\{y\})\cup\{xx'\}$ is a cycle containing e_1 and e_2 in G'_a with $e(C')<e(C)$. So $l(G'_a)\leq l_{(e_1,e_2)}(G'_a)<l_{(e_1,e_2)}(G_a)=l(G_a)$. If y is an endvertex of some $e_i\ (i=1,2)$, say $e_2\ (e_2=u_2v_2)$ and $y=u_2$, then by Lemma 2.1, $c_{e_1}(G'_a)\leq c_{e_1}(G_a)\leq k-1$. Considering the edge $e_2=yv_2$, it follows from Lemma 2.1 (b) that $c_{xv_2}(G'_a)\leq c_{yv_2}(G_a)=c_{e_2}(G_a)\leq k-1$ since $v_2\in N_{G_a}(y)\backslash\{x\}$. It is easy to see that $(C-\{y\})\cup\{xv_2\}$ is a cycle containing e_1 and xv_2 in G'_a. So $l(G'_a)\leq l_{(e_1,\ xv_2)}(G'_a)<l_{(e_1,e_2)}(G_a)=l(G_a)$. In either case, we have that $|F_{G'_a}|\geq 2$ and $l(G'_a)<l(G_a)$, which contradicts to our choice of G_a. Hence, $l(G_a)=3$.

Let $\mathcal{G}'=\{G|G\in\mathcal{G},\ e(G)=m^*$ and $l(G_a)=3\}$. By the discussion above, we know that $\mathcal{G}'\neq\emptyset$. For $G\in\mathcal{G}'$, define $q(G)=\max\{d(u)|u\in G$ and u is a common endvertex of e_i and e_j, where $e_i,e_j\in F_G$, and $l(e_i,e_j)=3, 1\leq i<j\leq l\}$. Choose $G_b\in\mathcal{G}'$ such that $q(G_b)$ is as large as possible. By Claim 1, we may assume that G_b has no vertex cut $\{u,v\}$ with $uv\in G_b$. We shall show that $q(G_b)=n-1$.

Without loss of generality, we may assume that $l(e_1, e_2) = l(G_b) = 3$, $u_1 = u_2 = u$, and $d_{G_b}(u) = q(G_b)$. Clearly, $v_1v_2 \in G_b$. If $d_{G_b}(u) < n-1$, then there exists a vertex z such that $uz \notin G_b$. Since $\{v_1, v_2\}$ is not a vertex cut of G_b by our assumption, there must exist a path from u to z, which doesn't pass through v_1 and v_2. Let $P = uz_2z_3 \ldots z_tz$ $(t \geq 2)$ be a shortest path from u to z with $v_i \notin P$ $(i = 1, 2)$. Clearly, $uz_3 \notin G_b$. If $N_{G_b}(u) \cap N_{G_b}(z_2) = \emptyset$, then by Claim 2, $e(G_b) \leq f(n, k)$; otherwise, let $G'_b = G_b[z_2 \to u]$. By Lemma 2.2 and our assumption, we get that G'_b is 2-connected. By Lemma 2.1 (a), $c_{e_i}(G'_b) \leq c_{e_i}(G_b) \leq k-1$, for $i = 1, 2$. That is, $|F_{G'_b}| \geq 2$. Since $v_1v_2 \in G'_b$, we have that $l_{(e_1,e_2)}(G'_b) = 3$, which implies that $l(G'_b) = 3$. It's also easy to see that $e(G'_b) = e(G_b) = m^*$, and $N_{G_b}(u) \subset N_{G'_b}(u)$ since $z_3 \in N_{G_b}(z_2) \backslash (N_{G_b}(u) \cup \{u\})$. Therefore, $G'_b \in \mathcal{G}'$ and $q(G'_b) \geq d_{G'_b}(u) > d_{G_b}(u) = q(G_b)$, a contradiction. Hence, $d_{G_b}(u) = n-1$.

Now G_b is a 2-connected graph of n vertices and m^* edges, $c_{uv_i}(G_b) \leq k-1$ $(i = 1, 2)$, $v_1v_2 \in G_b$ and $d_{G_b}(u) = n-1$. Let $G''_b = G_b - u$. If G''_b has a cut vertex w, then $\{u, w\}$ is a vertex cut of G_b with $uw \in G_b$. It contradicts to our assumption. So G''_b is 2-connected. Let C'' be a longest cycle of G''_b. We shall show that $e(C'') < k-1$. Let P'' be a path from v_1 to C'' in G''_b, and let w' be the first vertex of P'' on C''. Note that $w' = v_1$ when $v_1 \in C''$. Then $C''' = uv_1\overrightarrow{P''}w'\overrightarrow{C}w'^{-}u$ is a cycle containing $e_1 = uv_1$ with $e(C''') > e(C'')$ in G_b, where w'^{-} is the vertex on C'' immediately before w' according to the orientation of C''. Then $e(C'') < e(C''') \leq c_{uv_1}(G_b) \leq k-1$. By Theorem 3.1, we get that $e(G''_b) \leq t(n-1, k-2)$. Hence,

$$\begin{aligned} m^* = e(G_b) = e(G''_b) + d(u) &\leq t(n-1, k-2) + (n-1) \\ &= \max\{f_2(n,k), f_3(n,k)\} \leq f(n,k). \end{aligned}$$

This ends the proof of Theorem 1.2. □

Acknowledgements We are grateful for Reza Naserasr's kind help.
This research was supported by the Natural Science Foundation of Ningxia University under grant number ZR1421.

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