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ORIGINAL PAPER

Bounds for Judicious Balanced Bipartitions of Graphs

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Abstract
A bipartition of the vertex set of a graph is called balanced if the sizes of the sets in the bipartition differ by at most one. Bollobás and Scott proved that every regular graph with m edges admits a balanced bipartition V_1, V_2 of $V(G)$ such that $\max\{e(V_1), e(V_2)\} < \frac{m}{4}$. Only allowing $\Delta(G) - \delta(G)$ =1 and 2, Yan and Xu, and Hu, He and Hao, respectively showed that a graph G with n vertices and m edges has a balanced bipartition V_1, V_2 of $V(G)$ such that $\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} + O(n)$. In this paper, we give an upper bound for balanced bipartition of graphs G with $\Delta(G) - \delta(G) = t - 1$, $t \geq 2$ is an integer. Our result extends the conclusions above.

Keywords Bipartition · Balanced bipartition · Judicious bipartition · Graph and degree

Mathematics Subject Classification 05C35 · 05C75

1 Introduction

Graphs considered in this paper are finite and simple. For general theoretic notations, we follow Bondy and Murty [4]. Throughout the paper, the letter G denotes a graph. For $u \in V(G)$, denote by $N_G(u)$ and $d_G(u)$ the set of neighbors of u and the degree of u in G, respectively. The *maximum degree* of G is denoted by $\Delta(G)$ and *minimum*

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degree of G is denoted by $\delta(G)$ analogously. We use $e(G)$ to denote the number of edges of G.

Let G be a graph and k a positive integer. A *k-partition* of G is a partition of $V(G)$ into k pairwise disjoint nonempty sets. A 2-partition is usually referred to as a *bipartition*. Let $V_1, V_2, \ldots, V_k$ be a k-partition of G. For $1 \le i \le k$, we use $e(V_i)$ to denote the number of edges with both ends in V_i, and use $e(V_i, V_j)$ to denote the number of edges with one end in V_i and the other in V_j. For $\{i_1, i_2, \ldots, i_h\} \subseteq \{1, 2, \ldots, k\}$, let $e(V_{i_1}, V_{i_2}, \ldots, V_{i_h}) = \sum_{i \neq j \in \{i_1, i_2, \ldots, i_h\}} e(V_i, V_j)$, and accordingly we call $e(V_1, V_2, \ldots, V_k)$ the size of the partition.

The maximum bipartite subgraph (MBS) problem is a classic problem in graph theory. Given a graph G, the goal of MSB is to ask for a bipartition V_1, V_2 of $V(G)$ maximizing $e(V_1, V_2)$. In theory, this problem equivalently finding the minimum of $e(V_1)+e(V_2)$ over all partitions $V(G) = V_1 \cup V_2$. *Judicious partition* problem [2] asks for a bipartition of the vertex set of a graph into subsets such that several quantities are optimized simultaneously. The *Bottleneck Bipartition* problem [7] is such an example: Given a graph, find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$. Porter [6] proved that for any graph G with m edges there is a bipartition V_1, V_2 of $V(G)$ such that $\max\,\{e(V_1, V_2)\} \le \frac{m}{4} + O(\sqrt{m})$. Then, Xu and Yu [8] extended this result to k-partition.

Bollobás and Scott first studied Bottleneck problem with the additional requirement that the bipartition is balanced and posed the following conjecture.

Conjecture 1 [2] *Let G be a graph with minimum degree at least 2. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that* $\max\{e(V_1), e(V_2)\} \le e(G)/3$.

Xu and Yu [8] first made a lot of work for this conjecture [9,10] and then confirmed this conjecture [11].

However, Bollobás and Scott gave the following theorem which not only implies Conjecture 1 for *regular graphs*(every vertex has the same degree) but also reduces the upper bound $e(G)/3$ to $e(G)/4$.

Theorem 1 [3] *Let $d \ge 2$ be an integer and G a d-regular graph. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that*

(1) $\max\{e(V_1), e(V_2)\} \le \frac{1}{4}((d-1)/d)e(G)$ *when d is odd,*
(2) $\max\{e(V_1), e(V_2)\} \le \frac{1}{4}(d/(d+1))e(G)$ *when d is even and $|V(G)|$ is even, and*
(3) $\max\{e(V_1), e(V_2)\} \le \frac{1}{4}(d/(d+1))e(G) + d/4$ *when d is even and $|V(G)|$ is odd.*

Yan and Xu [12] generalized Theorem 1 to graphs with $\Delta(G) - \delta(G) = 1$ as the following theorem shows.

Theorem 2 [12] *Let $d \ge 2$ be an integer and G a graph with n_1 vertices of degree d and $n_2 = |V(G)| - n_1$ vertices of degree $d-1$. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that*

(1) $\max\{e(V_1), e(V_2)\} \le e(G)/4 - n_1/8$ *when d is odd and $|V(G)|$ is even,*
(2) $\max\{e(V_1), e(V_2)\} \le e(G)/4 - n_1/8 + (d-1)/8$ *when d is odd and $|V(G)|$ is odd,*

(3) $\max\{e(V_1), e(V_2)\} \le e(G)/4 + n_2/8$ *when* d *is even and* $|V(G)|$ *is even, and*
(4) $\max\{e(V_1), e(V_2)\} \le e(G)/4 + n_2/8 + d/8$ *when* d *is even and* $|V(G)|$ *is odd.*

Furthermore, in [5], Hu, He and Hao extended Yan's result to graphs with $\Delta(G) - \delta(G) = 2$.

Theorem 3 [5] *Let* $d \ge 2$ *be an integer and* G *a graph with* n_1 *vertices of degree* $d-1$*,* n_2 *vertices of degree* d *and* n_3 *vertices of degree* $d+1$*. Then* $V(G)$ *admits a balanced bipartition* V_1, V_2 *such that*

(1) $\max\{e(V_1), e(V_2)\} \le e(G)/4 - (n_2 + 2n_3)/8 + \alpha/2$ *when* $|V(G)|$ *is even and* d *is odd,*
(2) $\max\{e(V_1), e(V_2)\} \le e(G)/4 - (n_2 + 2n_3)/8 + \alpha/2 + (d-1)/8$ *when* $|V(G)|$ *is odd and* d *is odd,*
(3) $\max\{e(V_1), e(V_2)\} \le e(G)/4 + (n_1 - n_3)/8 + \alpha/2$ *when* $|V(G)|$ *is even and* d *is even, and*
(4) $\max\{e(V_1), e(V_2)\} \le e(G)/4 + (n_1 - n_3)/8 + \alpha/2 + d/8$ *when* $|V(G)|$ *is odd and* d *is even, where*

$$\alpha = \begin{cases} n_{13}, \text{ if } \max e\{(V_1), e(V_2)\} = e(V_1), \\ n_{23}, \text{ if } \max e\{(V_1), e(V_2)\} = e(V_2). \end{cases}$$

Here, n_{i3} *denotes the number of vertices in* V_i *with degree* $d+1$*,* $1 \le i \le 2$*.*

In this paper, we further generalize the result in [5] to general graphs with the following theorem. For convenience, let $\Delta(G) = \Delta$, $\delta(G) = \delta$ and $N_G(u) = N(u)$.

Theorem 4 *Let* G *be a graph with* n *vertices and* m *edges. Suppose that* $\Delta - \delta = t - 1, t \ge 2$ *and* n_i *is the number of vertices in* G *with degree* $\delta + i - 1$*,* $1 \le i \le t$*. Then* G *has a balanced bipartition* V_1, V_2*, such that*

(1) $\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{\sum_{i=2}^{t}(i-1)n_i}{8} + \frac{\alpha}{2}$ *when* n *is even and* δ *is even,*
(2) $\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{\sum_{i=2}^{t}(i-1)n_i}{8} + \frac{\alpha}{2} + \frac{\delta}{8}$ *when* n *is odd and* δ *is even,*
(3) $\max\{e(V_1), e(V_2)\} \le \frac{m}{4} + \frac{n_1 - \sum_{i=3}^{t}(i-2)n_i}{8} + \frac{\alpha}{2}$ *when* n *is even and* δ *is odd, and*
(4) $\max\{e(V_1), e(V_2)\} \le \frac{m}{4} + \frac{n_1 - \sum_{i=3}^{t}(i-2)n_i}{8} + \frac{\alpha}{2} + \frac{\delta+1}{8}$ *when* n *is odd and* δ *is odd, where*

$$\alpha = \begin{cases} \sum_{i=3}^{t}(i-2)n_{1,i}, \text{ if } \max\{e(V_1), e(V_2)\} = e(V_1), \\ \sum_{i=3}^{t}(i-2)n_{2,i}, \text{ if } \max\{e(V_1), e(V_2)\} = e(V_2), \end{cases}$$

and $n_{j,i}$ *is the number of vertices in* V_j *with degree* $\delta + i - 1$*,* $1 \le j \le 2$ *and* $1 \le i \le t$*.*

It is important to note that Theorem 4 is equivalent to Theorem 2 (resp. 3) when $t = 2$(resp. $t = 3$).

2 Proof of Theorem 4

In this section, we shall give a proof of Theorem 4. As described in this theorem, there are four cases to be handed. For the sake of clarity, we divide the proof into four parts.

Suppose V_1, V_2 is a balanced bipartition of $V(G)$ with $e(V_1, V_2)$ maximum among such partitions. Assume, without loss of generality, that $e(V_1) \geq e(V_2)$ and $t = 2k$ is even, since the other cases could be handled by the same way.

Part 1. n **is even and** δ **is even.**

Since n is even, $|V_1| = |V_2| = \frac{n}{2}$. We consider the following cases.

Case 1.1 $|N(v) \cap V_1| \leq |N(v) \cap V_2|$ for all $v \in V_1$.

In this case,

$$\begin{aligned} 2e(V_1) \leq & \frac{\delta}{2}n_{1,1} + \frac{(\delta+1)-1}{2}n_{1,2} + \frac{\delta+2}{2}n_{1,3} + \frac{(\delta+3)-1}{2}n_{1,4} + \cdots \\ & + \frac{\delta+(2k-2)}{2}n_{1,2k-1} + \frac{\delta+(2k-1)-1}{2}n_{1,2k} \\ = & \frac{\delta}{2}\cdot\frac{n}{2} + (n_{1,3}+n_{1,4}) + 2(n_{1,5}+n_{1,6}) + \cdots + (k-1)(n_{1,2k-1}+n_{1,2k}). \end{aligned}$$

It follows that

$$e(V_1) \leq \frac{\delta}{4}\cdot\frac{n}{2} + \frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2}. \tag{1}$$

By Handshaking Lemma,

$$\begin{aligned} 2m &= \delta\cdot n_1 + (\delta+1)\cdot n_2 + \cdots + (\delta+2k-1)\cdot n_{2k} \\ &= \delta\cdot n + n_2 + 2n_3 + \cdots + (2k-1)n_{2k} \\ &= \delta\cdot n + \sum_{j=2}^{2k}(j-1)n_j. \end{aligned} \tag{2}$$

According to Eqs. (1) and (2), we obtain that

$$\begin{aligned} e(V_1) &\leq \frac{2m - \sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2} \\ &\leq \frac{m}{4} - \frac{\sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2}. \end{aligned}$$

Case 1.2 There is a vertex v_1 in V_1 such that $|N(v_1) \cap V_1| > |N(v_1) \cap V_2|$.

We first assert that for all $w \in V_2$,

$$|N(w) \cap V_2| < |N(w) \cap V_1|. \tag{3}$$

Suppose, to the contrary, that there is $v_2 \in V_2$ such that $|N(v_2) \cap V_2| \geq |N(v_2) \cap V_1|$. Then, let $V_1^{'} = (V_1 \backslash \{v_1\}) \cup \{v_2\}$, $V_2^{'} = (V_2 \backslash \{v_2\}) \cup \{v_1\}$. We get a balanced bipartition $V_1^{'}, V_2^{'}$ of $V(G)$ with

$$\begin{aligned} e(V_1^{'}, V_2^{'}) &\geq e(V_1, V_2) + (|N(v_1) \cap V_1| - |N(v_1) \cap V_2|) \\ &\quad + (|N(v_2) \cap V_2| - |N(v_2) \cap V_1|) \\ &\geq e(V_1, V_2) + 1 \end{aligned}$$

This is a contradiction to the maximality of $e(V_1, V_2)$. Next, using Eq. (3), we deduce that

$$\begin{aligned} 2e(V_2) \leq &\frac{\delta - 2}{2} n_{2,1} + \frac{(\delta + 1) - 1}{2} n_{2,2} + \frac{(\delta + 2) - 2}{2} n_{2,3} + \cdots \\ &+ \frac{\delta + (2k-2) - 2}{2} n_{1,2k-1} + \frac{\delta + (2k-1) - 1}{2} n_{1,2k} \\ = &\frac{\delta}{2} \cdot \frac{n}{2} - n_{2,1} + (n_{2,4} + n_{2,5}) + 2(n_{2,6} + n_{2,7}) + \cdots + (k-1) n_{2,2k}, \end{aligned}$$

which yields that

$$e(V_2) \leq \frac{\delta}{4} \cdot \frac{n}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k} (j-2)(n_{2,2j-2} + n_{2,2j-1}) + (k-1) n_{2,2k}}{2}. \tag{4}$$

Based on Handshaking Lemma,

$$\begin{aligned} 2e(V_1) + e(V_1, V_2) &= \delta n_{1,1} + (\delta + 1) n_{1,2} + \cdots + (\delta + 2k - 1) n_{1,2k} \\ &= (\delta + 1)|V_1| + n_{1,3} + 2n_{1,4} + \cdots + (2k-2) n_{1,2k} - n_{1,1}; \quad (5) \\ 2e(V_2) + e(V_1, V_2) &= \delta n_{2,1} + (\delta + 1) n_{2,2} + \cdots + (\delta + 2k - 1) n_{2,2k} \\ &= (\delta + 1)|V_2| + n_{2,3} + 2n_{2,4} + \cdots + (2k-2) n_{2,2k} - n_{2,1}. \quad (6) \end{aligned}$$

Thereby,

$$e(V_1) - e(V_2) = \frac{\left(\sum_{j=3}^{2k} (j-2) n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k} (j-2) n_{2,j} - n_{2,1}\right)}{2}. \tag{7}$$

Combining Eqs. (2), (4) and (7), we obtain that

$$
\begin{aligned}
e(V_1) &= e(V_2) + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} \\
&\le \frac{\delta}{4}\cdot\frac{n}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1}) + (k-1)n_{2,2k}}{2} \\
&\quad + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} \\
&= \frac{\delta}{4}\cdot\frac{n}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1}) + (k-1)n_{2,2k} - \sum_{j=3}^{2k}(j-2)n_{2,j}}{2} \\
&\quad + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}}{2} \\
&\le \frac{\delta}{4}\cdot\frac{n}{2} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} \\
&= \frac{m}{4} - \frac{\sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2}.
\end{aligned}
$$

Part 2. n is odd and δ is even.

We distinguish this part into the following two cases.

Case 2.1 $|V_1| = \frac{n+1}{2}$, $|V_2| = \frac{n-1}{2}$.

It is claimed that for all $v \in V_1$, $|N(v)\cap V_1| \le |N(v)\cap V_2|$. On the contrary, assume that there is a vertex $v_1 \in V_1$ such that $|N(v_1)\cap V_1| > |N(v_1)\cap V_2|$. Then, we could increase the size of the partition by moving v_1 from V_1 to V_2. This is a contradiction to the maximality of $e(V_1, V_2)$. Therefore,

$$
\begin{aligned}
2e(V_1) &\le \frac{\delta}{2}n_{1,1} + \frac{(\delta+1)-1}{2}n_{1,2} + \frac{\delta+2}{2}n_{1,3} + \cdots \\
&\quad + \frac{\delta+(2k-2)}{2}n_{1,2k-1} + \frac{\delta+(2k-1)-1}{2}n_{1,2k} \\
&= \frac{\delta}{2}\cdot\frac{n+1}{2} + (n_{1,3}+n_{1,4}) + \cdots + (k-1)(n_{1,2k-1}+n_{1,2k}),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
e(V_1) &\le \frac{\delta}{4}\cdot\frac{n+1}{2} + \frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2} \\
&= \frac{2m - \sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2} + \frac{\delta}{8} \\
&\le \frac{m}{4} - \frac{\sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} + \frac{\delta}{8}.
\end{aligned}
$$

Case 2.2 $|V_1| = \frac{n-1}{2}, |V_2| = \frac{n+1}{2}$.

If there is a vertex $v_2' \in V_2$ such that $|N(v_2') \cap V_1| = |N(v_2') \cap V_2|$, then we set $V_1^{'} = V_1 \cup \{v_2'\}$ and $V_2^{'} = V_2 \backslash \{v_2'\}$. Thus, we get a balanced bipartition as depicted in Case 2.1.

Now, we assume that $|N(w) \cap V_1| \neq |N(w) \cap V_2|$ for all $w \in V_2$. In fact, $|N(w) \cap V_1| > |N(w) \cap V_2|$, for all $w \in V_2$. Otherwise, there is a vertex $v_2 \in V_2$ such that $|N(v_2) \cap V_1| < |N(v_2) \cap V_2|$. Then, we could increase the size of the partition by moving v_2 from V_2 to V_1. So,

$$e(V_2) \leq \frac{\delta}{4} \cdot \frac{n+1}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2} + n_{2,2j-1}) + (k-1)n_{2,2k}}{2}. \quad (8)$$

Again, by Eqs. (5) and (6), we have that

$$e(V_1) - e(V_2) = \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} - \frac{\delta+1}{2}. \quad (9)$$

By making use of Eqs. (2), (8) and (9), we have

$$\begin{aligned}
e(V_1) &= e(V_2) + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} - \frac{\delta+1}{2} \\
&\leq \frac{\delta}{4} \cdot \frac{n+1}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2} + n_{2,2j-1}) + (k-1)n_{2,2k}}{2} \\
&\quad + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} - \frac{\delta+1}{2} \\
&\leq \frac{\delta}{4} \cdot \frac{n+1}{2} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} \\
&= \frac{m}{4} - \frac{\sum_{j=2}^{2k}(j-1)n_j}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} + \frac{\delta}{8}.
\end{aligned}$$

Part 3. ***n* is even and δ is odd.**

We notice that $|V_1| = \frac{n}{2}, |V_2| = \frac{n}{2}$ since n is even. Also, there are two cases to be treated.

Case 3.1 $|N(v) \cap V_1| < |N(v) \cap V_2|$ for all $v \in V_1$.

Under this case,

$$\begin{aligned}2e(V_1) &\le \frac{\delta-1}{2}n_{1,1}+\frac{(\delta+1)-2}{2}n_{1,2}+\frac{(\delta+2)-1}{2}n_{1,3}+\cdots\\&\quad+\frac{\delta+(2k-2)-1}{2}n_{1,2k-1}+\frac{\delta+(2k-1)-2}{2}n_{1,2k}\\&=\frac{\delta-1}{2}\cdot\frac{n}{2}+(n_{1,3}+n_{1,4})+2(n_{1,5}+n_{1,6})+\cdots\\&\quad+(k-1)(n_{1,2k-1}+n_{1,2k})\\&\le\frac{\delta-1}{4}\cdot\frac{n}{2}+\frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2}\\&\le\frac{\delta+1-2}{4}\cdot\frac{n}{2}+\frac{\sum_{j=2}^{k}(j-1)(n_{1,2j-1}+n_{1,2j})}{2}.\end{aligned}$$

In addition,

$$\begin{aligned}2m &= \delta\cdot n_1+(\delta+1)\cdot n_2+\cdots+(\delta+2k-1)\cdot n_{2k}\\&=(\delta+1)\cdot n+n_3+\cdots+(2k-2)n_{2k}-n_1\\&=(\delta+1)\cdot n+\sum_{j=2}^{2k}(j-2)n_j-n_1.\end{aligned}\tag{10}$$

Thus,

$$e(V_1)\le\frac{m}{4}-\frac{\sum_{j=3}^{2k}(j-2)n_j-n_1}{8}+\frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2}.$$

Case 3.2 There is a vertex $v_1\in V_1$ such that $|N(v_1)\cap V_1|\ge|N(v_1)\cap V_2|$.

By an argument similar to Case 1.2, we obtain that for all $w\in V_2$,

$$|N(w)\cap V_1|\ge|N(w)\cap V_2|.\tag{11}$$

By Eq. (11),

$$\begin{aligned}2e(V_2)&\le\frac{\delta-1}{2}n_{2,1}+\frac{\delta+1}{2}n_{2,2}+\frac{(\delta+2)-1}{2}n_{2,3}+\frac{\delta+3}{2}n_{2,4}\\&\quad+\cdots+\frac{\delta+(2k-2)-1}{2}n_{2,2k-1}+\frac{\delta+(2k-1)}{2}n_{2,2k}\\&=\frac{\delta+1}{2}\frac{n}{2}-n_{2,1}+\cdots+(k-2)(n_{2,2k-2}+n_{2,2k-1})+(k-1)n_{2,2k}.\end{aligned}$$

As a consequence,

$$e(V_2)\le\frac{\delta+1}{4}\cdot\frac{n}{2}-\frac{n_{2,1}}{2}+\frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1})+(k-1)n_{2,2k}}{2}.\tag{12}$$

With the aid of Eqs. (7), (10) and (12), we obtain that

$$\begin{aligned} e(V_1) &= e(V_2) + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} \\ &\le \frac{\delta+1}{4}\cdot\frac{n}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1}) + (k-1)n_{2,2k}}{2} \\ &\quad + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} \\ &\le \frac{\delta+1}{4}\cdot\frac{n}{2} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} \\ &= \frac{m}{4} - \frac{\sum_{j=3}^{2k}(j-2)n_j - n_1}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2}. \end{aligned}$$

Part 4. n **is odd and** δ **is odd.**

Similarly, we consider the following two cases.

Case 4.1 $|V_1| = \frac{n+1}{2}$ and $|V_2| = \frac{n-1}{2}$.

According to the discussion of Case 2.1, we claim that $|N(v)\cap V_1| \le |N(v)\cap V_2|$ for all $v \in V_1$. Therefore,

$$\begin{aligned} 2e(V_1) &\le \frac{\delta-1}{2}n_{1,1} + \frac{\delta+1}{2}n_{1,2} + \frac{(\delta+2)-1}{2}n_{1,3} + \cdots \\ &\quad + \frac{\delta+(2k-2)-1}{2}n_{1,2k-1} + \frac{\delta+(2k-1)}{2}n_{1,2k} \\ &= \frac{\delta+1}{2}\cdot\frac{n+1}{2} - n_{1,1} + (n_{1,4}+n_{1,5}) + \cdots \\ &\quad + (k-2)(n_{1,2k-2}+n_{1,2k-1}) + (k-1)n_{1,2k}. \end{aligned}$$

It follows that

$$\begin{aligned} e(V_1) &\le \frac{\delta+1}{4}\cdot\frac{n+1}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{1,2j-2}+n_{1,2j-1}) + (k-1)n_{1,2k} - n_{1,1}}{2} \\ &\le \frac{m}{4} - \frac{\sum_{j=3}^{2k}(j-2)n_j - n_1}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} + \frac{\delta+1}{8}. \end{aligned}$$

Case 4.2 $|V_1| = \frac{n-1}{2}$ and $|V_2| = \frac{n+1}{2}$.

A similar argument to Case 2.2 deduces that for all $w \in V_2$

$$|N(w)\cap V_1| \ge |N(w)\cap V_2|.$$

Therefore,

$$e(V_2) \le \frac{\delta+1}{4}\cdot\frac{n+1}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1}) + (k-1)n_{2,2k}}{2},$$

which combining Eqs. (9) and (10) imply that

$$
\begin{aligned}
e(V_1) &= e(V_2) + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} - \frac{\delta+1}{2} \\
&\le \frac{\delta+1}{4}\cdot\frac{n+1}{2} - \frac{n_{2,1}}{2} + \frac{\sum_{j=3}^{k}(j-2)(n_{2,2j-2}+n_{2,2j-1}) + (k-1)n_{2,2k}}{2} \\
&\quad + \frac{\left(\sum_{j=3}^{2k}(j-2)n_{1,j} - n_{1,1}\right) - \left(\sum_{j=3}^{2k}(j-2)n_{2,j} - n_{2,1}\right)}{2} - \frac{\delta+1}{2} \\
&\le \frac{\delta+1}{4}\cdot\frac{n+1}{2} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} \\
&= \frac{m}{4} - \frac{\sum_{j=3}^{2k}(j-2)n_j - n_1}{8} + \frac{\sum_{j=3}^{2k}(j-2)n_{1,j}}{2} + \frac{\delta+1}{8}.
\end{aligned}
$$

Here, we establish the 4 Parts. Consequently, the proof of Theorem 4 is finished.

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