

On the Extendability of Quasi-Strongly Regular Graphs with Diameter 2

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Received: 21 August 2017 / Revised: 20 January 2018 / Published online: 30 May 2018
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Abstract Let ℓ denote a non-negative integer and let Γ be a connected graph of even order at least $2\ell + 2$. It is said that Γ is ℓ -extendable if it contains a matching of size ℓ and if every such matching is contained in a perfect matching of Γ . A connected regular graph Γ is *quasi-strongly regular* with parameters $(n, k, \lambda; \mu_1, \mu_2, \dots, \mu_s)$, if it is a k -regular graph on n vertices, such that any two adjacent vertices have exactly λ common neighbours and any two distinct and non-adjacent vertices have exactly μ_i common neighbours for some $1 \leq i \leq s$. The *grade* of Γ is the number of indices $1 \leq i \leq s$ for which there exist two distinct and non-adjacent vertices in Γ with μ_i common neighbours. In this paper we study the extendability of quasi-strongly regular

Štefko Miklavič, Primož Šparl: Supported in part by “Agencija za raziskovalno dejavnost Republike Slovenije”, research program P1-0285 and research projects J1-4010 and J1-4021.

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graphs of diameter 2 and grade 2. In particular, we classify the 2-extendable members of this class of graphs.

Keywords Regular graph · Quasi-strongly regular graph · Diameter 2 · ℓ -extendable

1 Introductory Remarks

Throughout this paper graphs are assumed to be finite and simple. We first recall the definition of ℓ -extendable graphs, introduced in 1980 by Plummer [14]. Let ℓ denote a non-negative integer and let Γ be a connected graph of even order at least $2\ell + 2$. It is said that Γ is ℓ -extendable if it contains a matching of size ℓ and if every such matching is contained in a perfect matching of Γ . We remark that in his definition from 1980, Plummer did not require an ℓ -extendable graph to have order at least $2\ell + 2$. But it turns out that this additional assumption is convenient since in this case ℓ -extendability of Γ implies $(\ell - 1)$ -extendability of Γ . Namely, it turns out that if a graph Γ of order at least $2\ell + 2$ is ℓ -extendable, then for any matching M of Γ of size $\ell - 1$ we can find an edge of Γ , which is not incident with any of the edges in M (see [14, Theorem 2.2] for details). Since its introduction in 1980, the family of ℓ -extendable graphs has been studied from various points of view, see for instance [16] for a survey of results on extendability prior to 1994, [1, 2, 12] for results on extendability of Cayley graphs, and [10, 15, 17–19] for various other results on the topic.

Considerable attention was given to the study of extendability of various families of highly regular graphs. For example, the extendability of the well known family of strongly regular graphs was considered in [3, 9, 11]. Recall that a connected graph Γ is *strongly regular* with parameters (n, k, λ, μ) if Γ is a k -regular graph on n vertices such that any two adjacent (distinct and nonadjacent, respectively) vertices have exactly λ (μ , respectively) common neighbours. It was proved in [9] that each strongly regular graph of even order $n \geq 4$ is 1-extendable. Moreover, it was proved in [9, 11] that there are only three strongly regular graphs of even order, which are not 2-extendable (see Proposition 2.3). The extendability of distance-regular graphs, which are a generalization of strongly regular graphs, was studied in [4].

The diameter of a connected strongly regular graph, which is not complete, is of course 2. Therefore, it seems natural to consider extendability of other regular graphs of diameter 2. For example, the extendability of Deza graphs, where we only insist that there exist two numbers such that for any pair of distinct vertices the number of their common neighbours equals one of those two numbers, was studied in [13] where it was proved that, apart from three strongly regular graphs of even order there is just one more non-2-extendable Deza graph of even order and diameter 2, namely the complement of the Möbius ladder on eight vertices.

The most important facts, on which the proofs of the results from [9, 11, 13] rely, are the regularity of the graph, the fact that its diameter equals 2 and the existence of the parameter λ . A regular graph is *edge-regular* if there exists a constant λ such that any pair of adjacent vertices share λ common neighbours. Therefore, each strongly regular graph is edge-regular. It thus seems natural to consider the following problem.

Problem 1.1 Classify the 2-extendable edge-regular graphs (of even order) and diameter 2.

The aim of this paper is to make the next step towards the solution of this problem. We focus on another generalization of strongly regular graphs, namely on the quasi-strongly regular graphs, which were studied to some extent in [7]. A *quasi-strongly regular graph* with parameters $(n, k, \lambda; \mu_1, \mu_2, \dots, \mu_s)$, where $\mu_1 > \mu_2 > \dots > \mu_s$, is a k -regular graph on n vertices such that any two adjacent vertices have λ common neighbours (and so Γ is edge-regular) and any two distinct and non-adjacent vertices have μ_i common neighbours for some $1 \leq i \leq s$. The *grade* of Γ is the number of indices $1 \leq i \leq s$, for which there exists a pair of distinct and non-adjacent vertices in Γ with μ_i common neighbours. A quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2, \dots, \mu_s)$ is *proper* if its grade equals s . Note that in such a case the graph is connected with diameter 2 if and only if $\mu_i \geq 1$ for all $1 \leq i \leq s$.

Note that each edge-regular graph is quasi-strongly regular of some grade. In this paper we study the extendability of quasi-strongly regular graphs of grade (at most) 2 with parameters $(n, k, \lambda; \mu_1, \mu_2)$, for which $\mu_1 > \mu_2 \geq 1$. Observe that such a graph has diameter 2 provided it is not complete, and is strongly regular if and only if it is of grade 1. Our main result is the following theorem.

Theorem 1.2 *Let Γ be a quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2)$, $n \geq 6$ even, $\mu_1 > \mu_2 \geq 1$. Then Γ is not 2-extendable if and only if it is isomorphic to one of the following graphs:*

- *the complete multipartite graph $K_{2,2,2}$ (which is strongly regular with parameters $(6, 4, 2, 4)$);*
- *the Möbius ladder on eight vertices (which is a proper quasi-strongly regular graph with parameters $(8, 3, 0; 2, 1)$);*
- *the Petersen graph (which is strongly regular with parameters $(10, 3, 0, 1)$);*
- *the lexicographic product $C_5[2K_1]$ of the cycle C_5 on five vertices with the edge-less graph on two vertices (which is a proper quasi-strongly regular graph with parameters $(10, 4, 0; 4, 2)$).*

The four graphs from Theorem 1.2 are depicted on Fig. 1.

2 Preliminaries

In this section we first fix some notation and then gather various results from the literature that will be used in the remainder of the paper.

Let Γ be a connected graph with vertex set $V = V(\Gamma)$ and let $v \in V$. The distance of v from a vertex $u \in V$ will be denoted by $d(u, v)$. For a non-negative integer i , we denote by $N_i(v) = \{u \in V : d(u, v) = i\}$ the set of all vertices of Γ at distance i from v . We abbreviate $N_1(v)$ by $N(v)$. The fact that the vertices u and v are adjacent in Γ will be denoted by $u \sim v$. For a subset $S \subseteq V$ we let $\Gamma - S$ be the subgraph of Γ induced on the set $V \setminus S$.

Let Γ_1 and Γ_2 be graphs. The *lexicographic product* of Γ_1 by Γ_2 , denoted by $\Gamma_1[\Gamma_2]$, is the graph whose vertex set is the cartesian product $V(\Gamma_1) \times V(\Gamma_2)$ with vertices

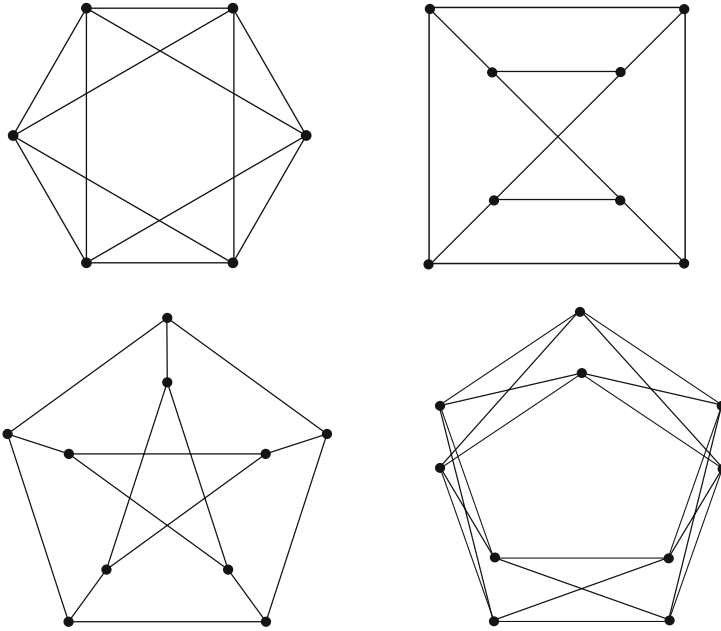


Fig. 1 The four non-2-extendable graphs from Theorem 1.2

(u_1, u_2) and (v_1, v_2) being adjacent if and only if either u_1 is adjacent with v_1 in Γ_1 , or $u_1 = v_1$ and u_2 is adjacent with v_2 in Γ_2 . We will denote the complete graph, the empty graph and the cycle on n vertices by K_n, E_n and C_n , respectively.

Let us now state some results from the literature that we will need in the remainder of the paper. The first two are about all regular graphs of diameter 2.

Lemma 2.1 ([13, Lemma 2.1]) *Let Γ be a regular graph of valency k , even order and diameter 2. Let $S \subseteq V$ be such that $\Gamma - S$ is not connected and let C be a component of $\Gamma - S$. Then there are at least k edges between C and S in Γ .*

Proposition 2.2 ([8, Theorem 1.3], [13, Theorem 2.2, Theorem 2.3]) *Let Γ be a regular graph of even order and diameter 2. Then Γ is both 0- and 1-extendable.*

We also record the result of [9, 11] on the extendability of strongly regular graphs.

Proposition 2.3 *Let Γ be a strongly regular graph of even order which is not 2-extendable. Then Γ is either the cycle C_4 on four vertices (of valency 2), or the Petersen graph (of valency 3), or the complete tripartite graph $K_{2,2,2}$ (of valency 4).*

The general idea of the proof of Theorem 1.2 is similar to the one used in [11] ([13], respectively) for strongly regular graphs (Deza graphs, respectively) of even order. One of the key factors in the proof of Theorem 1.2 is the classical result of Tutte from 1947 giving a necessary and sufficient condition for a graph to contain a perfect matching. To state it we first need to fix some additional notation. Connected components of a graph Γ will simply be called *components* of Γ . A component C of

Γ is called *even* (*odd*, respectively), if the cardinality of C is even (odd, respectively). The number of odd components of Γ will be denoted by $o(\Gamma)$. We can now state the above mentioned result of Tutte.

Theorem 2.4 ([5, Theorem 2.2.1]) *A graph Γ has a perfect matching if and only if for every subset $S \subseteq V(\Gamma)$ we have $o(\Gamma - S) \leq |S|$.*

When dealing with ℓ -extendability the following corollary of Theorem 2.4 is of use. The result was implicitly proved in [18, Theorem 2.2], but we include a short proof for the convenience of the reader.

Proposition 2.5 *Let $\ell \geq 1$ be an integer and let Γ be a connected graph of order at least $2\ell + 2$ containing a perfect matching. Then the graph Γ is not ℓ -extendable if and only if it contains a subset S of vertices such that the subgraph induced by S contains ℓ independent edges and $o(\Gamma - S) \geq |S| - 2\ell + 2$.*

Proof Suppose first there exists a set of ℓ independent edges of Γ that cannot be extended to a perfect matching of Γ and let S_0 be the set of their endvertices. Then $\Gamma - S_0$ does not contain a perfect matching, and so Theorem 2.4 implies there exists a subset $S' \subset V(\Gamma - S_0)$ such that $o((\Gamma - S_0) - S') > |S'|$. Setting $S = S_0 \cup S'$ we have $|S| = |S'| + 2\ell$, and so clearly $o(\Gamma - S) = o((\Gamma - S_0) - S') > |S| - 2\ell$ holds. As Γ is of even order $o(\Gamma - S)$ and $|S|$ are of the same parity, implying that $o(\Gamma - S) \geq |S| - 2\ell + 2$.

Conversely, suppose S is a set satisfying the assumptions of the proposition and write S as a disjoint union $S_0 \cup S'$, where S_0 consists of the endvertices of ℓ independent edges. Then $o((\Gamma - S_0) - S') = o(\Gamma - S) \geq |S'| + 2$, and thus $\Gamma - S_0$ does not have a perfect matching by Theorem 2.4. It follows that the chosen set of ℓ independent edges cannot be extended to a perfect matching of Γ . □

When searching for the largest ℓ for which the graph is still ℓ -extendable one can use the following corollary, which will be the main ingredient of our proofs throughout the paper.

Corollary 2.6 *Let $\ell \geq 1$ be an integer and let Γ be an $(\ell - 1)$ -extendable connected graph of order at least $2\ell + 2$. Then Γ is not ℓ -extendable if and only if it contains a subset S of vertices such that the subgraph induced by S contains ℓ independent edges and $o(\Gamma - S) = |S| - 2\ell + 2$.*

Proof If there exists $S \subseteq V(\Gamma)$ such that the subgraph induced by S contains ℓ independent edges and $o(\Gamma - S) = |S| - 2\ell + 2$, then Γ is not ℓ -extendable by Proposition 2.5. Assume now that Γ is not ℓ -extendable. By Proposition 2.5, there exists $S \subseteq V(\Gamma)$ such that the subgraph induced by S contains ℓ independent edges and $o(\Gamma - S) \geq |S| - 2\ell + 2$. On the other hand, since Γ is $(\ell - 1)$ -extendable, Proposition 2.5 implies that $o(\Gamma - S) < |S| - 2(\ell - 1) + 2 = |S| - 2\ell + 4$. But as Γ is of even order, $o(\Gamma - S)$ and $|S|$ are of the same parity, and so $o(\Gamma - S) = |S| - 2\ell + 2$ must hold. □

We finish this section with a nice result by Goldberg on quasi-strongly regular graphs of grade 2. Let Γ be a proper quasi-strongly regular graph with parameters

$(n, k, \lambda; \mu_1, \mu_2)$, where $\mu_1 > \mu_2 \geq 1$. Let v be a vertex of Γ and let $t_1(v)$, $t_2(v)$ denote the number of non-neighbours of v , different from v , that share μ_1 or μ_2 common neighbours with v , respectively.

Proposition 2.7 ([7, Theorem 2.1]) *Let Γ be a proper quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2)$, where $\mu_1 > \mu_2 \geq 1$. Then the numbers $t_1(v)$ and $t_2(v)$ do not depend on v and writing $t_1 = t_1(v)$ and $t_2 = t_2(v)$ we have:*

$$t_1 = \frac{k(k - \lambda - 1) - \mu_2(n - k - 1)}{\mu_1 - \mu_2}, \quad t_2 = \frac{\mu_1(n - k - 1) - k(k - \lambda - 1)}{\mu_1 - \mu_2}.$$

Moreover, $1 \leq t_1, t_2 \leq n - k - 2$.

3 The Graphs of Valency At Most 4

It proves convenient to analyze the regular graphs of diameter 2 and valency at most 4 separately. The cubic ones are particularly easy to deal with since there is only a handful of graphs to consider (see [13, Theorem 3.1]).

Theorem 3.1 *The five graphs from Fig. 2 are the only cubic graphs of diameter 2. The only 2-extendable graph among them is the complete bipartite graph $K_{3,3}$ and the only quasi-strongly regular graphs among them are the complete bipartite graph $K_{3,3}$, the Petersen graph (which are both strongly regular) and the Möbius ladder on eight vertices (which is a proper quasi-strongly regular graph with parameters $(8, 3, 0; 1, 2)$). As a consequence, the Möbius ladder on eight vertices is the only cubic non-2-extendable quasi-strongly regular graph of diameter 2, which is not strongly regular.*

Proof That the five graphs from Fig. 2 are the only cubic graphs of diameter 2 follows from [13, Theorem 3.1]. The other claims are easy to check. \square

We now focus on the more challenging case of quartic graphs of diameter 2. We in fact first classify all quasi-strongly regular graphs with parameters $(n, 4, \lambda; \mu_1, \mu_2)$ with n even and $\mu_1 > \mu_2 \geq 1$. We do this in two steps, the first of which is to show that apart from three well known graphs the only other possible such graphs are of order 14 with $\mu_1 = 2$ and $\mu_2 = 1$.

Lemma 3.2 *Let Γ be a quasi-strongly regular graph with parameters $(n, 4, \lambda; \mu_1, \mu_2)$ with n even and $\mu_1 > \mu_2 \geq 1$. Then Γ is the complete multipartite graph $K_{2,2,2}$, the complete bipartite graph $K_{4,4}$, the lexicographic product $C_5[E_2]$, or it is a proper quasi-strongly regular graph with parameters $(14, 4, 0; 2, 1)$.*

Proof Observe first that since Γ is of diameter 2 and valency 4, the fact that n is even implies $n \leq 16$. We divide our proof into three cases depending on the value of $0 \leq \lambda \leq 2$ (note that $\lambda = 3$ would imply $\Gamma = K_5$, contradicting the assumption that n is even).

Case $\lambda = 2$: Pick an edge uv of Γ , let $N(u) \cap N(v) = \{w_1, w_2\}$ and denote the remaining neighbours of u and v by x and y , respectively. Since x and u have two

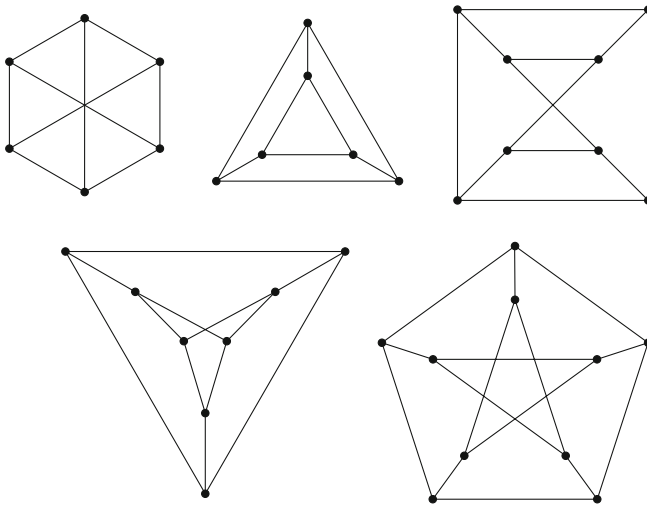


Fig. 2 The five cubic graphs of diameter 2

common neighbors $x \sim w_1, w_2$ and similarly $y \sim w_1, w_2$. It is now clear that for x and w_1 to have two common neighbors $x \sim y$ has to hold, and so $\Gamma = K_{2,2,2}$.

Case $\lambda = 1$: Note that this implies that for each $v \in V(\Gamma)$ the subgraph of Γ , induced on $N(v)$, is $2K_2$. Moreover, observe that for any $v \in V(\Gamma)$ each $w \in N_2(v)$ has at most two neighbours in $N(v)$ (otherwise there would be an adjacent pair of vertices in $N(v)$ having both v and w as common neighbors, contradicting $\lambda = 1$). Since the number of edges between $N(v)$ and $N_2(v)$ is clearly $4 \cdot 2 = 8$, this proves that $|N_2(v)| \leq 8 \leq 2|N_2(v)|$, and so the fact that $n = 1 + 4 + |N_2(v)|$ is even implies $n \in \{10, 12\}$. Then Γ cannot be strongly regular since counting the edges between $N(v)$ and $N_2(v)$ again would force $8 = \mu|N_2(v)|$, contradicting the fact that $|N_2(v)|$ must be odd. By [7, Section 7] it thus follows that no graph Γ , corresponding to Case 1, exists.

Case $\lambda = 0$: Pick a vertex $v \in V(\Gamma)$ and note that every vertex of $N(v)$ has three neighbours in $N_2(v)$, so that there are exactly 12 edges between $N(v)$ and $N_2(v)$. If all vertices of $N_2(v)$ have the same number of neighbours in $N(v)$ then the fact that n is even implies $|N_2(v)| = 3$, and so $\Gamma = K_{4,4}$. For the remainder of the proof we can thus assume that Γ is of grade 2 and that $n \geq 10$. Consequently, Proposition 2.7 implies that since $t_1 \geq 1$ and n is even, the only possibility for $\mu_2 > 1$ to hold is if $\mu_2 = 2, n = 10$ and $\mu_1 \in \{3, 4\}$, which by [7, Section 7] implies that $\mu_1 = 4$ and $\Gamma = C_5[E_2]$. We are thus left with the possibility that $\mu_2 = 1$. By [7, Section 7] we cannot have $n \leq 12$, and so $n \in \{14, 16\}$. It was proved in [6] that the only graph with diameter 2, maximal valency k and of order $n = k^2$ is isomorphic to C_4 . This shows that $n = 14$ has to hold. By Proposition 2.7, we have $t_1 = 3/(\mu_1 - 1)$, and so $\mu_1 \in \{4, 2\}$. If $\mu_1 = 4$, then $t_1 = 1$, and so there is a unique $w \in N_2(v)$ such that $N(w) = N(v)$. But then each vertex from $N(v)$ has at least three other vertices with which it shares at least two neighbors, contradicting the fact that $\mu_2 = 1$ and $t_1 = 1$. This finally proves that $\mu_1 = 2$, as claimed. \square

To complete the above mentioned classification we now prove that the last possibility from the above lemma in fact cannot occur.

Theorem 3.3 *The only quasi-strongly regular graphs of diameter 2 and valency 4 with parameters $(n, 4, \lambda; \mu_1, \mu_2)$ with n even and $\mu_1 > \mu_2 \geq 1$ are the complete multipartite graph $K_{2,2,2}$ (which is strongly regular with parameters $(6, 4, 2, 4)$), the complete bipartite graph $K_{4,4}$ (which is strongly regular with parameters $(8, 4, 0, 4)$) and the lexicographic product $C_5[E_2]$ (which is a proper quasi-strongly regular graph with parameters $(10, 4, 0; 4, 2)$).*

Proof By Lemma 3.2 and its proof we only need to prove that there is no proper quasi-strongly regular graph with parameters $(14, 4, 0; 2, 1)$. By way of contradiction suppose Γ is such a graph. By Proposition 2.7, we have $t_1 = 3$ and $t_2 = 6$. Pick a vertex $v \in V(\Gamma)$ and denote $U = N(v) = \{u_1, u_2, u_3, u_4\}$ and $N_2(v) = W \cup Z$, where $W = \{w_1, w_2, \dots, w_6\}$ and $Z = \{z_1, z_2, z_3\}$, where each of w_i has one common neighbour with v while each of z_j has two. Observe that since $\mu_1 = 2$ each pair of vertices from U can be adjacent to at most one of the vertices from Z . We distinguish three cases depending on whether each vertex of U is adjacent to at least one from Z and on whether there is some $u_i \in U$ which is adjacent to the whole Z .

Case 1: Some $u_i \in U$ is adjacent to the whole Z .

With no loss of generality assume $u_4 \sim z_j$ for all $1 \leq j \leq 3$, $u_i \sim z_i$ for $1 \leq i \leq 3$, and then $u_i \sim w_{2i-1}, w_{2i}$, for $1 \leq i \leq 3$. In this case u_1, u_2 and u_3 are the three vertices from $N_2(u_4)$, each having two common neighbours with u_2 , and so each of $w_i \in W$ has exactly one neighbour in Z . Since $w_1 \approx z_1$ (recall that $\lambda = 0$), we can further assume $w_1 \sim z_2$. For $d(u_1, z_3) \leq 2$ to hold we thus have $w_2 \sim z_3$. Since none of w_3 and w_4 can be adjacent to z_2 and only one neighbour of z_3 has not been determined yet, we can further assume $w_3 \sim z_1$. For $d(u_2, z_3) \leq 2$ to hold this forces $w_4 \sim z_3$. Hence $d(w_1, z_3), d(w_3, z_3) \leq 2$ forces $w_1 \sim w_4$ and $w_2 \sim w_3$. The only two vertices of W for which their unique neighbour in Z has not yet been determined are w_5 and w_6 , and so we can assume $w_5 \sim z_1$ and $w_6 \sim z_2$. But then $d(w_4, z_1), d(w_5, z_2) \leq 2$ force $w_4 \sim w_5$ and $w_1 \sim w_5$, respectively, giving rise to the 3-cycle (w_1, w_4, w_5) .

Case 2: Some $u_i \in U$ has no neighbours in Z .

With no loss of generality assume u_4 has no neighbours in Z while $u_1 \sim z_1, z_2$, $u_2 \sim z_2, z_3$ and $u_3 \sim z_1, z_3$. We can then also assume that $u_4 \sim w_i$ for $4 \leq i \leq 6$ and $u_j \sim w_j$ for all $1 \leq j \leq 3$. Now, $d(u_2, z_1), d(u_3, z_2), d(u_1, z_3) \leq 2$ force $w_2 \sim z_1, w_3 \sim z_2$ and $w_1 \sim z_3$, respectively. For $d(z_i, u_4) \leq 2$ to hold for each of $1 \leq i \leq 3$, each of the vertices from Z is adjacent to one of w_4, w_5 and w_6 , with no loss of generality assume $z_1 \sim w_4$. Moreover, the three vertices in $N_2(u_4)$ that each share two neighbours with u_4 can only be w_1, w_2 and w_3 , and so w_2 must be adjacent to w_5 and w_6 (since $w_2 \sim w_4$ would introduce the 3-cycle (w_2, z_1, w_4)). For $d(u_1, w_5), d(u_1, w_6) \leq 2$ to hold at least one of w_5, w_6 has to be adjacent to w_1 , so we can assume $w_1 \sim w_6$. Then $d(u_3, w_6) \leq 2$ implies $w_3 \sim w_6$ (as $z_3 \sim w_6$ would introduce a 3-cycle). It is now clear that the only vertices that could possibly have two common neighbours with w_6 are z_2, z_3, w_4 and w_5 , so at least one of them is z_2 or z_3 . But for each of them the remaining neighbour is one of w_4 and w_5 , contradicting $t_1 = 3$.

Case 3: For each $u_i \in U$ we have $|N(u_i) \cap Z| \in \{1, 2\}$.

Clearly two of the vertices from U have two neighbours in Z while the other two have one. With no loss of generality we can assume $N(u_1) \cap N_2(v) = \{w_1, w_2, z_1\}$, $N(u_2) \cap N_2(v) = \{w_3, z_1, z_2\}$, $N(u_3) \cap N_2(v) = \{w_4, z_2, z_3\}$ and $N(u_4) \cap N_2(v) = \{w_5, w_6, z_3\}$. For $d(u_1, z_2), d(u_4, z_2) \leq 2$ to hold z_2 must be adjacent to one of w_1, w_2 and to one of w_5, w_6 . We can assume $z_2 \sim w_1, w_6$. Then $d(w_2, z_2), d(w_5, z_2) \leq 2$ force $w_1 \sim w_5$ and $w_2 \sim w_6$. Since each of $w_2 \sim z_1$ and $w_5 \sim z_3$ would produce a 3-cycle, $d(u_2, w_2), d(u_3, w_5) \leq 2$ force $w_2 \sim w_3$ and $w_4 \sim w_5$, respectively. Observe that for $\lambda = 0$ to hold the remaining neighbour of w_1 must be one of w_3 and z_3 , while the remaining neighbour of w_6 must be one of z_1 and w_4 .

Now, if $w_1 \sim w_3$ then $d(w_1, z_3) \leq 2$ forces $w_3 \sim z_3$, and consequently the remaining neighbour of z_3 is z_1 . But then $d(u_3, w_2) \leq 2$ forces $w_2 \sim w_4$, so that $w_6 \approx w_4$, and so $w_6 \sim z_1$, which leaves w_4 of valency 3, a contradiction. We thus must have $w_1 \sim z_3$. Now, each of w_2, w_3 and z_1 must have a common neighbour with u_3 , and so one of them is adjacent to z_3 and two to w_4 . Since $w_2 \sim w_3$ this forces $z_1 \sim w_4$. Then $d(u_2, z_3) \leq 2$ implies that z_3 is adjacent to one of w_3 and z_1 . But $z_3 \sim w_3$ would imply that $d(z_1, z_3) > 2$ as $z_1 \approx w_1, w_3$, and so $z_1 \sim z_3$ must hold. However, then $d(u_3, w_2) \leq 2$ forces $w_2 \sim w_4$, leaving w_6 to be of valency 3, a contradiction. □

We can now classify all non-2-extendable quasi-strongly regular graphs with parameters $(n, 4, \lambda; \mu_1, \mu_2)$, where n is even and $\mu_1 > \mu_2 \geq 1$.

Theorem 3.4 *The only non-2-extendable quasi-strongly regular graphs with parameters $(n, 4, \lambda; \mu_1, \mu_2)$, where n is even and $\mu_1 > \mu_2 \geq 1$ are the complete bipartite graph $K_{2,2,2}$ (which is strongly regular with parameters $(6, 4, 2, 4)$) and the lexicographic product $C_5[E_2]$ (which is a proper quasi-strongly regular graph with parameters $(10, 4, 0; 4, 2)$).*

Proof By Theorem 3.3 we only need to consider the graphs $K_{2,2,2}, K_{4,4}$ and $C_5[E_2]$. It is easy to see that $K_{4,4}$ is 2-extendable while $K_{2,2,2}$ is not (see also [11]). As for $C_5[E_2]$ one can easily find a pair of disjoint edges that do not extend to a perfect matching, proving that this graph is not 2-extendable (note that, since $C_5[E_2]$ is isomorphic to the Cayley graph of \mathbb{Z}_{10} with respect to the connection set $\{\pm 1, \pm 4\}$ this also follows from [1]). □

4 Odd Components

For the rest of this paper let Γ denote a quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2)$, where n is even, $k \geq 5$ and $\mu_1 > \mu_2 \geq 1$. The proof that Γ is 2-extendable is by contradiction. From now on we thus assume that Γ is not 2-extendable (recall that, by Proposition 2.2, it is 1-extendable). Note that, by Proposition 2.3, Γ is not strongly regular, so we can assume in addition that Γ is of grade 2. It follows from Corollary 2.6 that we can find a vertex subset S containing two independent edges such that $o(\Gamma - S) = |S| - 2$. The aim of this section is to prove that each odd component of $\Gamma - S$ must be a singleton (see Proposition 4.5).

Now, take S as in the previous paragraph and suppose $\Gamma - S$ has an even component C . Pick $x \in C$ and set $S' = S \cup \{x\}$. Observe that each component of $\Gamma - S$, different from C , is also a component of $\Gamma - S'$. The remaining components of $\Gamma - S'$ are the components of the subgraph of Γ , induced on $C \setminus \{x\}$. Since this set is of odd size at least one of them is odd. Thus $o(\Gamma - S') \geq o(\Gamma - S) + 1 = |S| - 1 = |S'| - 2$. However, as Γ is 1-extendable and $o(\Gamma - S')$ and $|S'|$ are of the same parity, Proposition 2.5 implies that in fact $o(\Gamma - S') = |S'| - 2$ holds. Repeating this process of enlarging S until no even component of $\Gamma - S$ exists we can thus eliminate all even components of $\Gamma - S$. For the rest of the paper we can thus assume that $\Gamma - S$ has no even components. For future reference we also name the endvertices of the chosen two independent edges within the subgraph of Γ induced on S .

Hypothesis 4.1 Let Γ be a non-2-extendable proper quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2)$, where n is even, $k \geq 5$ and $\mu_1 > \mu_2 \geq 1$. Furthermore, let $S \subseteq V(\Gamma)$ be such that the subgraph of Γ induced on S contains independent edges u_1v_1 and u_2v_2 , that $\Gamma - S$ has no even components and that $o(\Gamma - S) = |S| - 2$.

Observe that, since $|S| \geq 4$, there are at least two (odd) components of $\Gamma - S$, and so each $v \in V(\Gamma) \setminus S$ has at least one neighbour in S (otherwise the diameter of Γ is not 2). We next prove that, in fact, v has at least two neighbours in S .

Lemma 4.2 *Under Hypothesis 4.1, every vertex in $\Gamma - S$ has at least two neighbours in S .*

Proof Suppose on the contrary that there is a vertex v in $\Gamma - S$, which has just one neighbour, say u , in S . Let C denote the odd component containing v . Since $N(v) \setminus \{u\} \subseteq C$ we have $|C| \geq 5$ (recall that $k \geq 5$). Let $C_1, C_2, \dots, C_{|S|-3}$ denote the other (odd) components of $\Gamma - S$ and let $m_i = |C_i|$ for $1 \leq i \leq |S| - 3$. Without loss of generality assume $m_1 \leq m_2 \leq \dots \leq m_{|S|-3}$. Since for each $1 \leq i \leq |S| - 3$ the vertex v must have a common neighbour with each $w \in C_i$, the vertex u must be adjacent to all of the vertices in $C_1 \cup C_2 \cup \dots \cup C_{|S|-3}$ and clearly $\mu_2 = 1$ holds. Moreover, u and v have λ common neighbours, which are of course all contained in C . We therefore have

$$k \geq |N(u) \cap S| + 1 + m_1 + m_2 + \dots + m_{|S|-3} + \lambda. \tag{1}$$

On the other hand, for every vertex $w \in C_1$ we have $N(w) \subseteq C_1 \cup S$, implying

$$k \leq m_1 - 1 + |S|. \tag{2}$$

We split our analysis into two cases depending on whether $|S| = 4$ or not.

Case 1: $|S| = 4$, that is, $S = \{u_1, v_1, u_2, v_2\}$.

Since u has at least one neighbour in S , (1) and (2) yield

$$2 + m_1 + \lambda \leq k \leq m_1 + 3. \tag{3}$$

It follows that each $w \in C_1$ has at least $3 + \lambda$ neighbours in S , and so is a common neighbour of u_1 and v_1 or of u_2 and v_2 (or both), implying that $\lambda \geq 1$. But then (3)

implies $\lambda = 1$, which in turn implies that each $w \in C_1$ is adjacent to entire S . Then $m_1 = 1$ (as otherwise u_1 and v_1 have more than one common neighbour), contradicting $k \geq 5$.

Case 2: $|S| \geq 5$.

Observe that (1) and (2) imply $|S| \geq 2 + m_2 + \dots + m_{|S|-3} + \lambda \geq |S| - 3 + m_{|S|-3} + \lambda$. This implies that if $m_{|S|-3} \neq 1$ we must have $m_{|S|-3} = 3, m_{|S|-4} = 1$ and $\lambda = 0$. But then (1) implies $k \geq |S|$, forcing the unique vertex of C_1 to be adjacent to entire S , which contradicts $\lambda = 0$. This shows that $m_1 = m_2 = \dots = m_{|S|-3} = 1$. Combining together (1) and (2) we thus get

$$|S| \geq k \geq |N(u) \cap S| + |S| - 2 + \lambda. \tag{4}$$

For each $1 \leq i \leq |S| - 3$ denote the unique vertex of C_i by w_i . Now, if $\lambda \neq 0$ then for u and w_1 to have a common neighbour, $|N(u) \cap S| \geq 1$ must hold, and so (4) implies $\lambda = |N(u) \cap S| = 1$ and $k = |S|$. But then both w_1 and w_2 are common neighbours of u_1 and v_1 , contradicting $\lambda = 1$. Thus $\lambda = 0$, and so (4) forces $k = |S| - 2$ (and consequently u has no neighbours in S), since otherwise one of u_1 and v_1 or u_2 and v_2 would both be adjacent to w_1 . We thus have $N(u) = \{v\} \cup \{w_i : 1 \leq i \leq |S| - 3\}$, implying that v is adjacent to all other vertices of C (otherwise u has no common neighbour with such a vertex of C). Thus $|C| = k$ and so $\lambda = 0$ implies that each vertex of C , other than v , has $k - 1$ neighbours in S . Finally, take $x \in S \setminus \{u, u_1, v_1, u_2, v_2\}$ (note that $|S| = k + 2 \geq 7$) and observe that $\lambda = 0$ implies that x is adjacent to all of $w_i, 1 \leq i \leq |S| - 3$. But since each $w \in C \setminus \{v\}$ can also be adjacent to at most one of u_1, v_1 and at most one of u_2, v_2 , the fact that it is not adjacent to u implies that also $x \sim w$. Thus x has at least $|S| - 3 + k - 1 = 2k - 2$ neighbours, a contradiction. \square

Lemma 4.3 *Under Hypothesis 4.1, suppose C is a component of $\Gamma - S$ which is not a singleton. Then there are more than $3k/2$ edges between C and S .*

Proof Denote $m = |C|$ and note that $m \geq 3$. Let t be the number of edges between C and S . By Lemma 4.2, we have $t \geq 2m$, and so if $k \leq m + 1$, the fact that $k \geq 5$ implies $t \geq 2m \geq 2(k - 1) > 3k/2$.

We are left with the possibility that $k \geq m + 2$. Since each vertex of C is adjacent to at least $k - (m - 1)$ vertices of S , we thus have $t \geq m(k - (m - 1))$, and so

$$\begin{aligned} t - \frac{3k}{2} &\geq mk - m(m - 1) - \frac{3k}{2} = \left(m - \frac{3}{2}\right)k - m(m - 1) \\ &\geq \left(m - \frac{3}{2}\right)(m + 2) - m(m - 1) = \frac{3m}{2} - 3 \geq \frac{9}{2} - 3 > 0, \end{aligned}$$

as claimed. \square

We can now finally prove that all of the components of $\Gamma - S$ are singletons. We do this in two steps.

Lemma 4.4 *Under Hypothesis 4.1, $\Gamma - S$ has at most one component which is not a singleton.*

Proof Suppose that $\Gamma - S$ has at least two (odd) components, say C_1 and C_2 with cardinalities at least 3. Without loss of generality we can assume that $m_1 = |C_1| \leq m_2 = |C_2|$. Now, each vertex of C_1 has at least $k - (m_1 - 1)$ neighbours in S , and so there are at least $km_1 - m_1(m_1 - 1)$ edges between C_1 and S . Let t denote the number of edges between S and $V(\Gamma) \setminus S$ and note that $t \leq k|S| - 4$. If $k \geq m_1 + 2$, then Lemmas 2.1 and 4.3 imply

$$\begin{aligned}
 t &> km_1 - m_1(m_1 - 1) + \frac{3k}{2} + k(|S| - 4) = k \left(m_1 - \frac{5}{2} \right) - m_1(m_1 - 1) + k|S| \\
 &\geq (m_1 + 2) \left(m_1 - \frac{5}{2} \right) - m_1(m_1 - 1) + k|S| = \frac{m_1}{2} + k|S| - 5 > k|S| - 4,
 \end{aligned}$$

a contradiction.

Therefore, $k \leq m_1 + 1$. Recall that, by Lemma 4.2, each vertex of $C_1 \cup C_2$ has at least two neighbours in S . Using this we find that

$$k|S| - 4 \geq t \geq 2m_1 + 2m_2 + k(|S| - 4) \geq 4m_1 + k(|S| - 4), \tag{5}$$

which implies $k \geq m_1 + 1$, and consequently $k = m_1 + 1$ (yielding $m_1 \geq 5$). Moreover, equalities must hold in (5), and so there are exactly two edges in S , $m_1 = m_2$ holds, each vertex in $C_1 \cup C_2$ has exactly two neighbours in S , and there are exactly k edges between any other component of $\Gamma - S$ and S . By Lemma 4.3, all other components of $\Gamma - S$ are singletons. Denote the corresponding vertices by $w_3, \dots, w_{|S|-2}$.

Now, observe that $k = m_1 + 1$ and the fact that each vertex of $C_1 \cup C_2$ has two neighbours in S implies that the components C_1 and C_2 are both complete graphs K_{m_1} , and so $\lambda \geq m_1 - 2 \geq 3$. This implies $|S| = 4$, since otherwise the singleton w_3 would have common neighbours with at least one u in $S \setminus \{u_1, v_1, u_2, v_2\}$ which would necessarily have to be in S , contradicting the fact that there are just two edges in S . Note that any pair of vertices of C_1 has the same number $\lambda_S = \lambda - (m_1 - 2)$ of common neighbours in S . Since $|S| = 4$ and $m_1 \geq 5 > 2$, we have $\lambda_S \neq 0$, and so it is easy to see that $m_1 \geq 5$ implies $\lambda_S = 2$. But then $\lambda = m_1 = k - 1$, and so Γ is a complete graph, contradicting the fact that Γ is of grade 2. \square

Proposition 4.5 *Under Hypothesis 4.1, all of the components of $\Gamma - S$ are singletons.*

Proof Suppose to the contrary that $\Gamma - S$ has a component with cardinality at least 3 (recall that $\Gamma - S$ has no even components). By Lemma 4.4, $\Gamma - S$ has exactly one such component. Denote the vertices of the singleton components by $w_1, \dots, w_{|S|-3}$, and denote the remaining component by C . Let $s \geq 2$ denote the number of edges contained in S and let t be the number of edges between S and C . Then, counting the edges between S and $V(\Gamma) \setminus S$ in two ways we get

$$k|S| - 2s = k(|S| - 3) + t = k|S| + t - 3k, \tag{6}$$

and so $t + 2s = 3k$. Since Lemma 4.3 implies that $t > 3k/2$, we thus get the inequality $s < 3k/4$. Since $N(w_1) \subseteq S$ and the number of edges contained in $N(w_1)$ is clearly

$k\lambda/2$, we have that $k\lambda/2 < 3k/4$, and so $\lambda \in \{0, 1\}$. Moreover, $|S| \geq k \geq 5$, so that there are at least two singleton components of $\Gamma - S$.

Pick w_i, w_j , where $1 \leq i < j \leq |S| - 3$, and suppose $|N(w_i) \cap N(w_j)| = \mu_2$. As $\mu_2 < \mu_1 \leq k$ there are thus $k - \mu_2 \geq 1$ vertices $w \in N(w_i) \setminus N(w_j)$. Each such w sends at least μ_2 edges to $N(w_j)$, and so $s \geq (k - \mu_2)\mu_2$. However, it is easy to see that $(k - \mu_2)\mu_2 \leq 3k/4$ implies $\mu_2 \in \{0, k\}$, which is impossible. Therefore, each pair w_i and w_j , where $1 \leq i < j \leq |S| - 3$, must have μ_1 common neighbours. We now consider the two cases depending on whether λ is 1 or 0 separately.

Case 1: $\lambda = 1$.

Note that in this case the neighbours of a vertex come in (adjacent) pairs, so k is even, implying that $|S| \geq k \geq 6$. Now, since $s < 3k/4$, not all vertices of S have more than one neighbour in S , so let $u \in S$ have a unique neighbor in S , say v (recall that each vertex of S has a common neighbour with w_1 which is of course in S).

Since u has at least one common neighbour with each of $w_i, 1 \leq i \leq |S| - 3$, this neighbour must be v , and so $k = |N(v)| \geq |S| - 2$. Observe that $|S| = k$ would contradict $\lambda = 1$, as then u and v would both be adjacent to all of w_i (and $|S| - 3 \geq 3$). On the other hand, $k = |S| - 2$ would imply that u is the unique neighbour of v in S , and so u would also be adjacent to all of w_i , also contradicting $\lambda = 1$. Thus $k = |S| - 1$. But since $|S| - 3 \geq 3$, there exist at least two of the vertices w_i which are both adjacent to u_1 and v_1 or are both adjacent to u_2 and v_2 , again contradicting $\lambda = 1$.

Case 2: $\lambda = 0$.

Since w_1 cannot be adjacent to both u_1 and v_1 nor to both u_2 and v_2 , we must have $k \leq |S| - 2$ and consequently $|S| \geq 7$. We first claim that $t \geq 2k$. Recall that, by Lemma 4.2, we have $|N(v) \cap S| \geq 2$ for each $v \in C$, and so $t \geq 2m$, where $m = |C|$. Our claim thus surely holds if $m \geq k$, so we can assume $k \geq m + 1$. Since there are no 3-cycles in C , the well-known result of Mantel (see [5, Theorem 7.1.1] for a more general Turán’s theorem) implies there are at most $(m^2 - 1)/4$ edges contained in C , and therefore $t \geq km - 2(m^2 - 1)/4$. Thus $k \geq m + 1$ implies

$$t - 2k \geq k(m - 2) - \frac{m^2 - 1}{2} \geq (m + 1)\frac{m - 3}{2} \geq 0,$$

and the claim follows. Observe that (6) now in fact implies $s \leq k/2$.

This shows that there must exist a vertex $u \in S$ having no neighbours in S , since otherwise $k/2 \geq s \geq |S|/2$, contradicting the fact that $k \leq |S| - 2$. For the diameter of Γ to be 2, the vertex u thus has to be adjacent to each $w_i, 1 \leq i \leq |S| - 3$, and to at least one vertex of C , implying that $k = |S| - 2$ and that u has a unique neighbour, say v , in C . But now $v \sim w$ for all $w \in C \setminus \{v\}$, while there are no other edges in C (recall that $\lambda = 0$), and so

$$t = k - (m - 1) + (m - 1)(k - 1) = k + (m - 1)(k - 2).$$

Thus (6) implies $2k = 2s + (m - 1)(k - 2)$, yielding $m = 3$ and $s = 2$.

Now, pick $w \in S \setminus \{u, u_1, v_1, u_2, v_2\}$ (recall that $|S| \geq 7$). As $\lambda = 0$, none of $w_i, 1 \leq i \leq |S| - 3$, is adjacent to both u_1 and v_1 nor to both u_2 and v_2 , and so they are all adjacent to w . Similarly, both vertices from $C \setminus \{v\}$ have $|S| - 3$ neighbours in S

and since none of them is u , they must also be adjacent to w . But then w has at least $|S| - 1 > k$ neighbours, a contradiction. \square

5 Proof of the Main Theorem

In this section we finally prove Theorem 1.2. Let Γ be as in Hypothesis 4.1. By Proposition 4.5, all components of $\Gamma - S$ are singletons. The corresponding vertices will be denoted by $w_1, w_2, \dots, w_{|S|-2}$ for the rest of this paper. We first determine the parameters λ and μ_2 and determine the subgraph of Γ , induced on S .

Lemma 5.1 *Under Hypothesis 4.1, we have $\lambda = 0$, $\mu_2 = 1$ and $|S| = 2k$. In addition, the subgraph of Γ , induced on S , is a disjoint union of k edges.*

Proof Let s denote the number of edges of the subgraph of Γ , induced on S . Counting the edges between S and $\Gamma - S$ in two ways we thus obtain

$$k|S| - 2s = k(|S| - 2) = k|S| - 2k,$$

implying that $s = k$.

Suppose now there exists a vertex $v \in S$ such that $N(v) \subseteq S$. Then the connected components of $\Gamma - S'$, where $S' = S \setminus \{v\}$, coincide with the ones of $\Gamma - S$, except for the new singleton component consisting of the vertex v . But then $o(\Gamma - S') = |S| - 1 = |S'|$, and so Corollary 2.6 implies that Γ is not 1-extendable (note that S' contains at least one edge), contradicting Proposition 2.2. Thus each $v \in S$ has a neighbour in $\Gamma - S$. Now, if some $v \in S$ has no neighbours in S , then it must be adjacent to each w_i (otherwise the diameter of Γ would be at least 3), and so $k = |S| - 2$ and $\lambda = 0$ hold (as w_1 and v have no common neighbours). But as $s = k \geq 5$, there is then at least one adjacency between the $|S| - 2$ neighbours of w_1 , contradicting $\lambda = 0$. Thus each vertex of S has a neighbour in S as well as in $\Gamma - S$.

If each $u \in S$ has at least two neighbours in S then $k = s \geq |S| \geq k$, implying that $k = |S|$ (so that each w_i is adjacent to entire S) and that each vertex of S has exactly two neighbours in S . But then u_1 and v_1 have at least $|S| - 2 = k - 2 \geq 3$ common neighbours, while any neighbour of w_1 can have at most two common neighbours with w_1 (as they all must be in S), contradicting the existence of the parameter λ . There thus exists some $u \in S$, having exactly one neighbour, say v , in S .

For any of the $k - 1 \geq 4$ neighbours w_i of u in $\Gamma - S$ the only possible common neighbour of u and w_i is v , showing that $\lambda \leq 1$. In fact, if $\lambda = 1$, then u has a common neighbour with each w_i , implying that v must be adjacent to all of them, and so $k \geq |S| - 2 + 1$, which then implies that u also is adjacent to each w_i and consequently $k = |S| - 1$. But then u and v have $k - 1 \geq 4$ common neighbours, a contradiction, which shows that in fact $\lambda = 0$ holds.

Observe next that u cannot be adjacent to all of w_i , since then the fact that each vertex of S has a neighbour in $\Gamma - S$ would imply that u and v have a common neighbour, contradicting $\lambda = 0$. Thus $k \leq |S| - 2$, and so letting $w_i \in \Gamma - S$ be one of the nonneighbours of u , we then see that v is the only possible common neighbour of u and w_i , and so the fact that Γ is of diameter 2 forces $\mu_2 = 1$.

Finally, since each of the $|S| - k$ nonneighbours of w_1 from S must have at least one common neighbour with w_1 , we have $k = s \geq |S| - k$, and so $|S| \leq 2k$. For each $1 \leq i \leq |S| - 2$ each of the k vertices in $N(w_i)$ has a neighbour in S , which in fact has to be in $S \setminus N(w_i)$ (as $\lambda = 0$), giving rise to at least k edges in S . But then $s = k$ implies that each vertex of $N(w_i)$ has exactly one neighbour in S , and so the fact that each vertex of S is adjacent to at least one w_i , actually implies that each vertex of S has exactly one neighbour in S . Consequently $|S| = 2k$ and the induced subgraph on S is a disjoint union of k edges. \square

Proof of Theorem 1.2. That the four graphs from the theorem are indeed not 2-extendable and are quasi-strongly regular of diameter 2, follows from Theorems 3.1 and 3.3.

For the converse assume that Γ is a quasi-strongly regular graph with parameters $(n, k, \lambda; \mu_1, \mu_2)$ with n even and $\mu_1 > \mu_2 \geq 1$, which is not 2-extendable. If $k \leq 4$, we can apply Theorems 3.1 and 3.3 (note that the only quasi-strongly regular graph with valency 2, diameter 2 and of even order is the 4-cycle C_4). We now show that k cannot be greater than 4.

Assume to the contrary that $k \geq 5$. By Proposition 2.3 the graph Γ must be of grade 2. We can thus adopt Hypothesis 4.1. By Proposition 4.5, all components of $\Gamma - S$ are singletons (we denote the corresponding vertices by $w_i, 1 \leq i \leq |S| - 2$) and by Lemma 5.1 we have that $\lambda = 0$ and $\mu_2 = 1$, while the induced subgraph of Γ on S is a disjoint union of k edges, implying that $|S| = 2k$.

Denote the vertices of S by u_i, v_i , where $1 \leq i \leq k$, in such a way that v_i is the unique neighbour of u_i in S . Since $\lambda = 0$, we have that

$$N(u_i) \cap N(v_i) = \emptyset \quad \text{and} \quad (N(u_i) \cup N(v_i)) \setminus \{u_i, v_i\} = V(\Gamma - S) \tag{7}$$

holds for each $1 \leq i \leq k$. Moreover, every $w_j, 1 \leq j \leq 2k - 2$, is adjacent to exactly one of u_i, v_i for each $1 \leq i \leq k$. Consider now u_2 , which is of course at distance 2 from both u_1 and v_1 . If $|N(u_2) \cap N(u_1)| = |N(u_2) \cap N(v_1)| = \mu_1$, then (7) implies that $k - 1 = 2\mu_1$ is even and we have $\mu_1 = (k - 1)/2$. By Proposition 2.7, it follows that $t_1 = 2(k - 1)$, and so there exist $2(k - 1)$ vertices, which have μ_1 common neighbours with w_1 . However, for each vertex x of S the vertex w_1 is either adjacent to x or has just one common neighbour with x , and so $2(k - 1) = t_1 \leq |S| - 3 = 2k - 3$, a contradiction.

It follows that one of the vertices u_1 and v_1 has a unique (recall that $\mu_2 = 1$) common neighbour with u_2 , and so (7) implies that the other has $k - 2 = \mu_1$ common neighbours with u_2 . By Proposition 2.7, we have $t_1 = k - 1$ and $t_2 = 2(k - 1)$. Let $2 \leq j \leq 2k - 2$ be such that w_1 and w_j have just one common neighbour (note that $t_1 = k - 1 < 2k - 3$, so that j exists). Without loss of generality assume $j = 2$. This means that, with one exception, for each $1 \leq i \leq k$ one of w_1 and w_2 is adjacent to u_i while the other is adjacent to v_i . With no loss of generality assume $w_1 \sim u_i$ for each $1 \leq i \leq k, w_2 \sim u_1$ and $w_2 \sim v_i$ for each $2 \leq i \leq k$. Now, since w_1 and w_2 have one common neighbour and each of w_1 and w_2 is adjacent to k vertices of S and has one common neighbour with each of the remaining k vertices of S , there are $t_2 - k - 1 = k - 3$ vertices $w_j, 3 \leq j \leq 2k - 2$, having one common neighbour with

w_1 , and there are $k - 3$ vertices $w_{j'}$, $3 \leq j' \leq 2k - 2$, having one common neighbour with w_2 . As $k \geq 5$, these two sets of $k - 3$ vertices are disjoint, and so there are $2k - 4 - 2(k - 3) = 2$ vertices, say w_3 and w_4 , which have μ_1 common neighbours with both w_1 and w_2 . But by the above remarks this can only happen if both w_3 and w_4 are adjacent to u_1 and $1 + (k - 1)/2 = \mu_1 = k - 2$ holds, implying that $k = 5$.

With no loss of generality assume $w_3 \sim u_2, u_3, v_4, v_5$. Then each of these four vertices has at least two common neighbours with u_1 (namely w_3 and one of w_1 and w_2), and so these are the $t_1 = 4$ vertices having $\mu_1 = 3$ common neighbours with u_1 . But as w_4 is the only remaining neighbour of u_1 (apart from v_1, w_1, w_2 and w_3), this implies that each of u_2, u_3, v_4 and v_5 is adjacent to w_4 , implying that w_3 and w_4 have $5 > \mu_1$ common neighbours. This contradiction finally shows that $k \leq 4$, as claimed. \square

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