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ORIGINAL PAPER

On Generalized Strongly Regular Graphs

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Abstract A generalized strongly regular graph of grade p, as a generalization of strongly regular graphs, is a regular graph such that the number of common neighbours of both any two adjacent vertices and any two non-adjacent vertices takes on p distinct values. In this paper, we study generalized strongly regular graphs of grade 2 and provide some inequalities for the eigenvalues of them. In particular, we investigate a special family of generalized strongly regular graphs of grade 2, i.e., semi-strongly regular graphs. We obtain a relation between the parameters and two inequalities for the eigenvalues of these graphs. We also present some constructions of generalized strongly regular graphs based on Cayley graphs, graph operations and association schemes, respectively.

Keywords Strongly regular graph · Quasi-strongly regular graph · Deza graph · Semi-strongly regular graph · Cayley graph · Association scheme

Mathematics Subject Classification 05E30 · 05C50 · 15A42

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1 Introduction

Throughout this paper graphs are assumed to be finite and simple.

A k-regular graph on n vertices is called a *strongly regular graph* with parameters (n, k, λ, μ), denoted by SRG(n, k, λ, μ), if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. We refer the reader to [3,4] for many beautiful properties of strongly regular graphs.

Erickson et al. [5] and Golightly et al. [8] generalized the concept of strongly regular graphs from distinct perspectives, which are Deza graphs and quasi-strongly regular graphs, respectively. A *Deza graph* with parameters (n, k, a, b) is a k-regular graph on n vertices and any two distinct vertices have a or b common neighbours. Note that the number of common neighbours of any two distinct vertices does not necessarily depend on the adjacency of the two vertices, which is the only difference between a Deza graph and a strongly regular graph. A *quasi-strongly regular graph* with parameters $(n, k, a; c_1, c_2, \dots, c_p)$ is a k-regular graph on n vertices such that any two adjacent vertices have a common neighbours and any two non-adjacent vertices have c_i common neighbours for some $1 \le i \le p$. Thus, a quasi-strongly regular graph with $c_1 = c_2 = \cdots = c_p$ is strongly regular. In [7], Goldberg studied quasi-strongly regular graphs and explored the properties of those with $p = 2$.

In this paper we consider a new generalization of strongly regular graphs, generalized strongly regular graph, which was proposed by Huo and Zhang [10] (one of the authors of this paper). Huo et al. [10] showed that some subconstituents of a family of finite geometric graphs are exactly generalized strongly regular graphs. Here, we will present several families of generalized strongly regular graphs based on other combinatorial objects.

For $1 \le i \le p$, let a_i, c_i be non-negative integers, and $a_i \ne a_j$, $c_i \ne c_j$ if $i \ne j$ $(1 \le i, j \le p)$. A k-regular graph on n vertices is called a *generalized strongly regular graph* with parameters $(n, k; a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_p)$, denoted by GSRG$(n, k; a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_p)$, if

(i) Any two adjacent vertices have a_i common neighbours and any two non-adjacent vertices have c_i common neighbours for some $1 \le i \le p$;
(ii) For each $1 \le i \le p$, there exist two adjacent vertices and two non-adjacent vertices which have exactly a_i and c_i common neighbours, respectively.

A generalized strongly regular graph can be represented in terms of its adjacency matrix. We denote the all-ones matrix by J, the zero matrix by O, and the identity matrix by I. Let G be a k-regular graph on n vertices with adjacency matrix M. Then G is a generalized strongly regular graph if and only if for some distinct integers a_i's and distinct integers c_i's $(i = 1, \dots, p)$,

$$M^2 = kI + a_1A_1 + a_2A_2 + \cdots + a_pA_p + c_1B_1 + c_2B_2 + \cdots + c_pB_p, \qquad (1)$$

where A_i, B_i are (0,1)-matrices, $A_i \ne O$, $B_i \ne O$ for all $1 \le i \le p$, $A_1 + A_2 + \cdots + A_p = M$ and $B_1 + B_2 + \cdots + B_p = J - M - I$.

It is straightforward to show that the complement of a GSRG$(n, k; a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_p)$ is also a generalized strongly regular graph with parameters

$(n, \overline{k}; \overline{a}_1, \overline{a}_2, \ldots, \overline{a}_p; \overline{c}_1, \overline{c}_2, \ldots, \overline{c}_p)$, where

$$\begin{aligned}\overline{k} &= n - k - 1,\\ \overline{a}_i &= n - 2 - 2k + c_i \ (1 \le i \le p),\\ \overline{c}_i &= n - 2k + a_i \ (1 \le i \le p).\end{aligned}$$

Therefore, for a GSRG$(n, k; a_1, a_2, \ldots, a_p; c_1, c_2, \ldots, c_p)$, the parameters satisfy $2k - n \le a_i < k$, $2k + 2 - n \le c_i \le k$.

Note that the number of common neighbours of both two adjacent vertices and two non-adjacent vertices takes on p distinct values in a generalized strongly regular graph. We call p the *grade* of a generalized strongly regular graph. It is easy to see that a generalized strongly regular graph of grade 1 is strongly regular. In this paper, we specially focus on the generalized strongly regular graphs of grade 2. In particular, we call a GSRG$(n, k; a, a+q; c, c+q)$ a *semi-strongly regular graph* with parameters $(n, k; a, c; q)$. Obviously, if a semi-strongly regular graph with parameters $(n, k; a, c; q)$ exists, then q is an integer and satisfies

$$\max\{2k - n - a, 2k + 2 - n - c, -a, -c\} \le q \le \min\{k - 1 - a, k - c\}.$$

By saying a semi-strongly regular graph with parameters $(n, k; a, c; q)$ we mean that $q \ne 0$. In particular, semi-strongly regular graphs with parameters $(n, k; a, c; q)$ are Deza graphs with parameters $(n, k, a, a + q)$ if $a = c$.

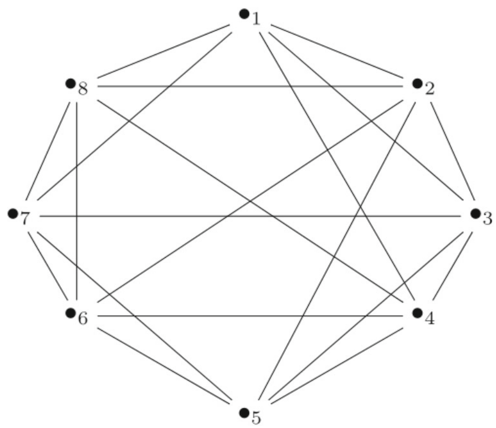
This paper is organized as follows. In Sect. 2, we study the structure of generalized strongly regular graphs of grade 2, and obtain a relation between the parameters of semi-strongly regular graphs. We also obtain some inequalities for the eigenvalues of generalized strongly regular graphs of grade 2. In Sect. 3, generalized strongly regular graphs of arbitrary grade are derived from a Cayley graph. The constructions of generalized strongly regular graphs based on some operations of graphs and association schemes are presented in Sects. 4 and 5, respectively.

2 Generalized Strongly Regular Graphs of Grade 2

In this section, we will restrict our attention to generalized strongly regular graphs of grade 2. Let G be a GSRG$(n, k; a_1, a_2; c_1, c_2)$. Without loss of generality, we assume that $a_1 > a_2$, $c_1 > c_2$. Let u be a vertex of G, $s_i(u)$ denote the number of vertices that are adjacent to u and share a_i common neighbours with u, and $t_i(u)$ denote the number of vertices that are non-adjacent to u and share c_i common neighbours with u, for $i = 1, 2$. Since G is k-regular, the vertex u has k neighbours, and $n - k - 1$ non-neighbours.

We will count the total number of edges between the neighbours and non-neighbours of u in two ways. Each of the k neighbours of u is adjacent to u itself, to either a_1 or a_2 neighbours of u, and thus to either $k - a_1 - 1$ or $k - a_2 - 1$ non-neighbours of u. Thus we have a total of $s_1(u)(k - a_1 - 1) + s_2(u)(k - a_2 - 1)$ edges. On the other hand, each non-neighbour of u is adjacent to either c_1 or c_2 neighbours of u. Hence

Fig. 1 GSRG (8,5;3,2;5,4)



we have a total of $c_1t_1(u) + c_2t_2(u)$ edges. Therefore, we obtain the following linear system of equations.

$$\begin{cases} s_1(u) + s_2(u) = k \\ t_1(u) + t_2(u) = n - k - 1 \\ s_1(u)(k - a_1 - 1) + s_2(u)(k - a_2 - 1) = c_1t_1(u) + c_2t_2(u). \end{cases} \tag{2}$$

By elementary transformations of (2), we obtain

$$\begin{cases} s_1(u) + s_2(u) = k \\ t_1(u) + t_2(u) = n - k - 1 \\ (a_2 - a_1)s_2(u) + (c_2 - c_1)t_2(u) = k(k - a_1 - 1) - c_1(n - k - 1). \end{cases} \tag{3}$$

From (2) or (3), we can not determine the values of $s_1(u)$, $s_2(u)$, $t_1(u)$ and $t_2(u)$. Indeed, these numbers may depend on the vertex u. For example, the graph G_0 shown in Fig. 1 is a GSRG(8,5;3,2;5,4), that is, a semi-strongly regular graph with parameters $(8, 5; 3, 5; -1)$. We can easily find that for vertices 1, 2, $s_1(1) = 2$, $s_2(1) = 3$, $t_1(1) = 0$, $t_2(1) = 2$, and $s_1(2) = 0$, $s_2(2) = 5$, $t_1(2) = 2$, $t_2(2) = 0$, which implies that the numbers s_1, s_2, t_1 and t_2 are not constant on the vertices of G_0. However the numbers $s_1 + t_1$ and $s_2 + t_2$ in G_0 are independent of the choice of the vertex, and satisfy $s_1 + t_1 = 2$, $s_2 + t_2 = 5$.

Let G_s be a semi-strongly regular graph with parameters $(n, k; a, c; q)$. Without loss of generality, we suppose that $q < 0$. The following theorem shows that both the number $s_1 + t_1$ and the number $s_2 + t_2$ are constant on the vertices of a semi-strongly regular graph .

Theorem 1 *Let G_s be a semi-strongly regular graph with parameters $(n, k; a, c; q)$, where $q < 0$. Then the numbers*

$$s_2(u) + t_2(u) = \frac{k(k - a - 1) - c(n - k - 1)}{q}$$

and

$$s_1(u) + t_1(u) = n - 1 - (s_2(u) + t_2(u)) = \frac{(n-1)(c+q) - k(k-a+c-1)}{q}$$

do not depend on the choice of u.

The theorem follows immediately from (2) or (3).

Obviously, both $s_1 + t_1$ and $s_2 + t_2$ must be positive integers. Therefore, Theorem 1 provides some constraints on the parameters of a semi-strongly regular graph. For example, there exist no semi-strongly regular graphs with parameters $(16, 8; 5, 6; -3)$. If otherwise, then $s_2 + t_2 = 26/3$, which is impossible.

From Theorem 1, it follows that in a semi-strongly regular graph with parameters $(n, k; a, a; q)$, which is a Deza graph, the numbers

$$s_1(u) + t_1(u) = \frac{(n-1)(a+q) - k(k-1)}{q}, \quad s_2(u) + t_2(u) = \frac{k(k-1) - a(n-1)}{q}$$

are independent of the choice of u. In fact, Erickson et al. [5] show that in a Deza graph with parameters (n, k, a_1, a_2), the number of vertices that share a_1 (or a_2) common neighbours with a vertex u does not depend on u.

Specially, the numbers s_1, s_2, t_1 and t_2 in some generalized strongly regular graphs of grade 2 are independent of the choice of the vertex, such as the semi-strongly regular graph with parameters $(9, 4; 1, 3; -1)$ shown in Fig. 2, the semi-strongly regular graph with parameters $(24, 12; 4, 12; -4)$ derived from Theorem 8 in Sect. 4, the semi-strongly regular graph with parameters $(64, 41; 30, 26; -6)$ and the GSRG(64,21;20,2;12,0) derived from Example 1 in Sect. 5.

In the rest of this section we study the eigenvalues of generalized strongly regular graphs of grade 2. Let G be a connected GSRG$(n, k; a_1, a_2; c_1, c_2)$. Let M be the adjacency matrix of G. Then M is real symmetric. Since G is k-regular, it follows that k is a simple eigenvalue of G with eigenvector $\mathbf{1}$. Thus any other eigenvectors of G are orthogonal to $\mathbf{1}$. Let θ be an eigenvalue of G and $\theta \neq k$. We obtain two inequalities for θ in the following theorem.

Theorem 2 *Let G be a connected GSRG*$(n, k; a_1, a_2; c_2, c_2)$, *where* $a_1 > a_2, c_1 > c_2$. *Let* $\theta \neq k$ *be an eigenvalue of* G. *Then*

$$\theta^2 < (a_2 - c_2)\theta + k(a_1 - a_2 + 1) + c_1(n-k-1) - c_2(n-k), \tag{4}$$
$$\theta^2 > (a_1 - c_1)\theta - k(a_1 - a_2 - 1) + c_2(n-k-1) - c_1(n-k). \tag{5}$$

Proof Let M be the adjacency matrix of G. Then from (1), we obtain

$$M^2 = kI + a_1A_1 + a_2A_2 + c_1B_1 + c_2B_2,$$

where the (i, j) entry of $A_m (B_m)$ for $m = 1, 2$ is 1 if the vertices i and j are adjacent (non-adjacent) and share a_m (c_m) common neighbours, and 0, otherwise. Hence $A_1 + A_2 = M$ and $B_1 + B_2 = J - M - I$.

Therefore, we have

$$M^2 = kI + a_1A_1 + a_2(M - A_1) + c_1B_1 + c_2(J - M - I - B_1), \tag{6}$$
$$M^2 = kI + a_1(M - A_2) + a_2A_2 + c_1(J - M - I - B_2) + c_2B_2. \tag{7}$$

Suppose that a column vector v is a unit eigenvector of M with eigenvalue $\theta \neq k$, and v^t is the transpose of v. From (6) and (7), it follows that

$$\begin{aligned}\theta^2 &= k + a_1v^tA_1v + a_2\theta - a_2v^tA_1v + c_1v^tB_1v - c_2\theta - c_2 - c_2v^tB_1v \\ &= (a_2 - c_2)\theta + k - c_2 + (a_1 - a_2)v^tA_1v + (c_1 - c_2)v^tB_1v \end{aligned}\tag{8}$$

and

$$\begin{aligned}\theta^2 &= k + a_1\theta - a_1v^tA_2v + a_2v^tA_2v - c_1\theta - c_1 - c_1v^tB_2v + c_2v^tB_2v \\ &= (a_1 - c_1)\theta + k - c_1 - (a_1 - a_2)v^tA_2v - (c_1 - c_2)v^tB_2v. \end{aligned}\tag{9}$$

Since the Rayleigh–Ritz and the Perron–Frobenius theorems,

$$\begin{aligned}v^tA_mv &\leq \rho(A_m) < \rho(M) = k, \\ v^tB_mv &\leq \rho(B_m) < \rho(J - M - I) = n - k - 1. \end{aligned}\tag{10}$$

Substituting (10) into (8) and (9), respectively, we have

$$\theta^2 < (a_2 - c_2)\theta + k - c_2 + (a_1 - a_2)k + (c_1 - c_2)(n - k - 1),$$
$$\theta^2 > (a_1 - c_1)\theta + k - c_1 - (a_1 - a_2)k - (c_1 - c_2)(n - k - 1).$$

The proof is completed. □

Notice that sometimes the inequality (5) is trivial, and provides no new information. For example, let λ be an eigenvalue of G_0 shown in Fig. 1 and $\lambda \neq 5$. It follows from (5) that $\lambda^2 + 2\lambda + 7 > 0$. But this holds for every real number λ. From (4), we have $\lambda^2 + 2\lambda - 8 < 0$, so $-4 < \lambda < 2$.

Recall that the numbers $s_1 + t_1$ and $s_2 + t_2$ do not depend on the choice of the vertex in a semi-strongly regular graph. It is valuable that we consider the eigenvalues of semi-strongly regular graphs.

Theorem 3 *Let G_s be a connected semi-strongly regular graph with parameters $(n, k; a, c; q)$, where $q < 0$. Let $\theta \neq k$ be an eigenvalue of G_s. Then*

$$\theta^2 \leq (a - c)\theta + k(k - a + c) - (c + q)n, \tag{11}$$
$$\theta^2 \geq (a - c)\theta + k(k - a + c) - cn. \tag{12}$$

Proof We follow the conventions in the proof of Theorem 2. The graph G_s is a GSRG$(n, k; a, a + q; c, c + q)$, where $q < 0$. Since $a_1 = a$, $a_2 = a + q$, $c_1 = c$ and

$c_2 = c + q$, it follows that (8) and (9) can be replaced respectively by

$$\theta^2 = (a - c)\theta + k - (c + q) + (-q)v^t(A_1 + B_1)v \tag{13}$$

and

$$\theta^2 = (a - c)\theta + k - c - (-q)v^t(A_2 + B_2)v. \tag{14}$$

From Theorem 1, the Rayleigh–Ritz and the Perron–Frobenius theorems, we have

$$v^t(A_1 + B_1)v \leq \rho(A_1 + B_1) = \frac{(n-1)(c+q) - k(k-a+c-1)}{q}, \tag{15}$$

$$v^t(A_2 + B_2)v \leq \rho(A_2 + B_2) = \frac{k(k-a-1) - c(n-k-1)}{q}. \tag{16}$$

Substituting (15) into (13), and (16) into (14) yields the assertion of the theorem. The proof is completed. □

For an eigenvalue λ ($\lambda \neq 5$) of G_0, according to (11), we have $-3 \leq \lambda \leq 1$, which is a better bound than that from (4). In fact, -3 is the smallest eigenvalue of G_0. Likewise, according to the inequality (11), it follows that 4 is an upper bound on the eigenvalues θ ($\theta \neq 12$) of the semi-strongly regular graph with parameters $(24, 12; 4, 12; -4)$ derived from Theorem 8 in Sect. 4. Indeed, the second largest eigenvalue of this graph is 4.

If the numbers s_1, s_2, t_1 and t_2 in a connected GSRG$(n, k; a_1, a_2; c_1, c_2)$ are independent of the choice of the vertex, then (10) can be replaced by

$$v^t A_1 v \leq \rho(A_1) = s_1, \quad v^t A_2 v \leq \rho(A_2) = s_2,$$
$$v^t B_1 v \leq \rho(B_1) = t_1, \quad v^t B_2 v \leq \rho(B_2) = t_2.$$

Thus we have

Theorem 4 *Let G' be a connected GSRG$(n, k; a_1, a_2; c_2, c_2)$, where $a_1 > a_2$, $c_1 > c_2$, and $\theta \neq k$ be an eigenvalue of G'. If the numbers s_1, s_2, t_1, t_2 in G' are independent of the choice of the vertex, then*

$$\theta^2 \leq (a_2 - c_2)\theta + k - c_2 + (a_1 - a_2)s_1 + (c_1 - c_2)t_1, \tag{17}$$

$$\theta^2 \geq (a_1 - c_1)\theta + k - c_1 - (a_1 - a_2)s_2 - (c_1 - c_2)t_2. \tag{18}$$

There really exist connected generalized strongly regular graphs of grade 2 such that some of their eigenvalues meet the equality in (17). For instance, in the connected GSRG(64,21;20,2;12,0), denoted by G'_0, derived from Example 1 in Sect. 5, we have $s_1 = 1$, $s_2 = 20$, $t_1 = 30$ and $t_2 = 12$. Thus from (17), the smallest eigenvalue of G'_0 is at least -19. In fact, according to the character table of Hamming scheme [2], it follows that -19 is the smallest eigenvalue of G'_0.

3 Generalized Strongly Regular Graphs from Cayley Graphs

We start this section by providing a brief description of Cayley graphs. Let K be a group and C be a subset of K that is closed under taking inverse and does not contain the identity. The *Cayley graph* $X(K, C)$ is the graph with vertex set K and two vertices $x, y \in K$ is adjacent if and only if $yx^{-1} \in C$. For a general overview of this subject, we refer the reader to [6].

Let $\mathbb{Z}_{4t+1} = \{0, 1, 2, \ldots, 4t\}$ be the ring of integers modulo $4t + 1$, and

$$C = \{4h + 2, 4h + 3 \,|h = 0, 1, 2, \ldots, t - 1\} \subseteq \mathbb{Z}_{4t+1}.$$

Then the Cayley graph $X(\mathbb{Z}_{4t+1}, C)$ is undirect and of degree $2t$. The following result shows that $X(\mathbb{Z}_{4t+1}, C)$ is a generalized strongly regular graph of grade t.

Theorem 5 *$X(\mathbb{Z}_{4t+1}, C)$ is a generalized strongly regular graph of grade t with parameters $n = 4t + 1$, $k = 2t$, and for $i = 1, 2, \ldots, t$,*

$$a_i = t - i; \quad c_i = 2t - i.$$

Proof For any two adjacent vertices x, y, without loss of generality, we assume that $y = x + 4h_1 + 2$ for some $h_1 \in H = \{0, 1, 2, \ldots, t - 1\}$, because if $y = x + 4h_1 + 3$, then $x = y + 4h' + 2$ for some $h' \in H$. Now we calculate the number of vertices $z \in \mathbb{Z}_{4t+1}$ adjacent to both x and y.

If $z = x + 4h_2 + 2 \equiv y + 4h_3 + 2 \pmod{4t + 1}$ for some $h_2, h_3 \in H$, then

$$4(h_3 + h_1 - h_2) + 2 \equiv 0 \pmod{4t + 1}.$$

Hence there exists an integer g such that

$$4(h_3 + h_1 - h_2) + 2 = g(4t + 1). \tag{19}$$

Since $h_1, h_2, h_3 \in H$, it follows that

$$-\ 4t + 4 \leq 4(h_3 + h_1 - h_2) \leq 8t - 8, \tag{20}$$

which implies that $g = 0$ if $t = 1$ and $g = 0$ or 1 if $t > 1$. However for both $g = 0$ and $g = 1$ and for any $h_1 \in H$, there exist no integers h_2, h_3 satisfying (19).

If $z = x + 4h_2 + 2 \equiv y + 4h_3 + 3 \pmod{4t + 1}$ for some $h_2, h_3 \in H$, then

$$4(h_3 + h_1 - h_2) + 3 \equiv 0 \pmod{4t + 1}.$$

It follows from (20) that there exist integers g, where $g = 0$ if $t = 1$ and $g = 0$ or 1 if $t > 1$, satisfying

$$4(h_3 + h_1 - h_2) + 3 = g(4t + 1). \tag{21}$$

But the Eq. (21) has no integer solutions for h_2, h_3 for both $g = 0$ and $g = 1$.

A similar argument indicates that there exist no vertices $z \in \mathbb{Z}_{4t+1}$ such that $z = x + 4h_2 + 3 \equiv y + 4h_3 + 3 \pmod{4t + 1}$ for some $h_2, h_3 \in H$.

If $z = x + 4h_2 + 3 \equiv y + 4h_3 + 2 \pmod{4t + 1}$, then

$$4(h_3 + h_1 - h_2) + 1 \equiv 0 \pmod{4t + 1}.$$

It follows from (20) that there exist integers g, where $g = 0$ if $t = 1$ and $g = 0$ or 1 if $t > 1$, satisfying

$$4(h_3 + h_1 - h_2) + 1 = g(4t + 1). \tag{22}$$

The Eq. (22) has no integer solutions for h_2, h_3 if $g = 0$. If $g = 1$, then $h_3+h_1-h_2 = t$. Let $h_1 = i$ for $i = 0, 1, 2, \ldots, t - 1$. Then the number of pairs (h_2, h_3) satisfying $h_3-h_2 = t-i$ is i for $i = 0, 1, 2, \ldots, t-1$. Thus the number of common neighbours of two adjacent vertices is i for some $i \in H$.

For two non-adjacent vertices x, y, we assume that $y = x + 4r$ where $r \in \{1, 2, \ldots, t\}$. Let $z \in \mathbb{Z}_{4t+1}$ be a common neighbour of x and y.

If $z = x + 4h_1 + 2 \equiv y + 4h_2 + 2 \pmod{4t + 1}$ for some $h_1, h_2 \in H$, then

$$4(r + h_2 - h_1) \equiv 0 \pmod{4t + 1}. \tag{23}$$

Since $1 \le r \le t, 0 \le h_1, h_2 \le t - 1$, we have

$$-4t + 4 \le 4(r + h_2 - h_1) \le 8t - 4. \tag{24}$$

Thus there exist integers g, where $g = 0$ if $t = 1$ and $g = 0$ or 1 if $t > 1$, such that

$$4(r + h_2 - h_1) = g(4t + 1). \tag{25}$$

There are no integers $h_1, h_2 \in H$ satisfying (25) if $g = 1$. For $g = 0$, we have $r + h_2 - h_1 = 0$. Let $r = i$, then the number of pairs (h_1, h_2) satisfying $h_1 - h_2 = i$ is $t - i$ for $i = 1, 2, \ldots, t$.

If $z = x + 4h_1 + 2 \equiv y + 4h_2 + 3 \pmod{4t + 1}$ for some $h_1, h_2 \in H$, then

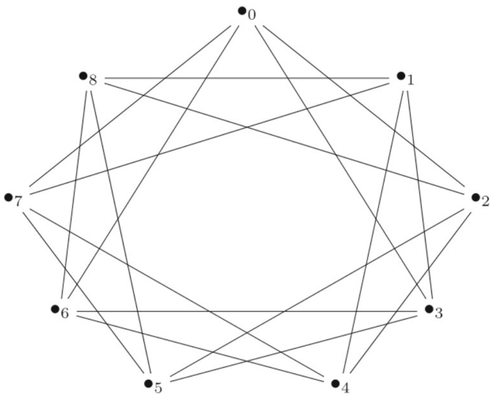
$$4(r + h_2 - h_1) + 1 \equiv 0 \pmod{4t + 1}.$$

Since (24), there exist integers $g = 0$ or 1 such that

$$4(r + h_2 - h_1) + 1 = g(4t + 1). \tag{26}$$

If $g = 0$, then there are no integers $h_1, h_2 \in H$ satisfying (26). If $g = 1$, then we have $r + h_2 - h_1 = t$. Suppose that $r = i$ for $i = 1, 2, \ldots, t$. Then the number of pairs (h_1, h_2) satisfying $h_2 - h_1 = t - i$ is i.

Fig. 2 A semi-strongly regular graph with parameters (9, 4; 1, 3; − 1)



If $z = x + 4h_1 + 3 \equiv y + 4h_2 + 2 \pmod{4t+1}$ for some $h_1, h_2 \in H$, then

$$4(r + h_2 - h_1) - 1 \equiv 0 \pmod{4t+1}. \tag{27}$$

The inequality (24) implies that there exist no integers $h_1, h_2 \in H$ satisfying (27) for any $r \in \{1, 2, \ldots, t\}$.

If $z = x + 4h_1 + 3 \equiv y + 4h_2 + 3 \pmod{4t+1}$ for some $h_1, h_2 \in H$, then (23) holds. By a similar argument, for $r = i$, the number of pairs (h_1, h_2) satisfying (23) is $t - i$.

Therefore, the number of common neighbours of two non-adjacent vertices is $2t - i$ for some $1 \leq i \leq t$. □

In particular, if $t = 1$, then $X(\mathbb{Z}_5, \{2, 3\})$ is strongly regular. For $t = 2$, $X(\mathbb{Z}_9, \{2, 3, 6, 7\})$ is the semi-strongly regular graph with parameters (9, 4; 1, 3; − 1) shown in Fig. 2.

4 Constructions from Graph Operations

We first introduce some operations of graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The *composition* $G_1[G_2]$ of G_1 and G_2 is a graph with vertex set $V_1 \times V_2$, and

$$(u_1, u_2) \sim (v_1, v_2) \text{ iff } u_1 \sim v_1 \text{ or } (u_1 = v_1 \text{ and } u_2 \sim v_2).$$

Define the *product* $G_1 \times G_2$ as a graph with vertex set $V_1 \times V_2$, and

$$(u_1, u_2) \sim (v_1, v_2) \text{ iff either } u_1 \sim v_1 \text{ or } u_2 \sim v_2.$$

The *join* of G_1 and G_2 is defined as a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1,\ v \in V_2\}$.

Refer to [9] for the above operations of graphs.

Based on the above operations, we can obtain new generalized strongly regular graphs from old ones.

Theorem 6 *Let G_1 be a GSRG$(n_1, k_1; \lambda_1, \lambda_2, \ldots, \lambda_p; \mu_1, \mu_2, \ldots, \mu_p)$, where $\lambda_i < k_1 - 1$ and $\mu_i < k_1 < n_1 - 1$ for $1 \leq i \leq p$. Let G_2 be a GSRG$(n_2, k_2; \lambda'_1, \lambda'_2, \ldots, \lambda'_q; \mu'_1, \mu'_2, \ldots, \mu'_q)$, where $0 < k_2 < n_2 - 1$. Then the composition $G_1[G_2]$ is a generalized strongly regular graph of grade $p + q$ with parameters $n = n_1 n_2$, $k = k_1 n_2 + k_2$, and*

$$a_i = \lambda_i n_2 + 2k_2 \text{ for } 1 \leq i \leq p, \quad a_{p+j} = \lambda'_j + k_1 n_2 \text{ for } 1 \leq j \leq q;$$
$$c_j = k_1 n_2 + \mu'_j \text{ for } 1 \leq j \leq q, \quad c_{q+i} = \mu_i n_2 \text{ for } 1 \leq i \leq p.$$

Proof Let $u = (u_1, u_2), v = (v_1, v_2) \in V(G_1[G_2])$. Let N_u denote the set of neighbours of u, and N_{uv} denote the set of common neighbours of u and v. Since the adjacency of $G_1[G_2]$, we have $|N_u| = k_1 n_2 + k_2$ for each $u \in V(G_1[G_2])$.

For any two vertices u, v, and $u \sim v$, we have

$$|N_{uv}| = \begin{cases} |N_{u_1 v_1}||G_2| + |N_{u_2}| + |N_{v_2}| & \text{if } u_1 \sim v_1, \\ |N_{u_2 v_2}| + |N_{u_1}||G_2| & \text{if } u_1 = v_1 \text{ and } u_2 \sim v_2. \end{cases}$$

If $u \nsim v$, then

$$|N_{uv}| = \begin{cases} |N_{u_1}||G_2| + |N_{u_2 v_2}| & \text{if } u_1 = v_1 \text{ and } u_2 \nsim v_2, \\ |N_{u_1 v_1}||G_2| & \text{if } u_1 \nsim v_1 \text{ and } u_1 \neq v_1. \end{cases}$$

It follows from $\lambda_i < k_1 - 1$ and $\mu_i < k_1$ that $\lambda_i n_2 + 2k_2 \neq \lambda'_j + k_1 n_2$ and $k_1 n_2 + \mu'_j \neq \mu_i n_2$ for any $1 \leq i \leq p$, $1 \leq j \leq q$. Notice that both $0 < k_1 < n_1 - 1$ and $0 < k_2 < n_2 - 1$ hold, which suggest that for each a_i and c_i, $1 \leq i \leq p + q$, there exist two adjacent vertices and two non-adjacent vertices which have exactly a_i and c_i common neighbours, respectively. Therefore, $G_1[G_2]$ is a generalized strongly regular graph. □

In particular, the composition of a SRG(n_1, k_1, λ, μ), where $\lambda < k_1 - 1$ and $\mu < k_1 < n_1 - 1$, and a SRG$(n_2, k_2, \lambda', \mu')$ with $0 < k_2 < n_2 - 1$ is a generalized strongly regular graph of grade 2. In what follows we consider the join of a generalized strongly regular graph and an empty graph.

Theorem 7 *Let G_1 be a GSRG$(n_1, k_1; \lambda_1, \lambda_2, \ldots, \lambda_p; \mu_1, \mu_2, \ldots, \mu_p)$, where $n_1 + \lambda_i \neq 2k_1$ and $\mu_i < k_1 < n_1 - 1$ for $1 \leq i \leq p$. Let G_2 be a $\overline{K}_{n_1 - k_1}$ (the graph on $n_1 - k_1$ vertices with no edges). Then the join of G_1 and G_2 is a generalized strongly regular graph of grade $p + 1$ with parameters $n = 2n_1 - k_1$, $k = n_1$, and*

$$a_i = \lambda_i + n_1 - k_1 \ \text{ for } \ 1 \leq i \leq p, \ a_{p+1} = k_1;$$
$$c_i = \mu_i + n_1 - k_1 \ \text{ for } \ 1 \leq i \leq p, \ \ c_{p+1} = n_1.$$

Proof The proof is straightforward. □

The graph G_0 in Fig. 1 is the join of C_5 and $\overline{K}_3$. Next, by the product of two strongly regular graphs, we obtain a class of generalized strongly regular graphs.

Theorem 8 *Let G_1 be a strongly regular graph with parameters $(n_1, k_1, \lambda_1, \mu_1)$, and G_2 be a strongly regular graph with parameters $(n_2, k_2, \lambda_2, \mu_2)$, where $0 < k_j < n_j - 1$ for $j = 1, 2$. Then the product $G_1 \times G_2$ is a k-regular graph on $n = n_1 n_2$ vertices, where $k = (k_1 n_2 + k_2 n_1 - 2k_1 k_2)$, and the number of common neighbours of two adjacent vertices in $V(G_1 \times G_2)$ is*

$$\begin{aligned}
a_1 &= \lambda_1 n_2 + k_2 n_1 - 2k_1 k_2,\\
a_2 &= \lambda_1 n_2 - 4\lambda_1 k_2 + 4\lambda_1 \mu_2 + n_1 \mu_2 - 4k_1 \mu_2 + 2k_1 k_2,\\
a_3 &= \lambda_2 n_1 + k_1 n_2 - 2k_1 k_2, \text{ or}\\
a_4 &= \lambda_2 n_1 - 4\lambda_2 k_1 + 4\lambda_2 \mu_1 + n_2 \mu_1 - 4k_2 \mu_1 + 2k_1 k_2;
\end{aligned}$$

the number of common neighbours of two non-adjacent vertices is

$$\begin{aligned}
c_1 &= \lambda_1 n_2 - 4\lambda_1 k_2 + 4\lambda_1 \lambda_2 - 4k_1 \lambda_2 + \lambda_2 n_1 + 2k_1 k_2,\\
c_2 &= \mu_2 n_1 + n_2 k_1 - 2k_1 k_2,\\
c_3 &= \mu_1 n_2 + n_1 k_2 - 2k_1 k_2, \text{ or}\\
c_4 &= \mu_1 n_2 + n_1 \mu_2 + 2k_1 k_2 - 4\mu_1 k_2 - 4\mu_2 k_1 + 4\mu_1 \mu_2.
\end{aligned}$$

For $1 \le i \le 4$, if both a_i's and c_i's take on $p' \le 4$ distinct values, then $G_1 \times G_2$ is a GSRG$(n, k; a_{1'}, \dots, a_{p'}; c_{1'}, \dots, c_{p'})$, where for $1' \le j' \le p'$, $a_{j'} = a_i$ for some $i \in \{1, 2, 3, 4\}$, and $c_{j'} = c_i$ for some $i \in \{1, 2, 3, 4\}$.

Proof It is easy to show that $G_1 \times G_2$ is $k_1(n_2 - k_2) + k_2(n_1 - k_1)$-regular according to the adjacency of $G_1 \times G_2$. Now we determine a_i and c_i for $i = 1, 2, 3, 4$.

Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in V(G_1 \times G_2)$. Then $u \sim v$ if and only if either (i) $u_1 \sim v_1$ and $u_2 \nsim v_2$ or (ii) $u_1 \nsim v_1$ and $u_2 \sim v_2$.

For (i), if $u_2 = v_2$, then let

$$\begin{aligned}
S_1 &= \{(w_1, w_2) | w_1 \sim u_1, w_1 \sim v_1, w_2 \nsim u_2 = v_2\}, \text{ and}\\
S_2 &= \{(w_1, w_2) | w_1 \nsim u_1, w_1 \nsim v_1, w_2 \sim u_2 = v_2\}.
\end{aligned}$$

Clearly, the common neighbours of u and v are exactly the vertices in $S_1 \cup S_2$. Therefore,

$$a_1 = |S_1| + |S_2| = \lambda_1(n_2 - k_2) + k_2(n_1 - 2k_1 + \lambda_1) = \lambda_1 n_2 + k_2 n_1 - 2k_1 k_2.$$

If $u_2 \neq v_2$, and $w = (w_1, w_2)$ is a common neighbour of u and v, then w must belong to one of the following four sets:

$$W_1 = \{(w_1, w_2) \in V(G_1 \times G_2) | w_1 \sim u_1, w_1 \sim v_1, w_2 \nsim u_2, w_2 \nsim v_2\},$$
$$W_2 = \{(w_1, w_2) \in V(G_1 \times G_2) | w_1 \nsim u_1, w_1 \nsim v_1, w_2 \sim u_2, w_2 \sim v_2\},$$
$$W_3 = \{(w_1, w_2) \in V(G_1 \times G_2) | w_1 \sim u_1, w_1 \nsim v_1, w_2 \nsim u_2, w_2 \sim v_2\},$$
$$W_4 = \{(w_1, w_2) \in V(G_1 \times G_2) | w_1 \nsim u_1, w_1 \sim v_1, w_2 \sim u_2, w_2 \nsim v_2\}.$$

Hence, we have

$$\begin{aligned} a_2 &= |W_1| + |W_2| + |W_3| + |W_4| \\ &= \lambda_1(n_2 - 2k_2 + \mu_2) + (n_1 - 2k_1 + \lambda_1)\mu_2 + 2(k_1 - \lambda_1)(k_2 - \mu_2) \\ &= \lambda_1 n_2 - 4\lambda_1 k_2 + 4\lambda_1\mu_2 + n_1\mu_2 - 4k_1\mu_2 + 2k_1k_2. \end{aligned}$$

For (ii), there still need to consider the case $u_1 = v_1$ and $u_1 \neq v_1$, respectively. The proof for a_3, a_4 is similar to that for a_1, a_2, so will be omitted.

For $u = (u_1, u_2), v = (v_1, v_2) \in V(G_1 \times G_2)$, $u \nsim v$ if and only if the vertices u and v satisfy one of the following four conditions:

(a) $u_1 \sim v_1$ and $u_2 \sim v_2$; (b) $u_1 = v_1$ and $u_2 \nsim v_2$;

(c) $u_1 \nsim v_1$ and $u_2 = v_2$; (d) $u_1 \nsim v_1, u_1 \neq v_1, u_2 \nsim v_2$ and $u_2 \neq v_2$.

It follows from the adjacency that the set of common neighbours of u and v is $W_1 \cup W_2 \cup W_3 \cup W_4$.

For (a), we have

$$\begin{aligned} c_1 &= |W_1| + |W_2| + |W_3| + |W_4| \\ &= \lambda_1(n_2 - 2k_2 + \lambda_2) + (n_1 - 2k_1 + \lambda_1)\lambda_2 + 2(k_1 - \lambda_1)(k_2 - \lambda_2) \\ &= \lambda_1 n_2 - 4\lambda_1 k_2 + 4\lambda_1\lambda_2 - 4k_1\lambda_2 + \lambda_2 n_1 + 2k_1k_2. \end{aligned}$$

For (b), $W_3 = W_4 = \emptyset$, so we have

$$\begin{aligned} c_2 &= |W_1| + |W_2| \\ &= k_1(n_2 - 2k_2 + \mu_2) + \mu_2(n_1 - k_1) \\ &= \mu_2 n_1 + n_2 k_1 - 2k_1k_2. \end{aligned}$$

For (c), we conclude that $W_3 = W_4 = \emptyset$, and

$$\begin{aligned} c_3 &= |W_1| + |W_2| \\ &= \mu_1(n_2 - k_2) + k_2(n_1 - 2k_1 + \mu_1) \\ &= \mu_1 n_2 + n_1 k_2 - 2k_1k_2. \end{aligned}$$

For (d), we have

$$\begin{aligned} c_4 &= |W_1| + |W_2| + |W_3| + |W_4| \\ &= \mu_1(n_2 - 2k_2 + \mu_2) + (n_1 - 2k_1 + \mu_1)\mu_2 + 2(k_1 - \mu_1)(k_2 - \mu_2) \\ &= \mu_1 n_2 + n_1\mu_2 + 2k_1k_2 - 4\mu_1k_2 - 4\mu_2k_1 + 4\mu_1\mu_2. \end{aligned}$$

Let u be a vertex of $G_1 \times G_2$, $s_i(u)$ denote the number of vertices that are adjacent to u and share a_i common neighbours with u, and $t_i(u)$ denote the number of vertices that are non-adjacent to u and share c_i common neighbours with u for $1 \le i \le 4$. It is obvious that $s_1 = k_1$, $s_2 = k_1(n_2 - k_2 - 1)$, $s_3 = k_2$ and $s_4 = k_2(n_1 - k_1 - 1)$. It is also obvious that $t_1 = k_1k_2$, $t_2 = n_2 - k_2 - 1$, $t_3 = n_1 - k_1 - 1$ and $t_4 = (n_1 - k_1 - 1)(n_2 - k_2 - 1)$. Note that k_j satisfies $0 < k_j < n_j - 1$ for $j = 1, 2$, which implies that both s_i's and t_i's are greater than 0 for $1 \le i \le 4$. Thus for each a_i and c_i, $1 \le i \le 4$, there exist two adjacent vertices and two non-adjacent vertices that have a_i and c_i common neighbours, respectively. Hence $G_1 \times G_2$ is a generalized strongly regular graph of grade p' if both a_i's and c_i's take on p' distinct values. □

For example, the product of a SRG(4,2,0,2) and a SRG(6,4,2,4) is a semi-strongly regular graph with parameters (24, 12; 4, 12; − 4); the product of a SRG(5,2,0,1) and a SRG(9,4,1,2) is a GSRG (45,22; 6,10,7;13,12,11); the product of a SRG(9,4,1,2) and a SRG(10,3,0,1) is a GSRG (90,43;13,19,16,20;22,25, 23,21).

If G_1 is a strongly regular graph with parameters (n_1, k_1, λ, μ), G_2 is a $\overline{K}_{n_2}$, then $G_1[G_2]$ and $G_1 \times G_2$ are equivalent and are quasi-strongly regular graphs with parameters $(n_1n_2, k_1n_2, \lambda n_2; k_1n_2, \mu n_2)$. Next we provide a similar result to Theorem 2.6 in [5]. We first introduce the following lemma in [7].

Lemma 1 [7] *Let G be a quasi-strongly regular graph with parameters $(n, k, a; c_1, c_2)$. Let u be some vertex of G and let $l_i(u)$ denote the number of vertices that are non-adjacent to u and share c_i common neighbours with u for $i = 1, 2$. Then $l_1(u)$, $l_2(u)$ do not depend on the choice of u and satisfy:*

$$l_1 = \frac{k(k-a-1) - c_2(n-k-1)}{c_1 - c_2}$$

and

$$l_2 = \frac{c_1(n-k-1) - k(k-a-1)}{c_1 - c_2}.$$

By Lemma 1 and the same method as in the proof of Theorem 2.6 in [5], we obtain

Theorem 9 *Let G be a quasi-strongly regular graph with parameters $(n, k, a; c_1, c_2)$. Then $c_1 = k$ if and only if G is isomorphic to $G_1[G_2]$, where G_1 is a SRG(n_1, k_1, λ, μ) and G_2 is a $\overline{K}_{n_2}$ with parameters satisfying*

$$n = n_1n_2,\ \ k = c_1 = k_1n_2,\ \ a = \lambda n_2,\ \ c_2 = \mu n_2 \text{ and } n_2 = \frac{k(k-a) - c_2(n-k)}{k - c_2}.$$

Proof Since the "if" part is clear, it is enough to prove the "only if" part. Assume that G is a quasi-strongly regular graph with parameters $(n, k, a; c_1, c_2)$, where $c_1 = k$. We define an equivalence relation R on the vertex set as: $(u, v) \in R$ iff u and v share c_1 common neighbours. Then Lemma 1 implies that each equivalence class has the same size $l_1 + 1$.

We define G_1 and G_2. Let $n_2 = l_1 + 1$, and $G_2 = \overline{K}_{n_2}$. The graph G_1 is defined to have the equivalence classes as its vertices, and two vertices C_1 and C_2 are defined to be adjacent if and only if there exists a vertex $u \in C_1$ and a vertex $v \in C_2$ such that u and v are adjacent in G. It is easy to show that G_1 is a strongly regular graph with parameters (n_1, k_1, λ, μ), where $n_1 = n/n_2, k_1 = k/n_2, \lambda = a/n_2$ and $\mu = c_2/n_2$.

Let the equivalence class C_i be $\{v_{i1}, v_{i2}, \ldots, v_{in_2}\}$, and let $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. We next show that f given by $f(C_i, u_j) = v_{ij}$ is a graph isomorphism from $G_1[G_2]$ to G.

Clearly, f is a bijection. Let (C_i, u_j) and $(C_{i'}, u_{j'})$ be vertices of $G_1[G_2]$. Since G_2 has no edges, it follows that $(C_i, u_j) \sim (C_{i'}, u_{j'})$ if and only if $C_i \sim C_{i'}$ in G_1. Thus we need to show that $v_{ij} \sim v_{i'j'}$ in G if and only if C_i is adjacent to $C_{i'}$ in G_1. If $i = i'$, then v_{ij} and $v_{i'j'}$ share $c_1 = k$ common neighbours. Since G has no loops, we have $v_{ij} \nsim v_{i'j'}$.

Suppose that $i \neq i'$. If $v_{ij} \sim v_{i'j'}$, then $C_i \sim C_{i'}$ in G_1. Conversely, if $C_i \sim C_{i'}$, then there exists a vertex $v_{il} \in C_i$ and a vertex $v_{i'l'} \in C_{i'}$ such that $v_{il} \sim v_{i'l'}$ in G. Since v_{il} and v_{ij} are in the same equivalence class, they share $c_1 = k$ common neighbours, which implies that v_{il} and v_{ij} have the same neighbourhood. Thus we have $v_{ij} \sim v_{i'l'}$. Similarly, $v_{ij} \sim v_{i'j'}$ since $v_{i'l'}$ and $v_{i'j'}$ are in the same equivalence class. Therefore, f is an isomorphism. □

5 Constructions Based on Association Schemes

In this section, we obtain a family of generalized strongly regular graphs from symmetric association schemes by merging some classes of an association scheme.

A d-class *symmetric association scheme* on a finite set Ω is a partition of $\Omega \times \Omega$ into sets $R_0, R_1, \ldots, R_d$, whose adjacency matrices are $A_0, A_1, \ldots, A_d$ respectively, such that:

(i) $A_0 = I_\Omega$;
(ii) A_i is symmetric for $i = 1, \ldots, d$;
(iii) none of the A_i's equals O, and $\sum_{i=0}^{d} A_i = J$;
(iv) for all i, j in $\{1, \ldots, d\}$, $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for some constants p_{ij}^k. (See [2] or [1] for more details.)

Theorem 10 *Let* $(\Omega, \{R_0, R_1, \ldots, R_d\})$ *be a symmetric association scheme, and* A_i *be the adjacency matrix of* R_i, *for* $0 \leq i \leq d$. *Let* $E \subseteq \{1, 2, \ldots, d\}$, $F = \{1, 2, \ldots, d\} \backslash E$, *and let G be the graph with adjacency matrix* $\sum_{i \in E} A_i$. *Then G is a generalized strongly regular graph of grade* $p \leq \lfloor d/2 \rfloor$ *if and only if* $\sum_{i,j \in E} p_{ij}^e$ *take on p distinct values as e ranges over E, and* $\sum_{i,j \in E} p_{ij}^f$ *take on p distinct values as f ranges over F.*

Proof For $u, v \in V(G)$, $u \sim v$ if and only if $(u, v) \in R_e$ for some $e \in E$. Hence G is regular of degree $\sum_{e \in E} p_{ee}^0$.

For two adjacent vertices u, v, if $(u, v) \in R_{e_i}$ for some $e_i \in E$, then the number of common neighbours of u and v is

$$a_i = \sum_{j,k \in E} p_{jk}^{e_i}.$$

If $u \nsim v$, then $(u, v) \in R_{f_i}$ for some $f_i \in F$. Therefore, the number of common neighbours of u and v is

$$c_i = \sum_{j,k \in E} p_{jk}^{f_i}.$$

Since $A_i \neq O$ for $i = 1, 2, \ldots, d$, G is a generalized strongly regular graph of grade $p \leq \lfloor d/2 \rfloor$ when both a_i's and c_i's take on p distinct values. □

We illustrate Theorem 10 by the following example, in which we obtain several generalized strongly regular graphs from a Hamming scheme.

Example 1 For a Hamming scheme H(6,2), $(\Omega, \{R_0, R_1, R_2, R_3, R_4, R_5, R_6\})$, let $\Omega = F_2^6$, and for $u, v \in \Omega$, $(u, v) \in R_i$ if u and v differ in exactly i positions, where $0 \leq i \leq 6$. Thus $|\Omega| = 64$, and for each $u \in \Omega$, $0 \leq i \leq 6$, we have $|\{v | (u, v) \in R_i\}| = \binom{6}{i}$.

From Theorem 10, we obtain:

a GSRG(64, 36; 30, 28, 10; 30, 12, 10) if $E = \{1, 2, 4\}$, $F = \{3, 5, 6\}$;

a GSRG(64, 21; 20, 2; 12, 0) if $E = \{3, 6\}$, $F = \{1, 2, 4, 5\}$;

a semi-strongly regular graph with parameters (64, 41; 30, 26; −6) if $E = \{2, 3, 5\}$, $F = \{1, 4, 6\}$.

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