Graphs and Combinatorics (2018) 34:489–499
https://doi.org/10.1007/s00373-018-1888-6

ORIGINAL PAPER

A Note on Non-jumping Numbers for r-Uniform Hypergraphs

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Received: 21 January 2017 / Revised: 27 February 2018 / Published online: 15 March 2018

Abstract A real number $\alpha \in [0, 1)$ is a jump for an integer $r \geq 2$ if there exists a constant $c > 0$ such that any number in $(\alpha, \alpha + c]$ cannot be the Turán density of a family of r-uniform graphs. Erdős and Stone showed that every number in [0,1) is a jump for $r = 2$. Erdős asked whether the same is true for $r \geq 3$. Frankl and Rödl gave a negative answer by showing the existence of non-jumps for $r \geq 3$. Recently, Baber and Talbot showed that every number in $[0.2299, 0.2316) \bigcup [0.2871, \frac{8}{27})$ is a jump for $r = 3$ using Razborov's flag algebra method. Pikhurko showed that the set of non-jumps for every $r \geq 3$ has cardinality of the continuum. But, there are still a lot of unknowns regarding jumps for hypergraphs. In this paper, we show that $1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$ is a non-jump for $r \geq 4$ and $l \geq 3$ which generalizes some earlier results. We do not know whether the same result holds for $r = 3$. In fact, when $r = 3$ and $l = 3$, $1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}} = \frac{2}{9}$, and determining whether $\frac{2}{9}$ is a jump or not for $r = 3$ is perhaps the most important unknown question regarding this subject. Erdős offered \$500 for answering this question.

Keywords Extremal problems in hypergraphs · Turán density · Erdős jumping constant conjecture · Lagrangians of uniform hypergraphs

Y. Peng: Partially supported by National Natural Science Foundation of China (no. 11671124).

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1 Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. An *r-uniform graph* G is a set $V(G)$ of vertices together with a set $E(G) \subseteq \binom{V(G)}{r}$ of edges. An r-uniform graph H is a *subgraph* of an r-uniform graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of an r-uniform graph G if $E(H) = E(G) \bigcap \binom{V(H)}{r}$. The *density* of an r-uniform graph G is defined to be $d(G) = |E(G)|/|\binom{V(G)}{r}|$. Let $\mathscr{F}$ be a family of r-uniform graphs. We say that an r-graph G is *$\mathscr{F}$-free* if G does not contain an isomorphic copy of any member of $\mathscr{F}$ as a subgraph. The *Turán density* of $\mathscr{F}$, denoted by $t_r(\mathscr{F})$ is the limit of the maximum density of an $\mathscr{F}$-free r-uniform graph of order n as $n \to \infty$. Finding good estimates of Turán densities in hypergraphs is believed to be one of the most challenging problems in extremal set theory. A real number $\alpha \in [0, 1)$ is a *jump* for an integer $r \geq 2$ if there exists a constant $c > 0$ such that any number in $(\alpha, \alpha + c]$ cannot be the Turán density of a family of r-uniform graphs. It is pointed out in [6] that it is also equivalent to the following definition.

Definition 1.1 A real number $\alpha \in [0, 1)$ is a jump for an integer $r \geq 2$ if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer m, $m \geq r$, there exists an integer $n_0(\epsilon, m)$ such that any r-uniform graph with $n \geq n_0(\epsilon, m)$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$.

Erdős et al. [3,4] showed that every $\alpha \in [0, 1)$ is a jump for 2. Erdős [2] proved that every $\alpha \in [0, \frac{r!}{r^r})$ is a jump for $r \geq 3$. Furthermore, Erdős proposed the well-known jumping constant conjecture: Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$. Frankl and Rödl [6] disproved this conjecture by showing that

Theorem 1.2 *For $r \geq 3$, $1 - \frac{1}{l^{r-1}}$ is a non-jump for r if $l > 2r$.*

Using a similar approach, more non-jumping numbers were obtained in [5,7,9–12] and some other papers. Recently, Baber and Talbot [1] showed that every number in $[0.2299, 0.2316) \bigcup [0.2871, \frac{8}{27})$ is a jump for $r = 3$ using Razborov's flag algebra method. Pikhurko [13] showed that the set of non-jumps for every $r \geq 3$ has cardinality of the continuum. However, there are still a lot of unknowns on determining whether a number is a jump for $r \geq 3$. Following the approach by Frankl and Rödl [6], we prove the following result.

Theorem 1.3 *Let $l \geq 3$ and $r \geq 4$ be integers. Then $1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$ is a non-jump for r.*

For $r = 4$ and $r = 5$, Theorem 1.3 implies the main result given in [7,9] respectively. We do not know whether the same result holds for $r = 3$. In fact, when $r = 3$ and $l = 3$, $1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}} = \frac{2}{9}$, and determining whether $\frac{2}{9}$ is a jump or not for $r = 3$ is perhaps the most important question regarding this subject. Erdős offered \$500 for answering this question.

2 Lagrangians and Other Tools

We first give a definition of the Lagrangian of an r-uniform graph.

Definition 2.1 For an r-uniform graph G with vertex set $\{1, 2, \ldots, n\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$$\lambda(G, \vec{x}) = \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \ldots, x_{i_r},$$

where x_i is called the weight of vertex i.

Definition 2.2 Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \ for$ $i = 1, 2, \ldots, n\}$. The Lagrangian of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{y} \in S$ is called an *optimum vector* of $\lambda(G)$ if $\lambda(G, \vec{y}) = \lambda(G)$.

Fact 2.3 *Let G_1, G_2 be r-uniform graphs and $G_1 \subset G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

We call two vertices i, j of an r-uniform graph G *equivalent* if for all $f \in \binom{V(G) - \{i, j\}}{r - 1}$, $f \cup \{j\} \in E(G)$ if and only if $f \cup \{i\} \in E(G)$.

Lemma 2.4 ([6]) *Suppose G is an r-uniform graph on vertex set $\{1, 2, \ldots, n\}$. If vertices $i_1, \ldots, i_t$ are pairwise equivalent, then there exists an optimum vector $\vec{y}$ $= (y_1, y_2, \ldots, y_n)$ of $\lambda(G)$ such that $y_{i_1} = y_{i_2} = \cdots = y_{i_t}$.*

We also introduce the blowup of an r-uniform graph which will allow us to construct r-uniform graphs with large number of vertices and densities close to $r!\lambda(G)$.

Definition 2.5 Let G be an r-uniform graph with $V(G) = \{1, 2, \ldots, m\}$ and $\vec{n}$ $= (n_1, n_2, \ldots, n_m)$ be a positive integer vector. Define the $\vec{n}$ blow-up of G, $\vec{n} \otimes G$ as an m-partite r-uniform graph with vertex set $V_1 \bigcup \cdots \bigcup V_m$, $|V_i| = n_i$, $1 \leq i \leq m$, and edge set $E(\vec{n} \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} : v_{i_k} \in V_{i_k} \ for \ 1 \leq k \leq r, \{i_1, i_2, \ldots, i_r\}$ $\in E(G)\}$.

We make the following easy remark proved in [8].

Remark 2.6 Let G be an r-uniform graph with m vertices and $\vec{y} = (y_1, y_2, \ldots, y_m)$ be an optimum vector of $\lambda(G)$. Then for any $\epsilon > 0$, there exists an integer $n_1(\epsilon)$, such that for any integer $n \geq n_1(\epsilon)$,

$$d\left((\lfloor ny_1 \rfloor, \lfloor ny_2 \rfloor, \ldots, \lfloor ny_m \rfloor) \otimes G\right) \geq r!\lambda(G) - \epsilon.$$

Let us also state a fact relating the Lagrangian of an r-uniform graph to the Lagrangian of its blow-up.

Fact 2.7 ([6]) *Let $\vec{n} = (n, n, \ldots, n)$, $n \geq 1$. Then for every r-uniform graph G and every integer n, $\lambda(\vec{n} \otimes G) = \lambda(G)$ holds.*

The following lemmma proved in [6] gives a necessary and sufficient condition for a number α to be a jump.

Lemma 2.8 ([6]) *The following two properties are equivalent.*

(i) α *is jump for* r.
(ii) *There exists some finite family* $\mathscr{F}$ *of* r*-uniform graphs satisfying* $\lambda(F) > \frac{\alpha}{r!}$ *for all* $F \in \mathscr{F}$ *and* $t_r(\mathscr{F}) \leq \alpha$.

We also need the following lemma from [6].

Lemma 2.9 ([6]) *For any* $\delta \geq 0$ *and any integer* $k \geq r$*, there exists* $t_0(k, \delta)$ *such that for every* $t > t_0(k, \delta)$*, there exists an* r*-uniform graph* A *satisfying:*

1. $|V(A)| = t$,
2. $|E(A)| \geq \delta t^{r-1}$,
3. *For all* $V_0 \subset V(A), r \leq |V_0| \leq k$*, we have* $|E(A) \bigcap \binom{V_0}{r}| \leq |V_0| - r + 1$.

The approach in proving Theorem 1.3 is sketched as follows: Let α be a number to be proved to be a non-jump. Assuming that α is a jump, we will derive a contradiction by the following steps.

Step 1. Construct an r-uniform graph with the Lagrangian close to but slightly smaller than $\frac{\alpha}{r!}$, then use Lemma 2.9 to add an r-uniform graph with enough number of edges but sparse and obtain an r-uniform graph with the Lagrangian $\geq \frac{\alpha}{r!} + \epsilon$ for some positive ϵ. Then we blow up this r-uniform graph to an r-uniform graph, say H with large enough number of vertices and density $> \alpha + \frac{\epsilon}{2}$ (see Remark 2.6). If α is a jump, by Lemma 2.8, $t_r(\mathscr{F}) \leq \alpha$ for some finite family $\mathscr{F}$ of r-uniform graphs with Lagrangians $> \frac{\alpha}{r!}$. So H must contain some member of $\mathscr{F}$ as a subgraph.

Step 2. We show that any subgraph of H with the number of vertices not greater than $max\{|V(F)|, F \in \mathscr{F}\}$ has the Lagrangian $\leq \frac{\alpha}{r!}$ and derive a contradiction.

3 Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Let $l \geq 3$ and $r \geq 4$ be integers. Let

$$\alpha = 1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}.$$

Suppose that α is a jump. By Lemma 2.8, there exists a finite family $\mathscr{F}$ of r-uniform graphs satisfying:

(i) $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathscr{F}$, and
(ii) $t_r(\mathscr{F}) \leq \alpha$.

Let t be a large enough integer determined later. Define an r-uniform hypergraph $G(r, l, t)$ on l pairwise disjoint sets $V_1, \ldots, V_l$, each with order t and $E(G(r, l, t)) = \{\{v_{i_1}, \ldots, v_{i_r}\} : \{v_{i_1}, \ldots, v_{i_r}\} \in \binom{V(G(r,l,t))}{r} \setminus \left(\bigcup_{i=1}^{l} \binom{V_i}{r} \bigcup \bigcup_{i=1}^{l} \bigcup_{j=1, j \neq i}^{l} \binom{V_i}{r-1}\right)$

$\times \binom{V_j}{1}\big)\}$. Note that

$$|E(G(r,l,t))| = \binom{lt}{r} - l\binom{t}{r} - l(l-1)t\binom{t}{r-1}$$
$$= \frac{\alpha}{r!}(lt)^r - c_0(l)t^{r-1} + o(t^{r-2}), \tag{3.1}$$

where $c_0(l) = \frac{\binom{r}{2}(l^{r-1}-l)}{r!} - \frac{l(l-1)\binom{r-1}{2}}{(r-1)!} > 0$.

It is easy to verify that $d(G(r,l,t))$ is close to α when t is large enough.

Take $\vec{x} = (x_1, \ldots, x_{lt})$, where $x_i = \frac{1}{lt}$ for each i, $1 \le i \le lt$. Then

$$\lambda(G(r,l,t)) \ge \lambda(G(r,l,t), \vec{x})$$
$$= \frac{|E(G(r,l,t))|}{(lt)^r}$$
$$= \frac{\alpha}{r!} - \frac{c_0(l)}{l^r t} + o\left(\frac{1}{t}\right),$$

which is close to $\frac{\alpha}{r!}$ when t is large enough.

Set $k_0 = max_{F\in\mathscr{F}}|V(F)|$ and $\delta_0 = 2c_0(l)$. Let $t_0(k_0, \delta_0)$ be given as in Lemma 2.9. Take an integer $t > t_0(k_0, \delta_0)$ and an r-uniform graph $A_{k_0,\delta_0(t)}$ satisfying the conditions in Lemma 2.9 with $V(A_{k_0,\delta_0(t)}) = V_1$. The r-uniform graph $H(r,l,t)$ is obtained by adding $A_{k_0,\delta_0(t)}$ to the r-uniform graph $G(r,l,t)$. Note that

$$\lambda(H(r,l,t)) \ge \frac{|E(H(r,l,t))|}{(lt)^r}.$$

In view of the construction of $H(r,l,t)$ and Eq. (3.1), we have

$$\frac{|E(H(r,l,t))|}{(lt)^r} = \frac{|E(G(r,l,t))| + \delta_0 t^{r-1}}{(lt)^r} \ge \frac{\alpha}{r!} + \frac{c_o(l)}{l^r t}$$

for sufficiently large t. Consequently,

$$\lambda(E(H(r,l,t)) \ge \frac{\alpha}{r!} + \frac{c_o(l)}{l^r t}.$$

Now suppose $\vec{y} = (y_1, y_2, \ldots, y_{lt})$ is an optimum vector of $\lambda(E(H(r,l,t))$. Let $\epsilon = \frac{c_0(l)}{2l^r t}$ and $n > n_1(\epsilon)$ as in Remark 2.6. Then the r-uniform graph $S_n = (\lfloor ny_1 \rfloor, \ldots, \lfloor ny_{lt} \rfloor) \otimes H(r,l,t)$ has density not less than $\alpha + \epsilon$. Since $t_r(\mathscr{F}) \le \alpha$, some member of $\mathscr{F}$ is a subgraph of S_n for $n \ge n_1(\epsilon)$. For such $F \in \mathscr{F}$, there exists a subgraph M of $H(r,l,t)$ with $|V(M)| \le |V(F)| \le k_0$ so that $F \subset \vec{n} \otimes M$. By Facts 2.3 and 2.7, we have

$$\lambda(F) \le \lambda(\vec{n} \otimes M) = \lambda(M). \tag{3.2}$$

Theorem 1.3 will follow from the following Lemma 3.1.

Lemma 3.1 *Let M be any subgraph of $H(r,l,t)$ with $|V(M)| \leq k_0$. Then*

$$\lambda(M) \leq \frac{\alpha}{r!}$$

holds.

Applying Lemma 3.1 to (3.2), we have

$$\lambda(F) \leq \frac{\alpha}{r!},$$

which contradicts the fact that $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathscr{F}$. □

To complete the proof of Theorem 1.3, it is sufficient to show Lemma 3.1.

3.1 Proof of Lemma 3.1

Define $U_i = V(M) \bigcap V_i$. Let $\vec{\xi} = (x_1, x_2, \ldots, x_{lt})$. Let a_i be the sum of the weights in U_i, $1 \leq i \leq l$ respectively. Define $M_1 = (U_1, E(M) \bigcap \binom{U_1}{r})$. Again, by Fact 2.3, it is enough to show Lemma 3.1 for the case $E(M_1) \neq \emptyset$. Thus we may assume $|V(M_1)| = r - 1 + d$ with d a positive integer. By Lemma 2.9, M_1 has at most d edges. Let $V(M_1) = \{v_1, v_2, \ldots, v_{r-1+d}\}$ and $\vec{\eta} = (x_1, x_2, \ldots, x_{r-1+d})$ be an optimum vector for $\lambda(M_1)$ with $x_1 \geq x_2 \geq \ldots \geq x_{r-1+d}$. The following Claim was proved in [6].

Claim 3.2

$$\sum_{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \in E(M_1)} x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_r}} \leq \sum_{r \leq i \leq r-1+d} x_1 x_2 \cdots x_{r-1} x_i.$$

By Claim 3.2, we may assume that

$$E(M_1) = \{\{v_1, v_2, \ldots, v_{r-1}, v_i\} : r \leq i \leq r - 1 + d\}.$$

Since $v_1, v_2, \ldots, v_{r-1}$ are equivalent, by Lemma 2.4, we may assume that $x_1 = x_2 = \cdots = x_{r-1} \stackrel{\text{def}}{=} \rho_0$, Notice that

$$\begin{cases} \sum_{i=1}^{l} a_i = 1, \\ \alpha_i \geq 0, 1 \leq i \leq l, \\ 0 \leq \rho_0 \leq \frac{\alpha_1}{r-1}. \end{cases}$$

Now we give an upper bound for $\lambda(M, \vec{\xi})$. Observing that each term in $\lambda(M, \vec{\xi})$ appears $r!$ times in the expansion $(x_1 + x_2 + \cdots + x_m)^r$ but this expansion contains lots of terms not appearing in $\lambda(M)$ as well. Since $E(M) = \{v_1, \ldots, v_{r-1}, v_i : v_i \in \{v_r, \ldots, v_{r-1+d}\} \subseteq U_1\} \bigcup \{\{v_{i_1}, \ldots, v_{i_r}\} : \{v_{i_1}, \ldots, v_{i_r}\} \in \binom{V(H(r,l,t))}{r} \setminus \left(\bigcup_{i=1}^{l} \binom{V_i}{r}\right)$

$\bigcup \bigcup_{i=1}^{l} \bigcup_{j=1, j\neq i}^{l} \binom{V_i}{r-1} \times \binom{V_j}{1})\}$, $r! \sum_{1\leq j\leq d} x_1 \dots x_{r-1}x_{r-1+j}$ will be added and $\sum_{j=1}^{l} \alpha_j^r$ and $r\sum_{i=1}^{l} \alpha_i^{r-1}(1-\alpha_i)$ should be subtracted in this expansion. Also note that $\{v_i, v_i, v_{i_3}, \dots, v_{i_{r-2}}, v_{s_2}, v_{s_3}\}$ is not an edge in M, where $1 \leq i \leq r-1$, and $\{i_3, \dots, i_{r-2}\}$ is an $(r-4)$-subset of $\{1, 2, \dots, r-1\} - \{i\}$ and s_2, s_3 (allow that $s_2 = s_3$) are any vertices in $\bigcup_{j=2}^{l} U_j$. Since each of the corresponding terms appears at least $\frac{r!}{4}$ times in the expansion, then $(r-1)\binom{r-2}{r-4}\frac{r!}{4}\rho_0^{r-2}(1-\alpha_1)^2 = \frac{(r-1)(r-2)(r-3)}{2}\frac{r!}{4}\rho_0^{r-2}(1-\alpha_1)^2 \geq (r-1)\frac{r!}{4}\rho_0^{r-2}(1-\alpha_1)^2$ should be subtracted from the expansion. Therefore,

$$\begin{aligned}\lambda(M,\vec{\xi}) \leq \frac{1}{r!}\Bigg\{&1-\sum_{i=1}^{l}\alpha_i^r + r!\sum_{1\leq j\leq d} x_1\dots x_{r-1}x_{r-1+j}\\ &-r\sum_{i=1}^{l}\alpha_i^{r-1}(1-\alpha_i) - (r-1)\frac{r!}{4}\rho_0^{r-2}(1-\alpha_1)^2\Bigg\}\\ \leq \frac{1}{r!}\Bigg\{&1-\sum_{i=1}^{l}[r-(r-1)\alpha_i]\alpha_i^{r-1}\\ &+r!\rho_0^{r-2}\left[\alpha_1\rho_0-(r-1)\rho_0^2-\frac{(r-1)}{4}(1-\alpha_1)^2\right]\Bigg\}.\end{aligned}$$

Lemma 3.1 follows directly from the following claim.

Claim 3.3 Let

$$\begin{aligned}f(\alpha_1,\alpha_2,\dots,\alpha_l,\rho_0) = 1 - &\sum_{i=1}^{l}[r-(r-1)\alpha_i]\,\alpha_i^{r-1}\\ &+ r!\rho_0^{r-2}\left[\alpha_1\rho_0-(r-1)\rho_0^2-\frac{(r-1)}{4}(1-\alpha_1)^2\right].\end{aligned}$$

Then

$$f(\alpha_1,\alpha_2,\dots,\alpha_l,\rho_0) \leq 1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$$

holds under the constraints

$$\begin{cases}\sum_{i=1}^{l} a_i = 1,\\ \alpha_i \geq 0, 1\leq i\leq l,\\ 0\leq \rho_0 \leq \frac{\alpha_1}{r-1}.\end{cases}$$

Proof of Claim 3.3. We consider three cases as follows.

Case 1. $\alpha_1 = 0$.
Note that $\rho_0 = 0$. We have

$$f(0, \alpha_2, \ldots, \alpha_l, 0) = 1 - \sum_{i=2}^{l}[r - (r-1)\alpha_i]\alpha_i^{r-1}.$$

Let $g(\alpha_2, \alpha_3, \ldots, \alpha_l) = 1 - \sum_{i=2}^{l}[r - (r-1)\alpha_i]\alpha_i^{r-1}$, where $\sum_{i=2}^{l}\alpha_i = 1, 0 \le \alpha_i \le 1, i = 2, 3, \ldots, l$. Let $L(\alpha_2, \alpha_3, \ldots, \alpha_l, \lambda) = g(\alpha_2, \alpha_3, \ldots, \alpha_l) + \lambda(\sum_{i=2}^{l}\alpha_i - 1)$, where λ is a real variable. By Lagrange multiplier method, an interior optimal point must satisfy

$$\begin{cases} L_{a_i} = -r(r-1)(1-a_i)a_i^{r-2} + \lambda = 0, \quad i = 2, 3, \ldots, l; \\ L_\lambda = \sum_{i=2}^{l}\alpha_i - 1 = 0. \end{cases}$$

Thus $\alpha_2 = \alpha_3 = \cdots = \alpha_l = \frac{1}{l-1}$ is the only possible interior optimal point and $1 + \frac{r-1}{(l-1)^{r-1}} - \frac{r}{(l-1)^{r-2}}$ is the corresponding possible optimal value for g. Similarly, for the boundary points with i zeros, $1 + \frac{r-1}{(l-1-i)^{r-1}} - \frac{r}{(l-1-i)^{r-2}}$ is the only possible optimal value for g.

Recall that $r \ge 4$. Let $h(x) = \frac{r-1}{x^{r-1}} - \frac{r}{x^{r-2}}$, where $x \in Z^+$. Then $h'(x) = \frac{-(r-1)^2 + r(r-2)x}{x^{r-2}}$. If $x \ge 2$, then $-(r-1)^2 + r(r-2)x \ge -(r-1)^2 + 2r(r-2) = r^2 - 2r - 1 \ge 7 > 0$ and $h'(x) > 0$. Also note that $h(1) < h(2)$. Thus $h(x)$ is monotonically increasing on Z^+. Therefore, for $0 \le i \le l-2$, $1 + \frac{r-1}{(l-1-i)^{r-1}} - \frac{r}{(l-1-i)^{r-2}} < 1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$. It settles this case.

Case 2. $\alpha_1 = 1$.
Note that

$$f(1, 0, \ldots, 0, \rho_0) = r!\rho_0^{r-1}[1 - (r-1)\rho_0].$$

Since the geometric mean is no more than the arithmetic mean, we obtain that

$$f(1, 0, \ldots, 0, \rho_0) \le r!\left[\frac{(r-1)\rho_0 + 1 - (r-1)\rho_0}{r}\right]^r = \frac{(r-1)!}{r^{r-1}}.$$

Recall that $h(l) = \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$ is monotonically increasing on $l \ge 3$. Thus

$$\frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}} \ge \frac{r-1}{3^{r-1}} - \frac{r}{3^{r-2}} = -\frac{2r+1}{3^{r-1}},$$
$$1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}} - \frac{(r-1)!}{r^{r-1}} \ge 1 - \left(\frac{2r+1}{3^{r-1}} + \frac{(r-1)!}{r^{r-1}}\right).$$

Let $h_1(r) = \frac{2r+1}{3^{r-1}}$ and $h_2(r) = \frac{(r-1)!}{r^{r-1}}$ for $r \ge 4$. Since $h_1'(r) = \frac{2-(2r+1)ln3}{3^{r-1}} < 0$ and $\frac{h_2(r+1)}{h_2(r)} = (\frac{r}{r+1})^r < 1$, $h_1(r)$ and $h_2(r)$ are both monotonically decreasing on

$r \geq 4$. Thus

$$1+\frac{r-1}{l^{r-1}}-\frac{r}{l^{r-2}}-\frac{(r-1)!}{r^{r-1}} \geq 1-\left(\frac{2r+1}{3^{r-1}}+\frac{(r-1)!}{r^{r-1}}\right)$$
$$\geq 1-\left(\frac{9}{3^3}+\frac{3!}{4^3}\right)=\frac{55}{96}.$$

Therefore,

$$f(1,0,\ldots,0,\rho_0) \leq \frac{(r-1)!}{r^{r-1}} < 1+\frac{r-1}{l^{r-1}}-\frac{r}{l^{r-2}}.$$

Case 3. $0 < \alpha_1 < 1$.

Let $g(\alpha_1,\alpha_2,\ldots,\alpha_l) = 1-\sum_{i=1}^{l}[r-(r-1)\alpha_i]\alpha_i^{r-1}$, where $\sum_{i=1}^{l}\alpha_i = 1, 0 \leq \alpha_i \leq 1, i = 1,2,\ldots,l$. Similar to case 1, we have

$$1-\sum_{i=1}^{l}[r-(r-1)\alpha_i]\alpha_i^{r-1} \leq 1+\frac{r-1}{l^{r-1}}-\frac{r}{l^{r-2}}.$$

If $\rho_0 = 0$, then $f(\alpha_1,\alpha_2,\ldots,\alpha_l,0) = 1-\sum_{i=1}^{l}[r-(r-1)\alpha_i]\alpha_i^{r-1} \leq 1+\frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$.

So we may assume that $\rho_0 > 0$. Also recall that $\rho_0 \leq \frac{\alpha_1}{r-1}$. We consider two subcases as follows.

Subcase 3.1. $0 < \alpha_1 \leq 1-\frac{1}{r}$.

Note that

$$f(\alpha_1,\alpha_2,\ldots,\alpha_l,\rho_0) \leq 1+\frac{r-1}{l^{r-1}}-\frac{r}{l^{r-2}} + r!\rho_0^{r-2}\left[\alpha_1\rho_0-(r-1)\rho_0^2-\frac{(r-1)}{4}(1-\alpha_1)^2\right].$$

Let $\Delta_1(\rho_0) = r!\rho_0^{r-2}\Delta_2(\rho_0)$, where $\Delta_2(\rho_0) = \alpha_1\rho_0-(r-1)\rho_0^2-\frac{(r-1)}{4}(1-\alpha_1)^2$. Then $\Delta_2'(\rho_0) = \alpha_1-2(r-1)\rho_0$, and $\Delta_2'(\rho_0) > 0$ when $0 < \rho_0 < \frac{\alpha_1}{2(r-1)}$ and $\Delta_2'(\rho_0) < 0$ when $\frac{\alpha_1}{2(r-1)} < \rho_0 \leq \frac{\alpha_1}{r-1}$. Thus $\Delta_1(\rho_0) = r!\rho_0^{r-2}\Delta_2(\rho_0) \leq r!\rho_0^{r-2}\Delta_2(\frac{\alpha_1}{2(r-1)}) = \frac{r!\rho_0^{r-2}}{4(r-1)}[\alpha_1^2-(r-1)^2(1-\alpha_1)^2] \leq 0$ since $\alpha_1 \leq 1-\frac{1}{r}$. Therefore,

$$f(\alpha_1,\alpha_2,\ldots,\alpha_l,\rho_0) \leq 1+\frac{r-1}{l^{r-1}}-\frac{r}{l^{r-2}}.$$

Subcase 3.2. $1-\frac{1}{r} \leq \alpha_1 < 1$.

Note that

$$f(\alpha_1,\alpha_2,\ldots,\alpha_l,\rho_0) \leq 1-[r-(r-1)\alpha_1]\alpha_1^{r-1}+r!\rho_0^{r-1}[\alpha_1-(r-1)\rho_0].$$

Let $\Delta_3(\alpha_1) = 1 - [r - (r-1)\alpha_1]\alpha_1^{r-1}$. Then

$$\Delta_3'(\alpha_1) = -r(r-1)\alpha_1^{r-2}(1-\alpha_1) < 0.$$

Thus $\Delta_3(\alpha_1)$ is monotonically decreasing on $[1 - \frac{1}{r}, 1)$.

To prove this subcase, now we need the following useful claim.

Claim 3.4 $(2 - \frac{1}{r})(1 - \frac{1}{r})^{r-1} \geq \frac{2}{e}$ for $r \geq 4$.

Proof of Claim 3.4. It is easy to verify that the claim is true for $r = 4$. Note that $(2 - \frac{1}{r})(1 - \frac{1}{r})^{r-1} \to \frac{2}{e}(r \to +\infty)$. Let $N > 0$ be a sufficiently large integer and $c_1(r) = (r-1)ln(1 - \frac{1}{r}) + ln(2 - \frac{1}{r})$ for $r \in [4, N]$. It is sufficient to proved that $c_1'(r) < 0$. Note that

$$\begin{aligned} c_1'(r) &= ln\left(1 - \frac{1}{r}\right) + (r-1) \cdot \frac{r}{r-1} \cdot \frac{1}{r^2} + \frac{r}{2r-1} \cdot \frac{1}{r^2} \\ &= ln\left(1 - \frac{1}{r}\right) + \frac{1}{r} + \frac{1}{r(2r-1)}. \end{aligned}$$

Let $c_2(r) = ln(1 - \frac{1}{r}) + \frac{1}{r} + \frac{1}{r(2r-1)}$ for $r \in [4, N]$. Then $c_2'(r) = \frac{r}{r-1} \cdot \frac{1}{r^2} - \frac{1}{r^2} - \frac{4r-1}{(2r-1)^2 r^2} = \frac{r}{(r-1)(2r-1)^2 r^2} > 0$. Thus $c_2(r)$ is monotonically increasing continuous function on $r \in [4, N]$. Clearly, $c_2(4) < 0$, $c_2(N) \to 0 (N \to +\infty)$. Hence $c_1'(r) < 0$ for $r \in [4, N]$. □

By Claim 3.4, $\Delta_3(\alpha_1) \leq \Delta_3(1 - \frac{1}{r}) \leq 1 - \frac{2}{e} < \frac{55}{96}$. From Case 2, we have

$$\begin{aligned} r!\rho_0^{r-1}[\alpha_1 - (r-1)\rho_0] &\leq \frac{(r-1)!}{r^{r-1}}, \\ 1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}} - \frac{(r-1)!}{r^{r-1}} &\geq \frac{55}{96}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(\alpha_1, \alpha_2, \ldots, \alpha_l, \rho_0) &\leq 1 - [r - (r-1)\alpha_1]\alpha_1^{r-1} + \frac{(r-1)!}{r^{r-1}} \\ &\leq 1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}. \end{aligned}$$

□

Remark 3.5 For $r = 5$ and $l = 2$, we can combine case 2 with subcase 3.2 in the proof of Claim 3.3, and verify that $1 + \frac{r-1}{l^{r-1}} - \frac{r}{l^{r-2}}$ is not jump for $r = 5, l \geq 2$. This result is given in [7].

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