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ORIGINAL PAPER

Degree Conditions for the Existence of Vertex-Disjoint Cycles and Paths: A Survey

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Abstract In this paper, we survey results and conjectures on degree conditions for the existence of vertex-disjoint cycles and paths. In particular, we focus on the type of degree conditions, the type of cycles or paths, and relations between results or conjectures.

Keywords Packing · Partitioning · Cycles · Paths · Degree conditions

Contents

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1 Introduction

In this paper, we give a survey on degree conditions for *packing cycles* (*paths*) *in a graph* and *partitions of a graph into cycles* (*paths*), i.e., finding a prescribed number of vertex-disjoint cycles (paths) and vertex-partitions into a prescribed number of cycles (paths) in graphs. It is known that the problem of determining whether a given graph has such partitions or not, is NP-complete. Therefore, many researchers have investigated degree conditions for packing and partitioning in terms of, for example, minimum degree; average degree; degree sum of independent vertices. In 2001, Enomoto gave a survey on this [78]. Since then, many results have been published. The purpose of this paper is to give an update survey. For the convenience of the readers, we will provide many results and conjectures not only since 2001 but also before then.

In this survey, we also focus on the following:

1. We investigate closely various kinds of degree conditions. The reason is that it is important to determine on which vertices we impose degree conditions. For example, Ore's theorem (Theorem 2.1.2) says that the degrees of two non-adjacent vertices, not all vertices, is important for hamiltonicity of graphs. Fan's theorem (Theorem 2.1.9) says that it is the degrees of two vertices whose distance is two. We are interested in such types and the sharpness of degree conditions for packing cycles (paths) and partitions into cycles (paths).
2. We are interested in to discover relations between results, in particular, to link together seemingly disparate results. We think that it is important to find such relations. For example, a result on partitions into cycles and degenerate cycles implies a result on spanning trees with few leaves (see Sect. 3.1.4); a result on directed hamiltonian cycles implies a result on partitions into paths whose end vertices belong to a pre-specified vertex set (see Sect. 5.2.2); a result on cycles passing through a perfect matching in bipartite graphs implies a result on directed cycles in digraphs (see Sect. 6.2.2).
3. We will mention theorems and conjectures which can lead to results on packing cycles (paths) and partitions into cycles (paths). For example, results on packing subgraphs with degree constraints, are sometimes useful tools to get results on packing cycles (see Sect. 4.1); the BEC-conjecture and the Pósa-Seymour's conjecture are related to the El-Zahár's conjecture deeply (see Sect. 4.3); the results on partitions into any fixed graphs (e.g., Alon–Yuster's result (Theorem 4.5.1)) are useful to get results on El-Zahár-type problems in some case (see Sect. 4.5); the concept of H-linked is a generalization of that of the connectivity (i.e., (X, Y)-paths) and k-linked (see Sect. 5.2.4).

Terminology and Notation

All graphs considered here are finite. Unless stated otherwise, "graph" means a simple undirected graph.

We now prepare terminology and notation which will be used in subsequent sections. For terminology and notation not defined in this paper, we refer the readers to [60]. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of

G, respectively. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For an edge e, $V(e)$ denotes the set of end vertices of e. For $M \subseteq E(G)$, we let $V(M) = \bigcup_{e \in M} V(e)$.

When a graph H is isomorphic to G, we write $H \simeq G$. When H is a subgraph of G, we write $H \subseteq G$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and let $G - X = G[V(G) \setminus X]$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, we write $G - H$ instead of $G - V(H)$ for a subgraph H of G.

Let now G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. We let $G_1 \cup G_2$ denote the union of G_1 and G_2, and let $G_1 + G_2$ denote the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by joining each vertex in $V(G_1)$ to all vertices in $V(G_2)$. For an integer $s \geq 1$, sG_1 denotes the union of s vertex-disjoint copies of G_1.

We denote by K^n the complete graph of order n. The complete bipartite graph with partite sets of cardinalities m and n is denoted by $K^{m,n}$. We denote by P^l and C^l, respectively, the path of order l and the cycle of order l.

An edge subset M of a graph G is called a *matching* if no two edges in M have a common end vertex. A matching M is said to be *perfect* if every vertex of G is contained in some edge of M, and a matching of size k is called a *k-matching*.

Invariants

We introduce graph invariants and we will consider conditions on them for packing cycles (paths) and partitions into cycles (paths). Let G be a graph. We denote by $\alpha(G)$ and $\kappa(G)$ the independence number and the connectivity of G, respectively. Let $\delta(G)$, $\Delta(G)$ and $d(G)$ be the minimum degree, the maximum degree and the average degree of G, respectively. Let $c(G)$ and $g(G)$ be the *circumference* (i.e., the length of a longest cycle) and the *girth* (i.e., the length of a shortest cycle) of a graph G, respectively. We define $\sigma_s(G)$, $\sigma_s^t(G)$, $\mu_2(G)$, $\mu(G)$ and $NU(G)$ as follows. Here, $d_G(x)$ and $N_G(x)$ denote the degree and the neighborhood of a vertex x in G, respectively, and $\mathrm{dist}_G(x, y)$ denotes the distance between two vertices x and y in G.

- For an integer $s \geq 1$, if $\alpha(G) \geq s$, then let

$$\sigma_s(G) = \min \left\{ \sum_{x \in X} d_G(x) : X \text{ is an independent set of } G \text{ with } |X| = s \right\};$$

  otherwise, $\sigma_s(G) = +\infty$.
- For a vertex subset X of a graph G with $|X| \geq s$, we define

$$\Delta_s(X) = \max \left\{ \sum_{x \in Y} d_G(x) : Y \subseteq X,\ |Y| = s \right\}.$$

  For integers $t \geq s \geq 1$, if $\alpha(G) \geq t$, then let

$$\sigma_s^t(G) = \min \left\{ \Delta_s(X) : X \text{ is an independent set of } G \text{ with } |X| = t \right\};$$

  otherwise, $\sigma_s^t(G) = +\infty$.

- For a connected graph G, if $\alpha(G) \geq 2$,

$$\mu_2(G) = \min\left\{d_G(x) + d_G(y) : x, y \in V(G),\ \mathrm{dist}_G(x, y) = 2\right\};$$

otherwise, $\mu_2(G) = +\infty$.
- For a connected graph G, if $\alpha(G) \geq 2$, then let

$$\mu(G) = \min\left\{\max\left\{d_G(x), d_G(y)\right\} : x, y \in V(G),\ \mathrm{dist}_G(x, y) = 2\right\};$$

otherwise, $\mu(G) = +\infty$.
- If $\alpha(G) \geq 2$, then let

$$NU(G) = \min\left\{|N_G(x) \cup N_G(y)| : x, y \in V(G), x \neq y, xy \notin E(G)\right\};$$

otherwise, $NU(G) = +\infty$.

By the definition of $\sigma_s(G)$, $\sigma_s^t(G)$, $\mu_2(G)$ and $\mu(G)$, we obtain the following relation.

Proposition 1 *Let r, s, t be positive integers with $r \leq s \leq t$, and let G be a graph.*

(1) $s \cdot \delta(G) \leq s \cdot \frac{\sigma_r(G)}{r} \leq \sigma_s(G) \leq s \cdot \frac{\sigma_t(G)}{t} \leq \sigma_s^t(G)$.
(2) $\frac{\sigma_2(G)}{2} \leq \min\left\{\sigma_1^2(G), \frac{\mu_2(G)}{2}\right\} \leq \max\left\{\sigma_1^2(G), \frac{\mu_2(G)}{2}\right\} \leq \mu(G)$ *(if G is connected).*

We next define graph invariants for a class of bipartite graphs. In this paper, we denote by $G[X, Y]$ a bipartite graph G with partite sets X and Y, and $G[X, Y]$ is *balanced* if $|X| = |Y|$. For a bipartite graph $G[X, Y]$, we define $\delta_{1,1}(G)$ and $\sigma_{1,1}(G)$ as follows.

- $\delta_{1,1}(G) = \min\left\{d_G(x) + d_G(y) : x \in X, y \in Y\right\}$.
- If G is not a complete bipartite graph, then let

$$\sigma_{1,1}(G) = \min\left\{d_G(x) + d_G(y) : x \in X, y \in Y, xy \notin E(G)\right\};$$

otherwise, let $\sigma_{1,1}(G) = +\infty$.

By the definition of $\delta_{1,1}(G)$ and $\sigma_{1,1}(G)$, we obtain the following relation.

Proposition 2 *Let G be a bipartite graph. Then*

$$2 \cdot \delta(G) \leq \min\{\delta_{1,1}(G), \sigma_2(G)\} \leq \max\{\delta_{1,1}(G), \sigma_2(G)\} \leq \sigma_{1,1}(G).$$

We further define a graph invariant for a class of digraphs. Let D be a digraph, and we denote by $V(D)$ and $A(D)$ the vertex set and the arc set of D. In this paper, unless stated otherwise, we consider only simple directed graphs. For a vertex x of D, let $d_D^+(x)$ and $d_D^-(x)$ denote the out-degree and the in-degree of x in D, respectively. We define $\sigma_{1^+,1^-}(D)$ as follows.

- If D is not a complete digraph, then let

$$\sigma_{1^+,1^-}(D) = \min\left\{d_D^+(x) + d_D^-(y) : x, y \in V(D), x \neq y, (x, y) \notin A(D)\right\};$$

otherwise, let $\sigma_{1^+,1^-}(D) = +\infty$.

In reading this paper, it should be noted the following:

Notes
- In Sects. 3–6, "disjoint" always means "vertex-disjoint" and "partition" always means "vertex-partition" .
- Unless stated otherwise, the lower bounds on degree conditions in results in Sects. 2–6 are sharp.
- In order to mention implications between results in Sects. 2–6, we will implicity use Propositions 1 and 2 throughout Sects. 2–6.

2 Hamiltonian Cycles

The researches on packing cycles and partitions into cycles are mostly motivated by results on hamiltonian cycles. In this section, we introduce results on hamiltonicity of graphs which are related to results on partitions into cycles (paths) in latter sections. If the reader is familiar with the research area on hamiltonian cycles, we recommend to skip this section.

2.1 Hamiltonian Cycles in Graphs

Erdős and Gallai (1959) gave a condition on the number of edges of graphs (i.e., an average degree condition).

Theorem 2.1.1 (Erdős and Gallai [85]) *Let G be a graph of order $n \geq 3$. If $|E(G)| \geq \binom{n-1}{2} + 2$, that is, $d(G) \geq \frac{(n-1)(n-2)+4}{n}$, then G contains a hamiltonian cycle.*

Like this theorem, the problem on determining of the maximum number of edges in a graph not containing a specified subgraph is called a Turán-type problem.

Ore (1960) obtained a degree sum condition for hamiltonicity, which is classical and well known in graph theory. (In 1952, Dirac [61] gave a minimum degree condition.)

Theorem 2.1.2 (Ore [200]) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G contains a hamiltonian cycle.*

If a graph satisfies the Erdős–Gallai condition, then the graph also satisfies the Ore condition. Hence, Theorem 2.1.2 is stronger than Theorem 2.1.1.

Proposition 2.1.3 *Let G be a graph of order n. If $d(G) \geq \frac{(n-1)(n-2)+4}{n}$, then $\sigma_2(G) \geq n$.*

On the other hand, Chvátal and Erdős (1972) gave a relation on the independence number and the connectivity for hamiltonicity of graphs.

Theorem 2.1.4 (Chvátal and Erdős [51]) *Let G be a graph of order* $n \geq 3$. *If* $\alpha(G) \leq \kappa(G)$, *then G contains a hamiltonian cycle.*

Bondy (1978) pointed out that the graph satisfying the Ore condition also satisfies the Chvátal–Erdős condition, that is, Theorem 2.1.4 is stronger than Theorem 2.1.2.

Theorem 2.1.5 (Bondy [25]) *Let G be a graph of order n. If* $\sigma_2(G) \geq n$, *then* $\alpha(G) \leq \kappa(G)$.

By Theorems 2.1.4 and 2.1.5, we should consider degree sum conditions for graphs G with $\alpha(G) \geq \kappa(G) + 1$. In fact, Bondy (1980) extended Theorem 2.1.4 by giving the following degree condition.

Theorem 2.1.6 (Bondy [26]) *Let G be a graph of order* $n \geq 3$. *If* $\sigma_{\kappa(G)+1}(G) > \frac{1}{2}(\kappa(G) + 1)(n - 1)$, *then G contains a hamiltonian cycle.*

Yamashita (2008) extended Theorem 2.1.6 as follows.

Theorem 2.1.7 (Yamashita [248]) *Let G be a graph of order* $n \geq 3$. *If* $\sigma_2^{\kappa(G)+1}(G) \geq n$, *then G contains a hamiltonian cycle.*

Note that Theorem 2.1.6 implies Theorem 2.1.2, and Theorem 2.1.7 impies Theorem 2.1.6, by Proposition 1 (1).

Ainouche and Christofides (1995), Jung (1978), Nara (1980) and Schmeichel and Hayes (1985), independently, characterized non-hamiltonian graphs G with $\sigma_2(G) = |G| - 1$.

Theorem 2.1.8 (Ainouche and Christofides [3], Jung [135], Nara [198], Schmeichel and Hayes [212]) *Let G be a graph of order* $n \geq 3$. *If* $\sigma_2(G) \geq n - 1$, *then one of the following holds:*

(i) *G contains a hamiltonian cycle,*
(ii) $K^{m,m+1} \subseteq G \subseteq K^m + (m+1)K^1$, *where* $m = \frac{n-1}{2}$ *and* $n \geq 5$ *is odd,*
(iii) $G \simeq K^1 + (K^p \cup K^q)$ *for some positive integers* p, q *with* $p + q = n - 1$.

Fan (1984) extended Theorem 2.1.2 by considering the maximum degree of two vertices with distance two.

Theorem 2.1.9 (Fan [88]) *Let G be a* 2*-connected graph of order n. If* $\mu(G) \geq n/2$, *then G contains a hamiltonian cycle.*

The condition $\sigma_2(G) \geq |G|$ (≥ 3) yields that G is 2-connected. Hence, by Proposition 1 (2), Theorem 2.1.9 implies Theorem 2.1.2.

In 1972, Jung showed that Theorem 2.1.2 admits a weaker degree sum condition for 1-tough graphs. A graph G is 1-*tough* if $|S| \geq w(G - S)$ for every vertex set S of G with $w(G - S) \geq 2$, where $w(G - S)$ denotes the number of components of $G - S$.

Theorem 2.1.10 (Jung [135]) *Let G be a* 1*-tough graph of order* $n \geq 11$. *If* $\sigma_2(G) \geq n - 4$, *then G contains a hamiltonian cycle.*

On the other hand, Ore (1963) obtained a σ_2 condition for the existence of a hamiltonian path joining specified two vertices.

Theorem 2.1.11 (Ore [201]) *Let G be a graph of order $n \geq 3$, and let $x, y \in V(G)$. If $\sigma_2(G) \geq n+1$, then G contains a hamiltonian path such that x and y are the end vertices.*

By choosing two adjacent vertices as x and y in this theorem, we can obtain the following corollary.

Corollary 2.1.12 *Let G be a graph of order $n \geq 3$, and let $e \in E(G)$. If $\sigma_2(G) \geq n+1$, then G contains a hamiltonian cycle passing through e.*

By considering an extension of a matching to a hamiltonian cycle, Kronk (1969) generalized Corollary 2.1.12. (Häggkvist (1979) gave a σ_2 condition for $n \leq 3k-1$, see [120].)

Theorem 2.1.13 (Kronk [162]) *Let k be a positive integer, G be a graph of order $n \geq 3k$ and M be a k-matching in G. If $\sigma_2(G) \geq n+k$, then G contains a hamiltonian cycle passing through every edge of M.*

This theorem implies the following corollary on the existence of a hamiltonian cycle passing through a pre-specified linear forest, by contracting each path of order at least 3 to an edge.

Corollary 2.1.14 *Let k be a positive integer, and let G be a graph of order $n \geq 3k$. Further, let F be a subgraph whose component is a path (possibly its order is one) in G with $|E(F)| = k$. If $\sigma_2(G) \geq n+k$, then G contains a hamiltonian cycle passing through every path in F.*

2.2 Hamiltonian Cycles in Bipartite Graphs

Moon and Moser (1963) considered a bipartite version of Ore's Theorem (Theorem 2.1.2) and they gave the following $\sigma_{1,1}$ condition for hamiltonicity of balanced bipartite graphs.

Theorem 2.2.1 (Moon and Moser [197]) *Let G be a balanced bipartite graph of order $2n \geq 4$. If $\sigma_{1,1}(G) \geq n+1$, then G contains a hamiltonian cycle.*

Ferrara et al. (2012) characterized non-hamiltonian balanced bipartite graphs G of order $2n$ such that $\sigma_{1,1}(G) = n$.

Theorem 2.2.2 (Ferrara et al. [97]) *Let G be a balanced bipartite graph of order $2n \geq 4$. If $\sigma_{1,1}(G) = n$, then (i) G contains a hamiltonian cycle, or (ii) G is one of two exceptional graphs of order 8, or (iii) G belongs to an exceptional class.*

Zamani and West (2012) considered a bipartite version of Corollary 2.1.14.

Theorem 2.2.3 (Zamani and West [255]) *Let m be a positive integer, and let G be a balanced bipartite graph of order n. Further, let F be a subgraph consisting of t_1 paths of odd length and t_2 paths of positive even length in G with $|E(F)| = k$. If*

$$2\sigma_{1,1}(G) \geq \begin{cases} n+k+2 & t_1 = 0 \text{ or } (t_1, t_2) \in \{(1,0), (2,0)\}, \\ n+k & \text{otherwise,} \end{cases}$$

then G contains a hamiltonian cycle passing through every path of F.

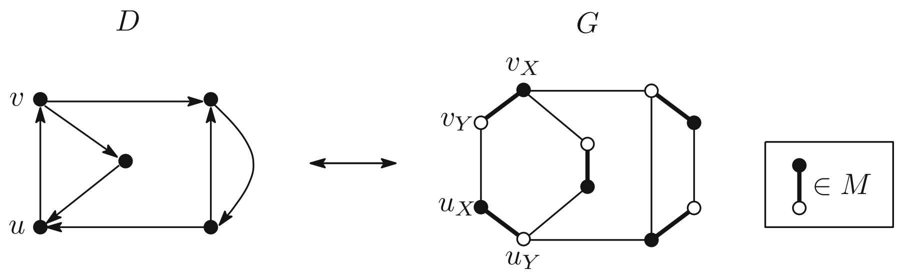

Fig. 1 The relation between digraphs and bipartite graphs with a perfect matching

The degree condition is sharp when $n \geq 3k + 1$.

In the rest of this section, we mention a relation between a directed cycle in digraphs and a cycle passing through a pre-specified perfect matching in bipartite graphs.

A directed cycle of a digraph is called a *directed hamiltonian cycle* if it contains all the vertices. Woodall (1972) gave a digraph version of Ore's Theorem (Theorem 2.1.2) as follows.

Theorem 2.2.4 (Woodall [247]) *Let D be a digraph of order* $n \geq 2$. *If* $\sigma_{1^+,1^-}(D) \geq n$, *then D contains a directed hamiltonian cycle.*

Considering the digraph obtained from a given graph G by replacing each edge uv in G with two arcs (u, v) and (v, u), we see that Theorem 2.2.4 implies Theorem 2.1.2.

Theorem 2.2.4 is related to degree conditions for the existence of a hamiltonian cycle passing through a pre-specified perfect matching in bipartite graphs. In fact, Las Vergnas (1972) rephrased Theorem 2.2.4 as follows.

Theorem 2.2.5 (Las Vergnas [167]) *Let G be a balanced bipartite graph of order* $2n \geq 4$, *and let M be a perfect matching in G. If* $\sigma_{1,1}(G) \geq n + 2$, *then G contains a hamiltonian cycle passing through every edge of M.*

Remark 2.2.6 (*see also* [49,120,258]) For a given digraph D, consider the following undirected simple graph G: We split each vertex v in D into two vertices v_X and v_Y and replace each arc (u, v) in $A(D)$ with a simple edge $u_X v_Y$, and we add the perfect matching $M = \{v_X v_Y : v \in V(D)\}$. Then, the resulting graph G is a balanced bipartite graph of order $2|D|$ with partite sets $\{v_X : v \in V(D)\}$ and $\{v_Y : v \in V(D)\}$ satisfying the following: $\sigma_{1,1}(G) = \sigma_{1^+,1^-}(D) + 2$; an alternating cycle with respect to M (i.e., the edges belong to M and not to M, alternately) of length $2l$ (≥ 4) in G corresponds to a directed cycle of length l in D (see also Fig. 1). Therefore, Theorem 2.2.5 implies Theorem 2.2.4. On the other hand, by considering the reverse of the above construction, we see that Theorem 2.2.4 implies Theorem 2.2.5.

3 Disjoint Cycles in Graphs

3.1 Cycles in Graphs

3.1.1 Packing Cycles

It is easy to see that if the minimum degree of a graph is at least two, then there exists a cycle in it.

Proposition 3.1.1 *Let G be a graph. If $\delta(G) \geq 2$, then G contains a cycle.*

In 1965, Lovász [178] characterized multigraphs G with $\delta(G) \geq 3$ that do not have two disjoint cycles. As a natural generalization of Proposition 3.1.1, Corrádi and Hajnal (1963) gave the following minimum degree condition for the existence of a prescribed number of disjoint cycles.

Theorem 3.1.2 (Corrádi and Hajnal [53]) *Let k be a positive integer, and let G be a graph of order at least $3k$. If $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Chiba et al. generalized Theorem 3.1.2 by considering a minimum degree condition for the existence of k disjoint cycles of even length (see Theorem 3.3.26). For other related generalizations, see Theorems 3.1.41 and 3.3.22.

In 1989, Justesen [136] extended Theorem 3.1.2 into a σ_2 version without a proof. But, the degree condition was not sharp. Later, Enomoto (1998) and Wang (1999), independently, gave a sharp σ_2 condition.

Theorem 3.1.3 (Enomoto [77], Wang [233]) *Let k be a positive integer, and let G be a graph of order at least $3k$. If $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.*

Kierstead et al. (2017) extended this theorem as follows.

Theorem 3.1.4 (Kierstead et al. [150]) *Let k be an integer with $k \geq 4$, and let G be a graph of order at least $3k + 1$. If $\sigma_2(G) \geq 4k - 3$ and $\alpha(G) \leq |G| - 2k$, then G contains k disjoint cycles.*

Note that the condition $\alpha(G) \leq |G| - 2k$ is necessary for a graph to contain k disjoint cycles. Note also that $\sigma_2(G) \geq 4k - 1$ implies $\alpha(G) \leq |G| - 2k$.

In fact, they characterized graphs G of order at least $3k + 1$ with $\sigma_2(G) \geq 4k - 3$ and $\alpha(G) \leq |G| - 2k$ that contain no k disjoint cycles for $k \geq 2$. Furthermore, they characterized graphs G of order at least $3k$ with $\delta(G) \geq 2k - 1$ that contain no k disjoint cycles. By using this result, in [151], they answered Dirac's question [63] on $(2k - 1)$-connected multigraphs without k disjoint cycles.

Fujita et al. (2006) gave a σ_3 condition as follows, which is a weaker degree condition than the ones of Theorems 3.1.2 and 3.1.3.

Theorem 3.1.5 (Fujita et al. [103]) *Let k be an integer with $k \geq 2$, and let G be a graph of order at least $3k + 2$. If $\sigma_3(G) \geq 6k - 2$, then G contains k disjoint cycles.*

In 2018, Gould et al. further extended Theorems 3.1.2, 3.1.3 and 3.1.5 by considering a σ_4 condition. (We do not know whether the order condition in Theorem 3.1.6 is sharp or not.)

Theorem 3.1.6 (Gould et al. [112]) *Let k be an integer with $k \geq 2$, and let G be a graph of order at least $7k+1$. If $\sigma_4(G) \geq 8k-3$, then G contains k disjoint cycles.*

They also posed the following more general conjecture.

Conjecture 3.1.7 (Gould et al. [112]) *Let k and t be integers with $k \geq 2$ and $t \geq 1$, and let G be a graph of sufficiently large order. If $\sigma_t(G) \geq 2kt-(t-1)$, then G contains k disjoint cycles.*

Theorems 3.1.2, 3.1.3, 3.1.5 and 3.1.6 support this conjecture for $1 \leq t \leq 4$. In [183], Ma and Yan announced that this conjecture is settled for $t \geq 5$.

Jiao et al. (2017) extended Theorem 3.1.2 for connected graphs by giving the following μ_2 condition.

Theorem 3.1.8 (Jiao et al. [133]) *Let k be a positive integer, and let G be a connected graph of order at least $3k$. If $\mu_2(G) \geq 4k$, then (i) G contains k disjoint cycles, or (ii) G is isomorphic to a graph obtained from $H_1 + H_2$ by adding one pendant edge to each vertex of H_2, where $(2k-1)K^1 \subseteq H_1 \subseteq K^{2k-1}$ and $H_2 \simeq lK^1$ for some integer l with $2l \geq k+1$.*

Yan et al. extended Theorem 3.1.2 by giving the following σ_1^2 condition.

Theorem 3.1.9 (Yan et al. [254]) *Let k be a positive integer, and let G be a graph of order at least $4k$. If $\sigma_1^2(G) \geq 2k$, then G contains k disjoint cycles.*

On the other hand, Dirac and Erdős (1963) extended Theorem 3.1.2 for graphs with large order by giving a condition on the number of vertices of high degree. Here, $V_{\geq 2k}$ and $V_{\leq 2k-2}$ are sets of vertices of degree at least $2k$ and at most $2k-2$, respectively.

Theorem 3.1.10 (Dirac and Erdős [64]) *Let k be an integer with $k \geq 3$, and let G be a graph. If $|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq k^2 + 2k - 4$, then G contains k disjoint cycles.*

The bound "$k^2 + 2k - 4$" in this theorem is not sharp. In 2017, Kierstead et al. significantly improved the bound and they generalized Theorem 3.1.2 as follows.

Theorem 3.1.11 (Kierstead et al. [149]) *Let k be an integer with $k \geq 2$, G be a graph of order at least $3k$, and t be the maximum number of disjoint triangles in G. If $|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 2k + t$, then G contains k disjoint cycles.*

Corollary 3.1.12 (Kierstead et al. [149]) *Let k be an integer with $k \geq 2$, and let G be a graph. If $|V_{\geq 2k}| - |V_{\leq 2k-2}| \geq 3k$, then G contains k disjoint cycles.*

The conditions in these two results are sharp when $|G| = 3k$.

Erdős and Pósa (1962) considered the Turán-type problem: for given positive integers n and k with $n \geq 3k$, what is the maximal graph of order n that contains no k disjoint cycles? The following result is also one of the classical results in graph theory.

Theorem 3.1.13 (Erdős and Pósa [86]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 24k$. If*

$$|E(G)| \geq \binom{2k-1}{2} + (2k-1)(n-2k+1), \text{ that is, } d(G) \geq 4k-2-\frac{2(2k^2-k)}{n},$$

then (i) G contains k disjoint cycles, or (ii) $G \simeq K^{2k-1} + (n-2k+1)K^1$.

They also conjectured that this theorem with a few modifications can be extended to $n \geq 3k$. In 1989, Justesen proved the conjecture.

Theorem 3.1.14 (Justesen [136]) *Let k be an integer with* $k \geq 2$, *and let G be a graph of order* $n \geq 3k$. *If*

$$|E(G)| \geq \max\left\{\binom{2k-1}{2} + (2k-1)(n-2k+1), \binom{3k-1}{2} + n - 3k + 2\right\},$$

then (i) G contains k disjoint cycles, or (ii) $G \simeq K^{2k-1} + (n-2k+1)K^1$.

In 1996, Andreae [10] characterized graphs G with $|E(G)| = \max\left\{\binom{2k-1}{2} + (2k-1)(n-2k+1), \binom{3k-1}{2} + n - 3k + 1\right\}$ which contain no k disjoint cycles.

Faudree and Gould (2005) considered a neighborhood union condition. (The $NU(G)$ condition in Theorem 3.1.15 is sharp when $|G| = 3k+1$ or $k = 1$.)

Theorem 3.1.15 (Faudree and Gould [90]) *Let k be a positive integer, and let G be a graph of order at least* $3k$. *If* $NU(G) \geq 3k$, *then G contains k disjoint cycles.*

Gould et al. (2013) showed that this neighborhood union condition can be weakened if the order of a graph is larger. (The $NU(G)$ condition in Theorem 3.1.16 is sharp when $k = 1, 2$, and the order condition is not sharp.)

Theorem 3.1.16 (Gould et al. [110]) *Let k be a positive integer, and let G be a graph of order at least* $30k$. *If* $NU(G) \geq 2k+1$, *then G contains k disjoint cycles.*

3.1.2 Partitions into Cycles

Brandt et al. (1997) investigated a σ_2 condition for graphs to be partitioned into k cycles and they generalized Ore's Theorem (Theorem 2.1.2) as follows.

Theorem 3.1.17 (Brandt et al. [27]) *Let k be a positive integer, and let G be a graph of order* $n \geq 4k-1$. *If* $\sigma_2(G) \geq n$, *then G can be partitioned into k cycles, i.e., G contains k disjoint cycles* $C_1, \ldots, C_k$ *satisfying* $V(G) = \bigcup_{1 \leq i \leq k} V(C_i)$.

Note that, in [27], the order condition is not "$n \geq 4k-1$" but "$n \geq 4k$" (see below for the detail). The condition $n \geq 4k-1$ is best possible. In 1996, Alon and Fischer [6] proved the asymptotic (minimum degree) version of Theorem 3.1.17 for sufficiently large graphs by using the regularity lemma. By Proposition 2.1.3 and Theorem 3.1.17, the Erdős-Gallai condition in Theorem 2.1.1 (and the order condition $n \geq 4k-1$) also guarantees the existence of a partition into k cycles.

The following two steps are often considered for problems of partitions into cycles (with some additional properties).

Step 1: To show the existence of k disjoint cycles (**Packing**).
Step 2: To show that the collection of cycles in Step 1 can be transformed into a collection of cycles forming a partition of G (**Partitioning**).

In fact, in order to prove Theorem 3.1.17, Brandt et al. first applied the Justesen's result [136] (Step 1), and then they constructed a partition into k cycles from the disjoint cycles (Step 2), see [27, Lemmas 1 and 2]. By applying Theorem 3.1.3 instead of the Justesen's result, we can improve the order condition into "$n \geq 4k - 1$".

We will introduce such type of results in latter sections (e.g., see Sects. 3.2.1, 3.2.2, 3.4.2, 3.4.4, 4.1.1, 5.2.4, 6.2.1 and 6.2.3).

Like Theorem 2.1.10, it is known that we can weaken the degree condition for 1-tough graphs. Faudree et al. (2004) proved the following result.

Theorem 3.1.18 (Faudree et al. [92]) *There exists an integer n_0 such that if G is a 1-tough graph of order $n \geq n_0$ with $\delta(G) \geq \frac{n}{2} - 2$, and k is a positive integer with $k \leq \frac{n}{4} - 4$, then G can be partitioned into k cycles.*

Moreover, in 2005, they also conjectured that the coefficient $\frac{1}{2}$ of n can be relaxed if hamiltonicity is assumed.

Conjecture 3.1.19 (Faudree et al. [93]) *For any integer $k \geq 2$, there are a positive real number $c_k < \frac{1}{2}$ and integers a_k and n_k such that if G is a hamiltonian graph of order $n \geq n_k$ with $\delta(G) \geq c_k n + a_k$, then G can be partitioned into k cycles.*

Sárközy (2008) settled this conjecture by using the regularity-blow-up method.

Theorem 3.1.20 (Sárközy [210]) *There exists a real number $\varepsilon > 0$ such that, for any integer $k \geq 2$, there is an integer $n_0 = n_0(k)$ depending on only k such that if G is a hamiltonian graph of order $n \geq n_0$ with $\delta(G) \geq \left(\frac{1}{2} - \varepsilon\right)n$, then G can be partitioned into k cycles.*

DeBiasio et al. (2014) improved this result without the use of the regularity lemma. (It is unknown that whether the degree condition in Theorem 3.1.21 is sharp or not.)

Theorem 3.1.21 (DeBiasio et al. [59]) *Let k be a positive integer, ε be a real number with $0 < \varepsilon < \frac{1}{10}$, and G be a hamiltonian graph of order $n \geq \frac{3k}{\varepsilon}$. If $\delta(G) \geq \left(\frac{2}{5} + \varepsilon\right)n$, then G can be partitioned into k cycles.*

As mentioned in the above, Step 1 and 2 are often considered in this type of problem. On the other hand, in the proof of Theorem 3.1.21, DeBiasio et al. transformed a hamiltonian cycle in a graph into k disjoint cycles that partition the graph. In this sense, the proof technique in [59] is interesting. In particular, by considering the following problem, we may be able to give an alternating proof of Theorem 3.1.17, and the proof technique may be useful for a similar type of problem.

Problem 3.1.22 *Can we improve the minimum degree condition in Theorem 3.1.21 into a σ_2 condition?*

3.1.3 The Independence Number and the Connectivity for Partitions

In this section, we consider the relationship between the independence number and the connectivity for partitions.

As mentioned in Sect. 3.1.2, the Ore condition in Theorem 2.1.2 guarantees the existence of a partition into k cycles (see Theorem 3.1.17). Considering this relation, it would be natural to conjecture that the condition for hamiltonicity of graphs in Theorem 2.1.4 also guarantees the existence of a partition into k cycles (see also [140, Problems 1.1–1.2] and [37, Conjecture 1]).

Conjecture 3.1.23 *Let k be a positive integer, and let G be a graph of order at least $4k-1$. If $\alpha(G) \leq \kappa(G)$, then G can be partitioned into k cycles.*

Note that by Theorem 2.1.5, this conjecture is stronger than Theorem 3.1.17. In 2003, Kaneko and Yoshimoto proved the case $k = 2$ for 4-connected graphs (see [140]). But, Egawa has pointed out that the proof misses one case to be considered (see [37,209]).

Chen et al. (2007) gave the following partial solution to Conjecture 3.1.23. Here, $r(l, m)$ denotes the Ramsey number, i.e., the smallest integer n such that every graph of order at least n contains a clique of size l or an independent set of size m.

Theorem 3.1.24 (Chen et al. [37]) *Let k and α be positive integers, and let G be a 2-connected graph with $\alpha(G) = \alpha \leq \kappa(G)$.*

(1) *If $|G| \geq k \cdot r(\alpha+4, \alpha+1)$, then G can be partitioned into k cycles.*
(2) *If $|G| \geq r(2\alpha+3, \alpha+1) + 3(k-1)$, then G can be partitioned into k cycles such that $k-1$ of them have length 3.*

On the other hand, in the proof of Chvátal–Erdős' Theorem (Theorem 2.1.4) of [51], by replacing the longest cycle C with maximum k disjoint cycles $C_1, \ldots, C_k$, we can obtain the following statement: If a graph G contains k disjoint cycles and $\alpha(G) \leq \lceil \kappa(G)/k \rceil$, then G can be partitioned into k cycles. Therefore, we should consider degree conditions for partitions into k cycles in a graph G with $\alpha(G) \geq \lceil \kappa(G)/k \rceil + 1$ (if the graph G contains k disjoint cycles). By considering this observation, Chiba (2017) gave the following degree condition for the existence of a partition into k cycles, which is a common generalization of Theorems 2.1.7 and 3.1.17.

Theorem 3.1.25 (Chiba [40]) *Let k be a positive integer, and let G be a graph of order $n \geq 5k-2$. If $\sigma_2^{\lceil \kappa(G)/k \rceil + 1}(G) \geq n$, then G can be partitioned into k cycles.*

As a corollary of this theorem, we can obtain the Bondy-type and the Chvátal–Erdős-type conditions as follows.

Corollary 3.1.26 *Let k be a positive integer, and let G be a graph of order $n \geq 5k-2$.*

(1) *If $\sigma_{\lceil \kappa(G)/k \rceil + 1}(G) > \frac{1}{2}(\lceil \kappa(G)/k \rceil + 1)(n-1)$, then G can be partitioned into k cycles.*
(2) *If $\alpha(G) \leq \lceil \kappa(G)/k \rceil$, then G can be partitioned into k cycles.*

The order condition "$n \geq 5k-2$" in Theorem 3.1.25 is required only to show the existence of k disjoint cycles (Step 1).

Finally, we remark a relation on the independence number and the minimum degree for partitions into unprescribed number of cycles. By using Tutte's factor theorem [225], Niessen proved that every graph G with $\delta(G) \geq 2$ and $\alpha(G) < \delta(G)$ has a 2-factor and characterized graphs G with $\delta(G) \geq 2$ and $\alpha(G) = \delta(G)$ which have no 2-factor (see [199, Theorems 1–2]).

3.1.4 Cycles and Degenerate Cycles

Enomoto and Li (2004) investigated a σ_2 condition for partitions into k cycles by regarding K^1 and K^2 as cycles, which are called *degenerate cycles*. They showed that a weaker condition than the one of Theorem 3.1.17 is sufficient. Recall that C^l denotes the cycle of order l.

Theorem 3.1.27 (Enomoto and Li [80]) *Let k be a positive integer, and let G be a graph of order $n \geq k$. If $\sigma_2(G) \geq n - k + 1$, then (i) G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 or K^2 for $1 \leq i \leq k$, or (ii) $k = 2$ and $G \simeq C^5$.*

The complete bipartite graph $K^{m,m+1}$ shows that K^1 is necessary in this conclusion. On the other hand, Hu and Li (2009) showed that if the order of a graph is large, then we do not need to consider K^2 as a degenerate cycle.

Theorem 3.1.28 (Hu and Li [128]) *Let k be a positive integer, and let G be a graph of order $n \geq \max\{k + 12, 10k - 9\}$. If $\sigma_2(G) \geq n - k + 1$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 for $1 \leq i \leq k$.*

Fujita (2005) showed that if the σ_2 condition strengthens, then we can reduce the number of K^1s.

Theorem 3.1.29 (Fujita [100]) *Let k and r be integers with $2 \leq r \leq k - 2$, and let G be a graph of order $n \geq 7k$. If $\sigma_2(G) \geq n - r$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 for $1 \leq i \leq r$, and H_i is a cycle for $r + 1 \leq i \leq k$.*

Kawarabayashi (2000) characterized graphs G with $\sigma_2(G) = |G| - 1$ that cannot be partitioned into k cycles, which corresponds to Theorem 2.1.8.

Theorem 3.1.30 (Kawarabayashi [144]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k$. If $\sigma_2(G) \geq n - 1$, then one of the following holds:*

(i) *G can be partitioned into k cycles,*
(ii) *$K^{\frac{n-1}{2}, \frac{n+1}{2}} \subseteq G \subseteq K^{\frac{n-1}{2}} + \frac{n+1}{2} K^1$,*
(iii) *$G \simeq K^1 + (K^1 \cup K^{n-2})$.*

Note that if (ii) or (iii) of this theorem holds, then G contains $k - 1$ disjoint cycles covering $n - 1$ vertices of G. In [100], Fujita pointed out that the following theorem is obtained from Theorems 3.1.17, 3.1.28, 3.1.29 and 3.1.30.

Theorem 3.1.31 *Let k and r be integers with $k \geq 1$ and $0 \leq r \leq k - 1$, and let G be a graph of order $n \geq 10k + 3$. If $\sigma_2(G) \geq n - r$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 for $1 \leq i \leq r$, and H_i is a cycle for $r + 1 \leq i \leq k$.*

On the other hand, Fujita (2009) improved Theorem 3.1.27 by giving a σ_1^2 condition as follows. (In [42], Chiba and Fujita considered a cyclable version of this theorem.)

Theorem 3.1.32 (Fujita [102]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq k$. If $\sigma_1^2(G) \geq \frac{n-k+1}{2}$, then (i) G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 or K^2 for $1 \leq i \leq k$, or (ii) $k = 3$ and $G \simeq K^1 \cup C^5$, or (iii) $k = 2$ and $G \simeq C^5$.*

We also remark a relation on the independence number and the minimum degree for partitions into unprescribed number of cycles and degenerate cycles. As mentioned in Sect. 3.1.3, the condition $\alpha(G) < \delta(G)$ implies the existence of a 2-factor. On the other hand, it is not always true that every graph G with $\alpha(G) \geq \delta(G)$ has a 2-factor. However, Bekkai and Kouider proved that in a graph G with $\alpha(G) \geq \delta(G)$, there is a spanning subgraph of G such that each component is a cycle or a degenerate cycle, and the number of degenerate cycles is at most $\alpha(G) - \delta(G) + 1$ (note that the total number of cycles and degenerate cycles are not specified), see [16].

We further introduce a result on packing cycles and degenerate cycles. Note that, in this case, we should not regard K^1 as a cycle. Andreae (1996) considered the Turán-type problem as follows.

Theorem 3.1.33 (Andreae [10]) *Let k and r be integers with $k \geq r \geq 1$ and $k \geq 2$, and let G be a graph of order $n \geq 3k - r$. If*

$$|E(G)| \geq \max\left\{\binom{2k-r-1}{2} + (2k-r-1)(n-2k+r+1), \binom{3k-r-1}{2}\right\}$$

then (i) G contains k disjoint subgraphs $H_1, \ldots, H_k$ such that $H_i \simeq K^2$ for $1 \leq i \leq r$, and H_i is a cycle for $r+1 \leq i \leq k$, or (ii) $G \simeq K^{2k-r-1} + (n-2k+r+1)K_1$, or (iii) $G \simeq K^{3k-r-1} \cup (n-3k+r+1)K^1$.

The case $k = r$ is a theorem of Erdős and Gallai [85] concerning the existence of a k-matching.

In the rest of this section, we mention a relation with spanning trees with at most k leaves. The results mentioned in the above are useful tools to get degree conditions for the existence of such spanning trees. In fact, Theorem 3.1.27 implies the following theorem obtained by Broersma and Tuinstra (1998).

Theorem 3.1.34 (Broersma and Tuinstra [28]) *Let k be an integer with $k \geq 2$, and let G be a connected graph of order $n \geq 2$. If $\sigma_2(G) \geq n-k+1$, then G has a spanning tree with at most k leaves.*

Proposition 3.1.35 *Theorem 3.1.27 implies Theorem 3.1.34.*

Proof Let k be an integer with $k \geq 2$, and let G be a connected graph of order $n \geq 2$ with $\sigma_2(G) \geq n-k+1$. We show that G has a spanning tree with at most k leaves. It is enough to consider for the case $n \geq k$. Then, by Theorem 3.1.27, G can be partitioned into k cycles and degenerated cycles (see Fig. 2). Since G is connected, we can obtain a spanning tree with at most k leaves from the partition by adding edges connecting components and by deleting one appropriate edge of each cycle. □

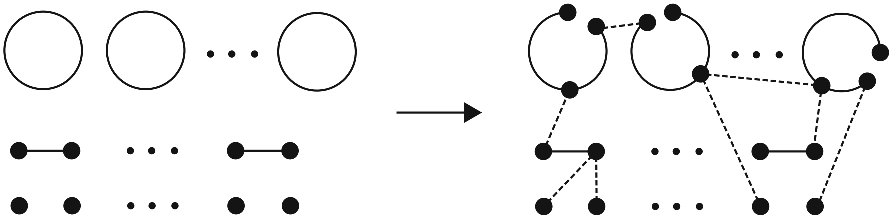

Fig. 2 The construction of a spanning tree with at most k leaves

3.1.5 Disjoint Cycles Covers

In 1976, Bermond and Linial, independently, gave an Ore-type condition for circumference, which is a generalization of Theorem 2.1.2. (The minimum degree condition was obtained by Dirac [61].)

Theorem 3.1.36 (Bermond [18], Linial [176]) *Let d be a positive integer, and let G be a 2-connected graph of order n. If $\sigma_2(G) \geq d$, then G has a cycle of length at least* $\min\{d, n\}$.

We consider a generalization of this theorem in terms of a vertex cover by k disjoint cycles. More precisely, we consider the following problem: How large is the order of the union of k disjoint cycles in a graph G if $\sigma_2(G) \geq d$? Egawa et al. (2005) and Egawa et al. (2003) gave an answer of this question for $k \geq 3$ and $k = 2$, respectively.

Theorem 3.1.37 (Egawa et al. [71,73]) *Let k and d be integers with $k \geq 2$ and $d \geq 4k - 1$, and let G be a graph of order $n \geq 3k$. If $\sigma_2(G) \geq d$, then G contains k disjoint cycles covering at least* $\min\{d, n\}$ *vertices of G, i.e., G contains k disjoint cycles $C_1, \ldots, C_k$ such that* $\left|\bigcup_{1 \leq i \leq k} C_i\right| \geq \min\{d, n\}$.

This is also a generalization of Theorems 3.1.3 and 3.1.17. By the sharpness of the degree condition in Theorem 3.1.3, the condition $d \geq 4k - 1$ is necessary in Theorem 3.1.37.

Theorem 3.1.37 says that there are k disjoint cycles in a graph G such that the average length of the cycles is large depending on $\sigma_2(G)$. On the other hand, are there k disjoint cycles in G such that "the length of each cycle" is large depending on $\sigma_2(G)$? Concerning a minimum degree version of this question, Wang (1994) proved the following result. (The minimum degree condition in Theorem 3.1.38 is sharp when $l = 2$.)

Theorem 3.1.38 (Wang [227]) *Let k and l be integers with $k \geq 1$ and $l \geq 2$, and let G be a graph of order $n \geq (l + 1)k$. If $\delta(G) \geq lk$, then G contains k disjoint cycles of length at least $l + 1$.*

In 2012, Wang also posed the following two conjectures.

Conjecture 3.1.39 (Wang [242]) *Let k and l be integers with $k \geq 2$ and $l \geq 3$, and let G be a graph of order $n \geq 2lk$. If $\delta(G) \geq lk$, then (i) G contains k disjoint cycles of length at least $2l$, or (ii) k is odd and $n = 2lk + r$ for some $1 \leq r \leq 2l - 2$.*

Conjecture 3.1.40 (Wang [242]) *Let k and l be integers with $k \geq 2$ and $l \geq 2$, and let G be a graph of order $n \geq (2l-1)k$. If $\delta(G) \geq lk$, then G contains k disjoint cycles of length at least $2l-1$.*

The graph $(lk-1)K^1 + (n-lk+1)K^1$ $(n \geq 2lk-2)$ shows the sharpness of the lower bounds on degree conditions.

In the same paper, Wang obtained the following result corresponding to the case $l = 2$ of Conjecture 3.1.39.

Theorem 3.1.41 (Wang [242]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k$. If $\delta(G) \geq 2k$, then (i) G contains k disjoint cycles of length at least 4, or (ii) G is a 4-regular graph of order 9, or (iii) G belongs to two exceptional classes.*

Chiba et al. showed that the same degree condition in this theorem implies the existence of k disjoint cycles of even length, except for two exceptional classes, for sufficiently large graphs (see Theorem 3.3.26).

In 2013, Wang showed that Conjecture 3.1.40 is true for $k = 2$ and $n \geq 9$, see [244].

Jiao et al. (2017) conjectured a μ_2 version of Conjecture 3.1.40 as follows, and they proved the case $l = 2$ (see Theorem 3.1.8).

Conjecture 3.1.42 (Jiao et al. [133]) *Let k and l be integers with $k \geq 1$ and $l \geq 2$, and let G be a graph of order $n \geq (2l-1)k$. If $\mu_2(G) \geq 2lk$, then (i) G contains k disjoint cycles of length at least $2l-1$, or (ii) $n-lk+1$ is even and G belongs to an exceptional class.*

On the other hand, Harvey and Wood (2015) gave the following average degree condition. They proved this result by using the concept on graph minors.

Theorem 3.1.43 (Harvey and Wood [126]) *Let k and l be integers with $k \geq 6$ and $l \geq 3$, and let G be a graph. If $d(G) \geq \frac{4}{3}lk$, then G contains k disjoint cycles of length at least l.*

In 2017, Csóka et al. improved this result as follows, which was originally conjectured by Harvey and Wood in [126].

Theorem 3.1.44 (Csóka et al. [56]) *Let k and l be integers with $k \geq 2$ and $l \geq 3$, and let G be a graph. If $d(G) > (l+1)k-2$, then G contains k disjoint cycles of length at least l.*

In fact, they gave a more general result and also proved the conjecture of Reed and Wood [208] ("every graph with average degree at least $\frac{4}{3}t-2$ contains every 2-regular graph of order t as a minor"), see [56, Theorem 3].

It is easy to show that every graph of average degree at least $2r$ contains a subgraph of minimum degree at least $r+1$. If we can show the existence of such a subgraph of sufficiently large order, then Conjecture 3.1.40 implies Theorem 3.1.44. However, it might be difficult to show the existence of such a subgraph.

The relations between the number of vertices covered by disjoint cycles and vertex cuts (resp., the cyclomatic number) have been investigated in [125] (resp., in [207]).

3.2 Cycles Passing Through Pre-specified Elements

3.2.1 Specified Edges

In this section, we focus on degree conditions for graphs to be partitioned into k cycles in which each cycle contains an edge in a pre-specified matching of size k. From the relation between Theorem 2.1.11 and Corollary 2.1.12, we can see that the existence of such disjoint cycles is deeply related to the concept of "k-linked". We will discuss it in Sect. 5.2.

Egawa et al. (2000) gave a σ_2 condition for the above mentioned partition, which was conjectured by Wang [231]. (They also gave δ and σ_2 conditions for $3k \leq n < 4k$ in [70]. See Theorem 3.2.11 for a δ condition in the case that n is large.)

Theorem 3.2.1 (Egawa et al. [70]) *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 4k - 1$ and M be a k-matching in G. If $\sigma_2(G) \geq n + 2k - 2$, then G can be partitioned into k cycles $C_1, \dots, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$.*

Note that Corollary 2.1.12 corresponds to the case $k = 1$, but the σ_2 condition is slightly different.

Ishigami and Wang (2002) showed that the degree condition in Theorem 3.2.1 also implies the following.

Theorem 3.2.2 (Ishigami and Wang [131]) *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 4k - 1$ and M be a k-matching in G. If $\sigma_2(G) \geq n + 2k - 2$, then (i) G can be partitioned into k cycles $C_1, \dots, C_{k-1}, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$, and $|C_i| \leq 4$ for $1 \leq i \leq k - 1$, or (ii) $uv \in E(G)$ for all $u \in V(M)$ and $v \in V(G) \setminus V(M)$, and $G - V(M) \simeq K^l \cup K^{n-2k-l}$ for some integer l with $2k - 1 \leq l \leq n - 4k + 1$.*

In order to show Theorems 3.2.1 and 3.2.2, they considered the same steps as the ones mentioned in Sect. 3.1, see Step 1 (Packing) and Step 2 (Partitioning). In particular, we here want to emphasize appropriate degree sum conditions for Step 1 and Step 2 in the proof of Theorem 3.2.1 (see Theorems 3.2.3 and 3.2.4).

In [70], Egawa et al. proved the following result for Step 1. They actually showed that it is important to consider the degree sum of a vertex in a k-matching M and a vertex in $V(G)\backslash V(M)$.

Theorem 3.2.3 (Egawa et al. [70]) *Let k be a positive integer, G be a graph of order $n \geq 4k - 1$ and M be a k-matching in G. If*

(M1) *$d_G(u) + d_G(v) \geq n + 2k - 2$ for $u \in V(M)$ and $v \in V(G) \setminus V(M)$ with $uv \notin E(G)$,*

then G contains k disjoint cycles $C_1, \dots, C_k$ such that $|E(C_i) \cap M| = 1$ and $|C_i| \leq 4$ for $1 \leq i \leq k$.

Note that the degree condition (M1) cannot be weakened even if we drop the condition "$|C_i| \leq 4$ for $1 \leq i \leq k$" in the conclusion.

On the other hand, in order to transform the disjoint cycles in Theorem 3.2.3 into a partition, it is important to consider the degree sum of two vertices in $V(G)\backslash V(M)$.

Theorem 3.2.4 (Egawa et al. [70]) *Let k be an integer with $k \geq 2$, G be a $(k+1)$-connected graph of order n and M be a k-matching in G. Suppose that G contains k disjoint cycles $D_1, \ldots, D_k$ such that $|E(D_i) \cap M| = 1$ for $1 \leq i \leq k$. If*

(M2) *$d_G(u) + d_G(v) \geq n + k$ for $u, v \in V(G) \setminus V(M)$ with $u \neq v$, $uv \notin E(G)$,*

then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$.

We now claim that the degree conditions (M1) and (M2) lead to the conclusion of Theorem 3.2.1 (see Corollary 3.2.6). To do that, we first remark the following corollary which is obtained from Theorem 3.2.4.

Corollary 3.2.5 *Let k be an integer with $k \geq 2$, G be a graph of order n and M be a k-matching in G. Suppose that $D_1, \ldots, D_k$ are the same k cycles as the ones in Theorem 3.2.4. If G satisfies* (M2) *and also satisfies that*

(M3) *$d_G(u) + d_G(v) \geq n + k - 1$ for $u \in V(M)$ and $v \in V(G) \setminus V(M)$ with $uv \notin E(G)$,*

then the same conclusion as Theorem 3.2.4 holds.

Proof If $\kappa(G) \geq k+1$, then by Theorem 3.2.4, it clearly holds. Thus, we may assume that $\kappa(G) \leq k$. Let S be a cut set of G with $|S| \leq k$.

Suppose first that there exist two vertices $u \in V(G)$ and $v \in V(G) \setminus V(M)$ which belong to different components of $G - S$. Then $|N_G(u) \cup N_G(v)| \leq |V(G) \setminus \{u, v\}| = n - 2$ and $|N_G(u) \cap N_G(v)| \leq |S| \leq k$. Thus, we have $d_G(u) + d_G(v) = |N_G(u)| + |N_G(v)| \leq |N_G(u) \cup N_G(v)| + |N_G(u) \cap N_G(v)| \leq n + k - 2$, which contradicts (M2) or (M3).

The above argument implies that $V(G) \setminus S \subseteq V(M)$, and hence $n = |G| = |S| + |V(G) \setminus S| \leq |S| + |V(M)| \leq 3k$. On the other hand, since G contains $D_1, \ldots, D_k$, it follows that $n = |G| \geq \sum_{i=1}^{k} |D_i| \geq 3k$. Thus, we have $n = \sum_{i=1}^{k} |D_i|$, that is, $D_1, \ldots, D_k$ form a partition of G. □

Note that if a graph G satisfies (M1), then G also satisfies (M3). Therefore, by Theorem 3.2.3 and Corollary 3.2.5, we can get the following.

Corollary 3.2.6 *Let k, G and M be the same as the ones in Theorem 3.2.1. If G satisfies* (M1) *and* (M2)*, then the same conclusion as Theorem 3.2.1 holds.*

The degree conditions (M1) and (M2) in Corollary 3.2.6 are sharp, respectively, and the conditions are weaker than the σ_2 condition in Theorem 3.2.1. In this sense, (M1) and (M2) are appropriate degree conditions for partitions into k cycles in which each cycle contains an edge in a pre-specified matching of size k.

In the next section, we also consider such degree conditions (see Propositions 3.2.15 and 3.2.16).

Egawa et al. [70] also showed that the degree condition (M2) in Theorem 3.2.4 guarantees the existence of the following partition into cycles and degenerate cycles.

Theorem 3.2.7 (Egawa et al. [70]) *Let k be an integer with $k \geq 2$, G be a $(k+1)$-connected graph of order n and M be a k-matching in G. If G satisfies* (M2)*, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^2, and $|E(H_i) \cap M| = 1$ for $1 \leq i \leq k$.*

By the similar argument as in the proof of Corollary 3.2.5, it follows that Theorem 3.2.7 also holds if we add (M3) (and the order condition) instead of the connectivity condition. Note that if the order of a graph G is large, then we can always take two vertices $u \in V(G)$ and $v \in V(G) \setminus V(M)$ as in the proof of Corollary 3.2.5.

Corollary 3.2.8 *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 3k+1$ and M be a k-matching in G. If G satisfies* (M2) *and* (M3)*, then the same conclusion as Theorem 3.2.7 holds.*

Considering the above situation, it would be natural to pose the following problem in order to study appropriate degree conditions for the partition in Theorem 3.2.2.

Problem 3.2.9 *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 4k-1$ and M be a k-matching in G. Determine sharp degree conditions which are similar types as the ones in Corollaries 3.2.6 and 3.2.8 for graphs to be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$, and $|C_i| \leq 4$ for $1 \leq i \leq k-1$.*

On the other hand, as another extension of Corollary 2.1.12, Kaneko and Yoshimoto (2002) gave the following result concerning the existence of a partition into k cycles passing through a pre-specfied edge.

Theorem 3.2.10 (Kaneko and Yoshimoto [139]) *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 4k+1$ and e be an edge of G. If $\sigma_2(G) \geq n$, then (i) G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $e \in E(C_1)$, or (ii) n is even and there exists a vertex subset S with $V(e) \subseteq S$ and $|S| = n/2$ such that $E(G-S) = \emptyset$.*

We finally introduce a result concerning minimum degree conditions. By Theorem 3.2.1, it follows that the minimum degree at least $\frac{n+2k-2}{2}$ guarantees the existence of a partition as in Theorem 3.2.1. However, a sharp minimum degree condition is slightly weaker than it. In fact, Matsumura (2006) proved the following result. (He actually proved a slightly stronger result than the following.)

Theorem 3.2.11 (Matsumura [194]) *Let k be a positive integer, G be a graph of order $n \geq \max\{6k+2, 4k+6\}$ and M be a k-matching in G. If $\delta(G) \geq \frac{n+2k-3}{2}$, then*

(1) *G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ and $|C_i| \leq 5$ for $1 \leq i \leq k$, and*
(2) *G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$.*

3.2.2 Specified Vertices

In this section, we consider vertex versions of results of Sect. 3.2.1.

We first consider σ_2 conditions for graphs to be partitioned into k cycles in which each cycle contains a vertex in pre-specified k vertices. To do that, we prepare the following proposition.

Proposition 3.2.12 *Let k be a positive integer, G be a graph of order $n \geq 3k$ and S be a set of k vertices in G. If $\sigma_2(G) \geq n + k - 2$, then G has a k-matching M such that each edge of M contains a vertex of S.*

Proof Let M be an l-matching in G such that each edge of M contains exactly one vertex in S. Choose M so that l is as large as possible (l may be 0, i.e., M may be an empty set). Suppose that $l \leq k - 1$. Let $G_1 = G[V(M)]$, $G_2 = G - G_1$, $S_1 = V(G_1) \cap S$ and $S_2 = V(G_2) \cap S$, and let $u \in S_2$ and $v \in V(G_2) \setminus S_2$. By the maximality of l, we have

$$d_{G_2-(S_2\setminus\{u\})}(u) = 0 \text{ and}$$
$$d_{G_2}(v) \leq |V(G_2) \setminus (S_2 \cup \{v\})| = |G_2| - (k - l + 1) = n - k - l - 1.$$

Moreover, again by the maximality of l, we also have

$$\left|N_G(u) \cap (V(e) \setminus S_1)\right| + \left|N_G(v) \cap V(e)\right| \leq 2 \text{ for } e \in M,$$

since otherwise, $G[V(e) \cup \{u, v\}]$ contains two independent edges f_1 and f_2 such that each f_i contains exactly one vertex in S, and so, replacing M with $(M \setminus \{e\}) \cup \{f_1, f_2\}$, this contradicts the maximality of l. This inequality implies that

$$\left|N_G(u) \cap (V(G_1) \setminus S_1)\right| + \left|N_G(v) \cap V(G_1)\right| \leq 2|M| = |G_1| = 2l.$$

Therefore, we get

$$\begin{aligned}
&d_G(u) + d_G(v) \\
&\quad = d_{G[S]}(u) + d_{G_2-(S_2\setminus\{u\})}(u) + d_{G_2}(v) \\
&\qquad + \left(\left|N_G(u) \cap (V(G_1) \setminus S_1)\right| + \left|N_G(v) \cap V(G_1)\right|\right) \\
&\quad \leq (k - 1) + 0 + (n - k - l - 1) + 2l \\
&\quad = n + l - 2 \leq n + k - 3,
\end{aligned}$$

a contradiction. □

Combining this proposition with Theorems 2.1.2 and 3.2.1, we can obtain the following corollary.

Corollary 3.2.13 (Ore [200], Egawa et al. [70]) *Let k be a positive integer, G be a graph of order $n \geq 4k - 1$ and S be a set of k vertices in G. If $\sigma_2(G) \geq n + 2k - 2$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$.*

Surprisingly, this σ_2 condition is sharp. In this sense, there is no difference between "specified k-matching" in Sect. 3.2.1 and "specified k vertices" in the above. However, looking more closely, they are different as shown in the following results.

Dong (2010) showed that the σ_2 condition of Corollary 3.2.13 also guarantees the existence of k disjoint cycles of small length, except at most one cycle, that partition a graph.

Theorem 3.2.14 (Dong [66]) *Let k be a positive integer, G be a graph of order $n \geq 3k$ and S be a set of k vertices in G. If $\sigma_2(G) \geq n + 2k - 2$, then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$ and $|C_i| \leq 4$ for $1 \leq i \leq k - 1$.*

Unlike Theorem 3.2.2, there are no exceptions in Theorem 3.2.14. This is one of the differences from "specified k-matching".

On the other hand, as a refinement of Theorem 3.2.14, Chiba and Yamashita (2017) confirmed that the following hold for Step 1 and Step 2, respectively (Propositions 3.2.15 and 3.2.16), which are analogous to Theorems 3.2.3 and 3.2.4.

Proposition 3.2.15 (Chiba and Yamashita [50]) *Let k be a positive integer, G be a graph of order $n \geq 3k$ and S be a set of k vertices in G. If*

(S1) $d_G(u) + d_G(v) \geq n + 2k - 2$ *for* $u \in S$ *and* $v \in V(G) \setminus S$ *with* $uv \notin E(G)$,
(S2) $d_G(u) + d_G(v) \geq 4k - 1$ *for* $u, v \in V(G) \setminus S$ *with* $u \neq v$, $uv \notin E(G)$,

then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \leq 4$ for $1 \leq i \leq k$.

Unlike the situation for Theorem 3.2.3, in Step 1, we need to consider the degree sum of two vertices which do not belong to the specified vertex set S. In fact, by considering the graph $G = K^{2k-1} + (n - 2k + 1)K^1$ and a vertex subset S of $V(K^{2k-1})$ with $|S| = k$, we see that the degree condition (S2) is necessary and sharp. Note that (S1) and (S2), respectively, cannot be weakened even if we drop the condition "$|C_i| \leq 4$ for $1 \leq i \leq k$" in the conclusion.

Proposition 3.2.16 (Chiba and Yamashita [50]) *Let k be a positive integer, G be a graph of order n and S be a set of k vertices in G. Suppose that G contains k disjoint cycles $D_1, \ldots, D_k$ such that $|V(D_i) \cap S| = 1$ and $|D_i| \leq 4$ for $1 \leq i \leq k$. If*

(S3) $d_G(u) + d_G(v) \geq n + k - 1$ *for* $u, v \in V(G) \setminus S$ *with* $u \neq v$, $uv \notin E(G)$,

then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \leq 4$ for $1 \leq i \leq k - 1$.

Note that if a graph G has order at least $3k$ and satisfies (S3), then G also satisfies (S2). Hence, by Propositions 3.2.15 and 3.2.16, we can get the following.

Corollary 3.2.17 *Let k, G and S be the same as the ones in Theorem 3.2.14. If G satisfies* (S1) *and* (S3)*, then the same conclusion as Theorem 3.2.14 holds.*

The degree conditions (S1) and (S3) in Corollary 3.2.17 are sharp, respectively, and the conditions are weaker than the σ_2 condition in Theorem 3.2.14. In this sense, (S1) and (S3) are appropriate degree conditions for partitions into k cycles in which each cycle contains a vertex in a pre-specified k vertices. In order to clarify the difference between "specified k-matching" and "specified k vertices", the study of these types of degree conditions may be important.

Table 1 summarizes the degree conditions in Theorem 3.2.3, Corollary 3.2.6, Proposition 3.2.15 and Corollary 3.2.17.

Matsubara and Sakai (2005) showed that the σ_2 condition of Corollary 3.2.13 can be weakened if we consider K^1 and K^2 as degenerate cycles.

Table 1 Comparisons of the degree conditions

	M is a matching of size k	S is a set of k vertices
Degree conditions for packing k cycles	(M1) (Theorem 3.2.3)	(S1), (S2) (Proposition 3.2.15)
Degree conditions for partitions into k cycles	(M1), (M2) (Corollary 3.2.6)	(S1), (S3) (Corollary 3.2.17)

(M1) $d_G(u) + d_G(v) \geq n + 2k - 2$ for $u \in V(M)$, $v \in V(G) \setminus V(M)$, $uv \notin E(G)$.
(M2) $d_G(u) + d_G(v) \geq n + k$ for $u, v \in V(G) \setminus V(M)$, $u \neq v$, $uv \notin E(G)$.
(S1) $d_G(u) + d_G(v) \geq n + 2k - 2$ for $u \in S$, $v \in V(G) \setminus S$, $uv \notin E(G)$.
(S2) $d_G(u) + d_G(v) \geq 4k - 1$ for $u, v \in V(G) \setminus S$, $u \neq v$, $uv \notin E(G)$.
(S3) $d_G(u) + d_G(v) \geq n + k - 1$ for $u, v \in V(G) \setminus S$, $u \neq v$, $uv \notin E(G)$

Theorem 3.2.18 (Matsubara and Sakai [192]) *Let k be a positive integer, G be a graph of order $n \geq k$ and S be a set of k vertices in G. If $\sigma_2(G) \geq n$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 or K^2, and $|V(H_i) \cap S| = 1$ for $1 \leq i \leq k$.*

Considering Theorem 3.2.7 and Corollary 3.2.8, we can consider the following problem.

Problem 3.2.19 *Let k be a positive integer, G be a graph of order $n \geq k$ and S be a set of k vertices in G. Determine sharp degree conditions which are similar types as the ones in Proposition 3.2.15 and Corollary 3.2.17 for graphs to be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K^1 or K^2, and $|V(H_i) \cap S| = 1$ for $1 \leq i \leq k$.*

Inspiring Theorems 3.1.27, 3.1.28 and 3.1.31, we can also consider the following problem.

Problem 3.2.20 *Can we omit K^2 in Theorem 3.2.18? Moreover, can we control the number of K^1s by the σ_2 condition like Theorem 3.1.31?*

On the other hand, Chiba et al. (2010) showed that the σ_2 condition of Theorem 3.2.14 can be weakened and "$|C_i| \leq 4$" in the conclusion can be replaced with "$|C_i| = 3$" when the pre-specified vertex set S is independent.

Theorem 3.2.21 (Chiba et al. [41]) *Let k be a positive integer, G be a graph of order $n \geq 3k$ and S be an independent set of k vertices in G. If $\sigma_2(G) \geq n + k - 1$, then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$ and $|C_i| = 3$ for $1 \leq i \leq k - 1$.*

Chiba and Yamashita (2017) proved the following result which is a common generalization of Theorems 3.2.14 and 3.2.21. (They actually gave degree conditions which are similar types as the ones in Proposition 3.2.15 and Corollary 3.2.17.)

Theorem 3.2.22 (Chiba and Yamashita [50]) *Let k be a positive integer, G be a graph of order $n \geq 3k$ and S be a set of k vertices in G. If $\sigma_2(G) \geq n+k-1+\Delta(G[S])$, then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$, $|C_i| = 3$ for $1 \leq i \leq k - 1 - \Delta(G[S])$, and $|C_i| \leq 4$ for $k - \Delta(G[S]) \leq i \leq k - 1$.*

We next consider minimum degree conditions. In 2003, Egawa et al. gave the following result, which corresponds to Theorem 3.2.11. (In fact, they also gave minimum degree conditions for $3k \le n \le 6k - 4$.)

Theorem 3.2.23 (Egawa et al. [69]) *Let k be a positive integer, G be a graph of order $n \ge 6k - 3$ and S be a set of k vertices in G. If $\delta(G) \ge \frac{n}{2}$, then,*

(1) *G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \le 5$ for $1 \le i \le k$, and*
(2) *G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \le i \le k$.*

Unlike the situation for σ_2 conditions, there is clear difference between "specified k-matching" and "specified k vertices" in terms of sharp minimum degree conditions (see Theorem 3.2.11).

Ishigami (2001) gave a minimum degree condition for the existence of k disjoint cycles of length at most 4.

Theorem 3.2.24 (Ishigami [129]) *Let k be a positive integer, G be a graph of order $n \ge 3k$ and S be a set of k vertices in G. If $\delta(G) \ge \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$, then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \le 4$ for $1 \le i \le k$.*

Moreover, Ishigami and Jiang (2003) gave the following minimum degree condition for the existence of k disjoint cycles of length at most 6 in large graphs, which was originally conjectured by Enomoto [78].

Theorem 3.2.25 (Ishigami and Jiang [130]) *For any positive integer k, there exists an integer $n_0 = n_0(k)$ depending on only k such that, if G is a graph of order $n \ge n_0$ and $\delta(G) \ge \left\lfloor \sqrt{n + \frac{9}{4}k^2 - 4k + 1} + \frac{3}{2}k - 1 \right\rfloor$, then for any set S of k vertices, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \le 6$ for $1 \le i \le k$.*

What happens if we specify both edges and vertices in graphs? Concerning this problem, Enomoto and Matsumura (2009) gave the following σ_2 condition as a common generalization of Theorem 3.2.1 and Corollary 3.2.13. (They also gave a minimum degree condition.)

Theorem 3.2.26 (Enomoto and Matsumura [81]) *Let k, p and q be integers with $k \ge p + q \ge 1$, $p \ge 0$ and $q \ge 0$, and let G be a graph of order $n \ge 10k$. Further, let S be a set of p vertices and M be a q-matching in G such that $S \cap V(M) = \emptyset$. If $\sigma_2(G) \ge \max\{n + q, n + 2p + 2q - 2\}$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \le i \le p$ and $|E(C_{p+i}) \cap M| = 1$ for $1 \le i \le q$.*

As a more general case, in [190], Matsubara and Matsumura considered the case where we specify not only vertices and edges but also paths of order at least three.

We finally consider packing k cycles in which each cycle contains at least a prescribed number of vertices in a pre-specified vertex set. Concerning this problem, Wang (2015) gave the following result. (The degree condition is sharp when $n = 3k$.)

Theorem 3.2.27 (Wang [246]) *Let k be a positive integer, G be a graph of order n and S be a set of vertices of G such that $|S| \geq 3k$. If $d_G(x) \geq \frac{2}{3}n$ for $x \in S$, then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $S \subseteq \bigcup_{1 \leq i \leq k} V(C_i)$, and $|V(C_i) \cap S| \geq 3$ for $1 \leq i \leq k$.*

In [246], Wang also conjectured the following. Note that Theorem 3.2.27 implies that this conjecture is true for the case $n_i = 3$ $(1 \leq i \leq k)$.

Conjecture 3.2.28 (Wang [246]) *Let k be a positive integer, G be a graph of order n and S be a set of vertices of G such that $|S| = \sum_{i=1}^{k} n_i$, where $n_i \geq 3$ for $1 \leq i \leq k$. If $d_G(x) \geq \frac{2}{3}n$ for $x \in S$, then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = n_i$ for $1 \leq i \leq k$.*

The case $S = V(G)$ in this conjecture is a weaker version of a conjecture of El-Zahár (Conjecture 3.3.1 in the next section).

We should also consider degree conditions for the case $n_i \geq 1$ in Conjecture 3.2.28 as a generalization of Theorem 3.2.27.

Problem 3.2.29 *Let k be a positive integer, G be a graph of order n and S be a set of vertices of G such that $|S| \geq \sum_{i=1}^{k} n_i$, where $n_i \geq 1$ for $1 \leq i \leq k$. Determine a sharp degree condition to guarantee that a graph G contains k disjoint cycles covering S such that each of them contains at least n_i vertices in S.*

3.3 Cycles with Length Constraints

3.3.1 The El-Zahár's Conjecture

The following conjecture is well known due to El-Zahár (1984).

Conjecture 3.3.1 (El-Zahár [75]) *Let k be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 3$ for $1 \leq i \leq k$. If $\delta(G) \geq \sum_{i=1}^{k} \lceil \frac{n_i}{2} \rceil$, then G can be partitioned into k cycles of lengths $n_1, n_2, \ldots, n_k$.*

El-Zahár proved the case $k = 2$ and Abbasi, in his Ph.D. Thesis [1], settled this conjecture for sufficiently large graphs by using the regularity lemma. In Sects. 4.3 and 4.5, we will discuss some generalizations which lead to this type of results.

Wang (1995) considered a problem of packing two cycles with a pre-specfied length.

Theorem 3.3.2 (Wang [229]) *Let n_1 and n_2 be integers with $n_1 \geq 3$ and $n_2 \geq 3$, and let G be a graph of order $n \geq n_1 + n_2$. If $\delta(G) \geq \frac{n+1}{2}$, then (i) G contains two disjoint cycles of lengths n_1 and n_2, respectively, or (ii) n, n_1 and n_2 are all odd and $G \simeq K^{(n-1)/2,(n-1)/2} + K^1$.*

This result is weaker than the result of El-Zahár [75] for the case $n = n_1 + n_2$ and each n_i is even. So, Wang also gave the following result for the case where each n_i is even.

Theorem 3.3.3 (Wang [229]) *Let n_1 and n_2 be even integers with $n_1 \geq 4$ and $n_2 \geq 4$, and let G be a graph of even order $n \geq n_1 + n_2$. If $\delta(G) \geq \frac{n}{2}$, then G contains two disjoint cycles of lengths n_1 and n_2, respectively.*

Now, we focus on cycles with a pre-specified small length for a while. In particular, we consider the case $3 \le n_i \le 5$ in Conjecture 3.3.1.

The case $n = 3k$ in Theorem 3.1.2 is the case $n_i = 3$ $(1 \le i \le k)$ in Conjecture 3.3.1. The case $n = 3t$ in the following theorem is also the case $n_i = 3$ $(1 \le i \le k)$ in Conjecture 3.3.1. (In 1989, Justesen [136] obtained a σ_2 version of Theorem 3.3.4.)

Theorem 3.3.4 (Dirac [62]) *Let t be a positive integer, and let G be a graph of order $n \ge 3t$. If $\delta(G) \ge \frac{n+t}{2}$, then G contains t disjoint cycles of length* 3.

In 1990, Erdős and Faudree conjectured the following minimum degree condition for partitions into quadrilaterals, which is the case $n_i = 4$ $(1 \le i \le k)$ in Conjecture 3.3.1.

Conjecture 3.3.5 (Erdős and Faudree [84]) *Let q be a positive integer, and let G be a graph of order $n = 4q$. If $\delta(G) \ge \frac{n}{2}$, then G can be partitioned into q cycles of length* 4.

Concerning this conjecture, in [7], Alon and Yuster (1996) proved that for any $\varepsilon > 0$, there exists a positive integer q_0 such that for any integer q with $q \ge q_0$, every graph G of order $n = 4q$ with $\delta(G) \ge (\frac{1}{2} + \varepsilon)n$ can be partitioned into q cycles of length 4. In fact, they proved a more general result. We mention it in Sect 4.5.

In 2010, Wang settled Erdős-Faudree's conjecture.

Theorem 3.3.6 (Wang [241]) *Let q be a positive integer, and let G be a graph of order $n = 4q$. If $\delta(G) \ge \frac{n}{2}$, then G can be partitioned into q cycles of length* 4.

Wang (2012) also gave a minimum degree condition for partitions into pentagons, which is the case $n_i = 5$ $(1 \le i \le k)$ in Conjecture 3.3.1.

Theorem 3.3.7 (Wang [243]) *Let p be a positive integer, and let G be a graph of order $n = 5p$. If $\delta(G) \ge \frac{n+p}{2}$, then G can be partitioned into p cycles of length* 5.

Wang (1995) also considered the case $n_i = 3$ for $1 \le i \le k-1$ in Conjecture 3.3.1 (all n_i's, except one, are exactly 3) and proved the following result.

Theorem 3.3.8 (Wang [228]) *Let t be a non-negative integer, and let G be a graph of order $n \ge 3t + 3$. If $\delta(G) \ge \frac{n+t}{2}$, then G can be partitioned into $t + 1$ cycles $C_1, \dots, C_t, C_{t+1}$ such that $|C_i| = 3$ for $1 \le i \le t$.*

Note that if $n_i = 3$ for $1 \le i \le k-1$ in Conjecture 3.3.1, then $\sum_{i=1}^{k} \lceil n_i/2 \rceil = \lceil (n + k - 1)/2 \rceil$, and hence Theorem 3.3.8 is the case $n_i = 3$ $(1 \le i \le k-1)$. Note also that Theorem 3.3.8 is a generalization of Theorem 3.3.4.

In 2001, Enomoto improved the minimum degree condition in this theorem into a σ_2 condition. More generally, he conjectured the following.

Conjecture 3.3.9 (Enomoto [78]) *Let t and q be non-negative integers, and let G be a graph of order $n \ge 3t + 4q + 3$. If $\sigma_2(G) \ge n + t$, then G can be partitioned into $t + q + 1$ cycles $C_1, \dots, C_{t+q}, C_{t+q+1}$ such that $|C_i| = 3$ for $1 \le i \le t$ and $|C_i| \le 4$ for $t + 1 \le i \le t + q$.*

As a related result to this conjecture, Brandt et al. proved the following result on the packing problem.

Theorem 3.3.10 (Brandt et al. [27]) *Let t and q be non-negative integers with $t+q \geq 1$, and let G be a graph of order $n \geq 3t+4q$. If $\sigma_2(G) \geq n+t$, then G contains $t+q$ disjoint cycles $C_1, \ldots, C_{t+q}$ such that $|C_i| = 3$ for $1 \leq i \leq t$ and $|C_i| \leq 4$ for $t+1 \leq i \leq t+q$.*

Zhang et al. (2011) investigated the case where each n_i is 3 or 4 in Conjecture 3.3.1. They showed that "$|C_i| \leq 4$" in the conclusion of Theorem 3.3.10 can be improved into "$|C_i| = 4$" if $n = 3t + 4q$ and t is large compared with q. (Wang [240] gave a minimum degree version of Theorem 3.3.11, and another related result can be found in [249].)

Theorem 3.3.11 (Zhang et al. [256]) *Let t and q be positive integers with $t \geq 2q-2$, and let G be a graph of order $n = 3t+4q$. If $\sigma_2(G) \geq n+t$, then G can be partitioned into $t+q$ cycles $C_1, \ldots, C_{t+q}$ such that $|C_i| = 3$ for $1 \leq i \leq t$ and $|C_i| = 4$ for $t+1 \leq i \leq t+q$.*

To complete the case where each n_i is 3 or 4, we should consider the following problem.

Problem 3.3.12 *Can we omit the condition $t \geq 2q-2$ in Theorem 3.3.11?*

Along this line, we can also consider σ_2 conditions for other cases. In 2010, Hayashi proved the following, which is related to a σ_2 version of Theorem 3.3.7.

Theorem 3.3.13 (Hayashi [127]) *Let p be an integer with $p \geq 3$, and let G be a graph of order $n = 5p$. If $\sigma_2(G) \geq n+p-2$, then (i) G can be partitioned into p subgraphs $H_1, \ldots, H_{p-1}, H_p$ such that H_i is a cycle of length 5 for $1 \leq i \leq p-1$, and H_p is a path of order 5, or (ii) $(p-2)K^1 + K^{2p+1,2p+1} \subseteq G \subseteq K^{p-2} + K^{2p+1,2p+1}$.*

Considering the degree condition in Theorem 3.3.7, we can pose the following problem, which is the case $n_i = 5$ $(1 \leq i \leq k)$ in Conjecture 3.3.1.

Problem 3.3.14 *Let p be a positive integer, and let G be a graph of order $n = 5p$. Is it true that, if $\sigma_2(G) \geq n+p$, then G can be partitioned into p cycles of length 5?*

As a related result to this problem, Bauer and Wang (2010) gave the following result.

Theorem 3.3.15 (Bauer and Wang [14]) *Let t and p be integers with $t \geq 1$ and $p \geq 0$, and let G be a graph of order $n = 3t+5p$. If $\delta(G) \geq \frac{n+t+p}{2}$, then G can be partitioned into $t+p$ cycles $C_1, \ldots, C_{t+p}$ such that $|C_i| = 3$ for $1 \leq i \leq t$ and $|C_i| = 5$ for $t+1 \leq i \leq t+p$.*

Note that, in this theorem, t is a positive integer and the degree condition is not a σ_2 condition. Therefore, we can consider a more general problem than Problem 3.3.14 as follows, which is the case where each n_i is 3 or 5 in Conjecture 3.3.1.

Problem 3.3.16 *Can we replace the conditions "t and p are integers with $t \geq 1$ and $p \geq 0$" and "$\delta(G) \geq \frac{n+t+p}{2}$" in Theorem 3.3.15 with "t and p are non-negative integers with $t+p \geq 1$" and "$\sigma_2(G) \geq n+t+p$", respectively?*

Moreover, considering Theorem 3.3.11, we can also pose the following problem.

Problem 3.3.17 *Let p and h be non-negative integers with $p + h \geq 1$, and let G be a graph of order $n = 5p + 6h$. Is it true that, if $\sigma_2(G) \geq n + p$, then G can be partitioned into $p + h$ cycles $C_1, \ldots, C_{p+h}$ such that $|C_i| = 5$ for $1 \leq i \leq p$ and $|C_i| = 6$ for $p + 1 \leq i \leq p + h$?*

In the last of this section, we introduce an El-Zahár-type problem of partitions into k cycles passing through pre-specified elements, such as considered in Sect. 3.2.

In 2012, Magnant and Ozeki posed the following conjecture.

Conjecture 3.3.18 (Magnant and Ozeki [187]) *Let k be an integer with $k \geq 2$ and G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 3$ for $1 \leq i \leq k$. Let S be a set of k vertices in G. If $\sigma_2(G) \geq n + 2k - 2$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| = n_i$ for $1 \leq i \leq k$.*

They proved that for a sufficiently large graph G, the degree condition in this conjecture guarantees the existence of a partition into k cycles of approximately pre-specified lengths such that each cycle contains a vertex in a pre-specified k vertices (see [187, Theorem 6]).

As a version of "pre-specified k-matchings", such as considered in Sect. 3.2.1, we pose the following conjecture.

Conjecture 3.3.19 *Let k be an integer with $k \geq 2$ and G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 5$ for $1 \leq i \leq k$. Let M be a k-matching in G. If $\sigma_2(G) \geq n + 2k - 1$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ and $|C_i| = n_i$ for $1 \leq i \leq k$.*

The degree condition in this conjecture is sharp in the following sense if it is true. Consider the graph $G = K^{2k} + 2K^{(n-2k)/2}$, and let M be a k-matching in G such that $V(M) = V(K^{2k})$. Then we have $\sigma_2(G) = n+2k-2$. But, for integers $n_1, \ldots, n_k$ such that $n = \sum_{i=1}^{k} n_i$, $n_i \geq 3$ $(1 \leq i \leq k)$ and there exists no subset J of $\{1, 2, \ldots, k\}$ with $\sum_{j \in J} n_j = (n - 2k)/2 + 2|J|$, it follows that G does not contain the partition in Conjecture 3.3.19.

Moreover, the condition "$n_i \geq 5$ for $1 \leq i \leq k$" is necessary in the following sense. Let k and n be integers such that $k \geq 4$ and $\frac{n+3k-1}{2(3k-2)}$ is an integer at least 2, and consider the graph $G = \frac{n-3k+1}{2} K^1 + lK^{3k-2}$, where $l = \frac{n+3k-1}{2(3k-2)}$. Then, we can check that the graph G has order n and $\delta(G) = \frac{n+3k-5}{2} \geq \frac{n+2k-1}{2}$, and hence $\sigma_2(G) \geq n + 2k - 1$. But, if M is a k-matching in some component of lK^{3k-2}, then G does not contain the partition in Conjecture 3.3.19 for $n_1 = \cdots = n_{k-1} = 4$ and $n_k = n - 4k + 4$.

In 2014, Wang considered a partition into k cycles with the following stronger property. He proved this conjecture for $k = 2$, but the other cases are still open.

Conjecture 3.3.20 (Wang [245]) *Let k be an integer with $k \geq 2$ and G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 5$ for $1 \leq i \leq k$. Let $\{e_1, \ldots, e_k\}$ be a k-matching in G. If $\delta(G) \geq \frac{n+2k}{2}$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| = n_i$ for $1 \leq i \leq k$.*

In [245], Wang assumed $n_i \geq 4$ in this conjecture. But this is not true, and $n_i \geq 5$ is necessary. In fact, consider the graph G in the second paragraph following Conjecture 3.3.19 and assume $k \geq 5$, and then we have $\delta(G) = \frac{n+3k-5}{2} \geq \frac{n+2k}{2}$. Moreover, he showed that the degree condition in Conjecture 3.3.20 is sharp in the sense that there exists a counterexample for the case $\delta(G) = \frac{n+2k}{2} - 1$ (see [245] for more details). However, considering this minimum degree, the following conjecture would be more natural. (The degree condition is weaker than the one in Conjecture 3.3.20 if n is odd.)

Conjecture 3.3.21 *Let k be an integer with $k \geq 2$ and G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 5$ for $1 \leq i \leq k$. Let $\{e_1, \ldots, e_k\}$ be a k-matching in G. If $\delta(G) \geq \frac{n+2k-1}{2}$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| = n_i$ for $1 \leq i \leq k$.*

In Sect. 5.2.3, we will discuss relations between these conjectures and other conjectures on k-linkedness (see also Fig. 5 in Sect. 5.2.3). By using the relation, Theorem 5.2.19 implies that for a sufficiently large graph G, the minimum degree condition in Conjecture 3.3.21 guarantees the existence of a partition into k cycles of approximately pre-specified lengths such that each cycle contains an edge in a pre-specified k-matching.

3.3.2 Cycles of the Same Length

In 1983, Thomassen [220] considered a minimum degree condition for the existence of disjoint cycles of the same length but not specifying the length of each cycle. He conjectured that for a sufficiently large graph G, the minimum degree condition "$\delta(G) \geq 2k$" in Corrádi-Hajnal's Theorem (Theorem 3.1.2) guarantees the existence of k disjoint cycles of the same length. In 1985, Häggkvist [121] proved that every sufficiently large graph with average degree at least 12 contains two disjoint cycles of the same length. In 1996, the conjecture of Thomassen for $k \geq 3$ was proved by Egawa. Here, $o(k)$ denotes some function $f(k)$ such that $\lim_{k\to\infty} f(k)/k = 0$.

Theorem 3.3.22 (Egawa [68]) *Let k be an integer with $k \geq 3$, and let G be a graph of order at least $17k + o(k)$. If $\delta(G) \geq 2k$, then G contains k disjoint cycles of the same length.*

Versträete (2003) proved the conjecture of Thomassen for all $k \geq 2$ as follows. The integer $n(k)$ in this result is much larger than $17k + o(k)$ in Theorem 3.3.22, but the proof is simpler than the one of Theorem 3.3.22.

Theorem 3.3.23 (Versträete [226]) *For any integer $k \geq 2$, there exists an integer $n = n(k)$ depending on only k such that if G is a graph of order at least n and $\delta(G) \geq 2k$, then G contains k disjoint cycles of the same length.*

For graphs with large order and large girth, a much weaker degree condition guarantees the existence of such disjoint cycles.

Theorem 3.3.24 (Thomassen [220]) *Let k be a positive integer, and let G be a graph of order n such that $n/(\log n)^4 > 2^{13}(k-1)^2$ and $g(G) \geq 5$. If $\delta(G) \geq 4$, then G contains k disjoint cycles of the same length.*

The complete bipartite graph $K^{4,m}$ ($m \geq 4$) does not contain 3 disjoint cycles, and hence $g(G) \geq 5$ is necessary in the sense. Moreover, it is known that there are infinitely many graphs G with $g(G) = 6$ and $\delta(G) = 3$ such that it does not contain 5 disjoint cycles. However, if we assume $g(G) \geq 7$, then the following holds.

Theorem 3.3.25 (Thomassen [220]) *For any positive integer k, there exists an integer $n = n(k)$ depending on only k such that, if G is a graph of order at least n, $g(G) \geq 7$ and $\delta(G) \geq 3$, then G contains k disjoint cycles of the same length.*

3.3.3 Cycles Whose Lengths Have the Same Parity

In this section, we consider a sharp minimum degree condition for packing cycles whose lengths have the same parity (odd or even).

We first consider the odd cycle case (i.e., a cycle of odd length). To do that, consider the graph $G = (k-1)K^1 + K^{\frac{n-k+1}{2},\frac{n-k+1}{2}}$. Then we can easily check that $\delta(G) = \frac{n+k-1}{2}$, and G does not contain k disjoint odd cycles (because every odd cycle contains a vertex in $(k-1)K^1$). Therefore, in order to guarantee the existence of k disjoint odd cycles for a graph G of order n, the minimum degree condition should be at least $\frac{n+k}{2}$. On the other hand, Theorem 3.3.4 says that the minimum degree at least $\frac{n+k}{2}$ implies the existence of k disjoint odd cycles (we actually have the existence of k disjoint triangles). It follows from this observation that a sharp minimum degree condition for packing k odd cycles, is $\delta(G) \geq \frac{n+k}{2}$.

On the other hand, the situation is quite different in terms of the lower bound of the minimum degree condition if we consider the even cycle case (i.e., a cycle of even length). In fact, Chiba et al. (2014) proved the following result.

Theorem 3.3.26 (Chiba et al. [44]) *For any positive integer k, there exists an integer $n = n(k)$ depending on only k such that, if G is a graph of order at least n and $\delta(G) \geq 2k$, then (i) G contains k disjoint cycles of even length, or (ii) $(2k-1)K^1 + pK^2 \subseteq G \subseteq K^{2k-1} + pK^2$ ($p \geq k \geq 2$), or (iii) $k = 1$ and each block in G is either a K^2 or an odd cycle, especially, each end block in G is an odd cycle.*

Note that the graphs in (ii) and (iii) have minimum degree $2k$ and contain k disjoint cycles, respectively. Therefore, this theorem is a generalization of Theorem 3.1.2 for sufficiently large graphs.

Unlike the situation for odd cycles, the minimum degree condition which depends on only k, guarantees the existence of k disjoint even cycles. In Sect. 4.1.2, we show that cycles whose lengths are congruent 0 mod 3, relaxed structures of triangles, have the same situation as even cycles (see Theorem 4.1.15).

In 2014, Egawa et al. [72] announced a same length version of Theorem 3.3.26 for graphs G with $\delta(G) \geq 2k + 1$.

3.4 Chorded Cycles

A *chord* of a cycle is an edge between two vertices on the cycle that is not an edge of the cycle. A cycle with at least c chords is called a *c-chorded cycle*, and we simply say *chorded cycle* instead of "1-chorded cycle".

3.4.1 Chorded Cycles

As mentioned in Sect. 3.1, it is easily shown that any graph G with $\delta(G) \geq 2$ contains a cycle. We can easily see that $\delta(G) \geq 3$ guarantees the existence of a chorded cycle. (Posa [203] proposed this problem in 1961, and Czipszer [57] published a solution in 1963. See also Lovász [182, problem 10.2].) Finkel (2008) gave a natural generalization of this fact, which is analogous to Corrádi-Hajnal's Theorem (Theorem 3.1.2).

Theorem 3.4.1 (Finkel [99]) *Let k be a positive integer, and let G be a graph of order at least $4k$. If $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.*

Gao and Li (2011) improved this theorem into a σ_2 condition.

Theorem 3.4.2 (Gao and Li [106]) *Let k be a positive integer, and let G be a graph of order at least $4k$. If $\sigma_2(G) \geq 6k - 1$, then G contains k disjoint chorded cycles.*

Molla et al. (2017) characterized graphs G with $\sigma_2(G) = 6k - 2$ that do not contain k disjoint chorded cycles.

Theorem 3.4.3 (Molla et al. [196]) *Let k be an integer with $k \geq 2$, and let G be a graph of order at least $4k$. If $\sigma_2(G) \geq 6k - 2$, then (i) G contains k disjoint chorded cycles, or (ii) $G \simeq K^{3k-1,n-3k+1}$ $(n \geq 6k - 2)$, or (iii) $G \simeq K^{3k-2,3k-2} + K^1$.*

In 2013, Gao et al. and Gould et al., independently obtained a neighborhood union condition, which is analogous to Theorem 3.1.15. (The $NU(G)$ condition in Theorem 3.4.4 is sharp when $|G| = 4k + 2$ or $k = 1$.)

Theorem 3.4.4 (Gao et al. [108], Gould et al. [110]) *Let k be a positive integer, and let G be a graph of order at least $4k$. If $NU(G) \geq 4k + 1$, then G contains k disjoint chorded cycles.*

3.4.2 Cycles and Chorded Cycles

In 2008, Bialostocki et al. [20] considered packing cycles and chorded cycles, and they proposed a conjecture as a common generalization of Theorems 3.1.2 and 3.4.1. Chiba et al. (2010) settled a σ_2 version of the conjecture.

Theorem 3.4.5 (Chiba et al. [43]) *Let r and s be non-negative integers with $r+s \geq 1$, and let G be a graph of order at least $3r+4s$. If $\sigma_2(G) \geq 4r+6s-1$, then G contains $r + s$ disjoint cycles such that s of them are chorded cycles.*

Balister et al. showed that the minimum degree at least $2r + 3s$ guarantees the existence of $r + s$ disjoint cycles and chorded cycles with a stronger property, see Theorem 3.4.10.

Qiao and Zhang (2012) proved the following result for Step 2 (Partitioning) mentioned in Sect. 3.1.2.

Theorem 3.4.6 (Qiao and Zhang [206]) *Let r and s be non-negative integers with $r + s \geq 1$, and let G be a graph of order n. Suppose that G contains $r + s$ disjoint cycles such that s of them are chorded cycles. If $\delta(G) \geq \frac{n}{2}$, then G can be partitioned into $r + s$ cycles such that s of them are chorded cycles.*

By Theorems 3.4.5 and 3.4.6, we get the following corollary. This is a generalization of a minimum degree version of Theorem 3.1.17.

Corollary 3.4.7 *Let r and s be non-negative integers with $r + s \geq 1$, and let G be a graph of order $n \geq 4r + 6s$. If $\delta(G) \geq \frac{n}{2}$, then G can be partitioned into $r + s$ cycles such that s of them are chorded cycles.*

As a neighborhood union condition, Qiao (2012) obtained a common generalization of Theorems 3.1.15 and 3.4.4 as follows. (The $NU(G)$ condition in Theorem 3.4.8 is sharp when $s = 0$ and $|G| = 3r + 1$, or $r = 0$ and $|G| = 4s + 2$, or $r + s = 1$.)

Theorem 3.4.8 (Qiao [204]) *Let r and s be non-negative integers with $r + s \geq 1$, and let G be a graph of order at least $3r + 4s$. If $NU(G) \geq 3r + 4s + 1$, then G contains $r + s$ disjoint cycles such that s of them are chorded cycles.*

3.4.3 Cycles with Many Chords

In 2010, Qiao and Zhang [205] considered the same problem as Theorem 3.4.1 for disjoint 2-chorded cycles. They showed that the minimum degree at least $\lceil 7k/2 \rceil$ guarantees the existence of k disjoint 2-chorded cycles, but the minimum degree condition was not sharp. Gould et al. (2015) showed that if the order of a graph is large compared with k, then the degree condition in Theorem 3.4.1 also implies the existence of k disjoint 2-chorded cycles. (Note that the order condition in Theorem 3.4.9 is not sharp.)

Theorem 3.4.9 (Gould et al. [111]) *Let k be a positive integer, and let G be a graph of order at least $6k$. If $\sigma_2(G) \geq 6k - 1$, then G contains k disjoint 2-chorded cycles.*

Balister et al. (2018) considered packing cycles, chorded cycles and 2-chorded cycles, and they proved a stronger result than the minimum degree version of Theorem 3.4.5.

Theorem 3.4.10 (Balister et al. [12]) *Let r and s be non-negative integers with $r + s \geq 1$, and let G be a graph of order at least $3r + 4s$. If $\delta(G) \geq 2r + 3s$, then G contains $r + s$ disjoint cycles such that each of s of them is a 2-chorded cycle, or a chorded cycle of length 4.*

We consider the existence of disjoint cycles with more chords. In 1970, Hajnal and Szemerédi gave the following minimum degree condition for partitions into k complete graphs K^{c+1}. Note that a hamiltonian cycle of a complete graph of order $c + 1$ is a $\frac{(c+1)(c-2)}{2}$-chorded cycle, and hence a $\frac{(c+1)(c-2)}{2}$-chorded cycle of order at least $c + 1$ is one of the relaxed structures of a complete graph of order $c + 1$.

Theorem 3.4.11 (Hajnal and Szemerédi [122]) *Let k and c be integers with $k \geq 1$ and $c \geq 2$, and let G be a graph of order $n = (c + 1)k$. If $\delta(G) \geq \frac{c}{c+1}n$, then G can be partitioned into k subgraphs isomorphic to K^{c+1}.*

In Sect. 4.5, we will discuss generalizations of this theorem. In 2008, Kierstead and Kostochka improved this into a σ_2 condition.

Theorem 3.4.12 (Kierstead and Kostochka [148]) *Let k and c be integers with $k \geq 1$ and $c \geq 2$, and let G be a graph of order $n = (c+1)k$. If $\sigma_2(G) \geq \frac{2c}{c+1}n - 1$, then G can be partitioned into k subgraphs isomorphic to K^{c+1}.*

These theorems are related to an *equitable k-coloring*, i.e., a proper k-vertex-coloring such that any two color classes differ in size by at most one, and many researchers have investigated it. Refer to [152] in detail.

In 1980, Gupta et al. proved the following result for the existence of a $\frac{(c+1)(c-2)}{2}$-chorded cycle in a graph, which is a generalization of the case $k = 1$ in Theorem 3.4.11 (see also [4, 113, 142]).

Theorem 3.4.13 (Gupta et al. [118]) *Let c be an integer with $c \geq 2$, and let G be a graph. If $\delta(G) \geq c$, then G contains a $\frac{(c+1)(c-2)}{2}$-chorded cycle.*

In 2014, Gould et al. proposed the following conjecture, which is a generalization of Hajnal and Szemerédi's Theorem (Theorem 3.4.11). Note that this conjecture is also a generalization of Corrádi and Hajnal's Theorem (Theorem 3.1.2) and Finkel's Theorem (Theorem 3.4.1).

Conjecture 3.4.14 (Gould et al. [113]) *Let k and c be integers with $k \geq 1$ and $c \geq 2$, and let G be a graph of order at least $(c+1)k$. If $\delta(G) \geq ck$, then G contains k disjoint $\frac{(c+1)(c-2)}{2}$-chorded cycles.*

The minimum degree condition is best possible when $|G| = (c+1)k$ (in this case, the conjecture is Theorem 3.4.11). They showed that this conjecture is true for sufficiently large graphs compared with c and k.

Theorem 3.4.15 (Gould et al. [113]) *For any integers $k \geq 1$ and $c \geq 2$, there exists a positive integer $n = n(k, c)$ depending on only k and c such that if G is a graph of order at least n and $\delta(G) \geq ck$, then G contains k disjoint $\frac{(c+1)(c-2)}{2}$-chorded cycles.*

We will mention other partial solutions to Conjecture 3.4.14 later (see Corollary 4.1.13).

As another related result, Babu and Diwan (2009) gave the following result.

Theorem 3.4.16 (Babu and Diwan [11]) *Let k and c be integers with $k \geq 1$ and $c \geq 2$, and let G be a graph of order at least $(c+1)k$. If $\sigma_2(G) \geq 2ck - 1$, then G contains k disjoint cycles with $c - 2$ chords incident with a common vertex.*

The complete bipartite graph $K^{ck-1, n-ck+1}$ shows the sharpness of the lower bound on the degree condition. They actually proved a stronger result as follows.

Theorem 3.4.17 (Babu and Diwan [11]) *Let k be a positive integer, $n_1, n_2, \ldots, n_k$ be integers with $n_i \geq 3$ for $1 \leq i \leq k$ and $n = \sum_{i=1}^{k} n_i$, and let $H_1, H_2, \ldots, H_k$ be a cycle or a tree such that $|H_i| = n_i$ for $1 \leq i \leq k$. Then every graph G of order at least n with $\sigma_2(G) \geq 2(n-k) - 1$ contains k disjoint subgraphs $H'_1, H'_2, \ldots, H'_k$ such that $H'_i \simeq H_i$ if H_i is a tree, and H'_i is a cycle with $n_i - 3$ chords incident with a common vertex if H_i is a cycle.*

Applying Theorem 3.4.17 with $H_i = C^{c+1}$ for $1 \leq i \leq k$ implies Theorem 3.4.16 as a corollary.

3.4.4 Chorded Cycles Passing Through Pre-specified Elements

Cream et al. (2016) investigated chorded cycle versions of the results in Sects. 3.2.1 and 3.2.2.

The following result is a chorded cycle version of Theorem 3.2.23 (1).

Theorem 3.4.18 (Cream et al. [54]) *Let k be a positive integer, G be a graph of order $n \geq 16k - 5$ and S be a set of k vertices in G. If $\delta(G) \geq \frac{n}{2}$, then G contains k disjoint chorded cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \leq 6$ for $1 \leq i \leq k$.*

The following result is a chorded cycle version of Theorem 3.2.3.

Theorem 3.4.19 (Cream et al. [54]) *Let k be a positive integer, G be a graph of order $n \geq 18k - 3$ and M be a k-matching in G. If $\delta(G) \geq \frac{n}{2} + k - 1$, then G contains k disjoint 2-chorded cycles $C_1, \ldots, C_k$ such that C_i contains an exactly one edge in M as a cycle edge and $|C_i| \leq 6$ for $1 \leq i \leq k$.*

For partitions into k chorded cycles passing through a pre-specified k-matching, they gave the following theorem for Step 2 (Partitioning).

Theorem 3.4.20 (Cream et al. [54]) *Let k be a positive integer, G be a graph of order $n \geq 4k$ and M be a k-matching in G. Suppose that G contains k disjoint chorded cycles $D_1, \ldots, D_k$ such that D_i contains exactly one edge in M as a cycle edge for $1 \leq i \leq k$. If $\delta(G) \geq \frac{n+k}{2}$, then G can be partitioned into k chorded cycles $C_1, \ldots, C_k$ such that C_i contains exactly one edge in M as a cycle edge for $1 \leq i \leq k$.*

Combining this with Theorem 3.4.19, we can obtain the following corollary, which is a chorded cycle version of Theorem 3.2.1.

Corollary 3.4.21 (Cream et al. [54]) *Let k be an integer with $k \geq 2$, G be a graph of order $n \geq 18k - 3$ and M be a k-matching in G. If $\delta(G) \geq \frac{n}{2} + k - 1$, then G can be partitioned into k chorded cycles $C_1, \ldots, C_k$ such that C_i contains exactly one edge in M as a cycle edge for $1 \leq i \leq k$.*

They also considered packing k chorded cycles containing a pre-specified k-matching as chords, and they posed the following conjecture concerning a σ_2 condition, which also corresponds to Theorem 3.2.3.

Conjecture 3.4.22 (Cream et al. [54]) *Let k be a positive integer, G be a graph of order $n \geq 6k$ and M be a k-matching in G. If $\sigma_2(G) \geq n + 3k - 2$, then G contains k disjoint chorded cycles $C_1, \ldots, C_k$ such that C_i contains exactly one edge in M as a chord and $|C_i| \leq 6$ for $1 \leq i \leq k$.*

In the same paper, they showed that this conjecture is settled if we add the condition $\delta(G) \geq 6k - 3$.

3.4.5 Chorded Cycles of the Same Length

Chen et al. (2015) considered a minimum degree condition for disjoint chorded cycles of the same length, which is analogous to theorems of Egawa and Versträete (Theorems 3.3.22 and 3.3.23). They actually considered disjoint chorded cycles with stronger property as follows.

Theorem 3.4.23 (Chen et al. [35]) *For any positive integer k, there exists an integer $n = n(k)$ depending on only k such that if G is a graph of order at least n and $\delta(G) \geq 3k + 8$, then G contains k disjoint isomorphic chorded cycles.*

They conjectured that the minimum degree at least $3k$ is sufficient, that is, the degree condition in Theorem 3.4.1 also guarantees the existence of such disjoint cycles for sufficiently large graphs.

Conjecture 3.4.24 (Chen et al. [35]) *For any positive integer k, there exists an integer $n = n(k)$ depending on only k such that if G is a graph of order at least n and $\delta(G) \geq 3k$, then G contains k disjoint isomorphic chorded cycles.*

For disjoint c-chorded cycles, they also gave the following result.

Theorem 3.4.25 (Chen et al. [35]) *For any integers $k \geq 1$ and $c \geq 0$, there exist integers $n = n(k, c)$ and $t = t(c)$ such that if G is a graph of order at least n and $\delta(G) \geq \lceil \sqrt{c+1} + 1 \rceil k + t$, then G contains k disjoint isomorphic c-chorded cycles.*

They proved it as $t(c) = 12 \cdot (9/2)^c$. The coefficient $\lceil \sqrt{c+1} + 1 \rceil$ of k in the lower bound on $\delta(G)$ is best possible.

We also mention the El-Zahár-type problem (Conjecture 3.3.1), that is, the degree condition for partitions into chorded cycles with a pre-specified length. As a chorded cycle version of this problem, Theorem 3.4.17 leads to the following result. (Apply Theorem 3.4.17 with $|G| = n$ and $H_i = C^{n_i}$ for $1 \leq i \leq k$.) We do not know whether the σ_2 condition in Theorem 3.4.26 is sharp or not.

Theorem 3.4.26 (Babu and Diwan [11]) *Let k be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 3$ for $1 \leq i \leq k$. If $\sigma_2(G) \geq 2(n-k) - 1$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|C_i| = n_i$ and C_i has $n_i - 3$ chords incident with a common vertex for $1 \leq i \leq k$.*

We will also mention other related results to this problem for cycles of short length in the next section.

3.4.6 Chorded Cycles of Short Length

Kawarabayashi (2002) gave the following minimum degree condition for partitions into chorded quadrangles.

Theorem 3.4.27 (Kawarabayashi [146]) *Let k be a positive integer, and let G be a graph of order $n = 4k$. If $\delta(G) \geq \frac{n+k}{2}$, then G can be partitioned into k chorded cycles of length four.*

In the same paper, he conjectured a packing version of this theorem on a σ_2 condition. Fujita (2005) settled this conjecture. (Note that, in [101], the condition was not "$k \geq 1$" but "$k \geq 2$". However, we can check that the case $k = 1$ is also true.)

Theorem 3.4.28 (Fujita [101]) *Let k be a positive integer, and let G be a graph of order $n \geq 4k + 1$. If $\sigma_2(G) \geq n + k$, then G contains k disjoint chorded cycles of length four.*

The order condition "$n \geq 4k + 1$" cannot be replaced with "$n \geq 4k$" for $k \geq 4$.

By combining Theorems 3.4.27 and 3.4.28, it follows that if G is a graph of order $n \geq 4k$ and $\delta(G) \geq \frac{n+k}{2}$, then G contains k disjoint chorded cycles of length four. Note that a chorded cycle of length four contains a triangle, and hence this theorem is a generalization of Dirac's Theorem (Theorem 3.3.4).

Gao et al. (2011) showed that the degree condition in Theorem 3.4.28 also guarantees the existence of the following partition.

Theorem 3.4.29 (Gao et al. [107]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k + 1$. If $\sigma_2(G) \geq n + k$, then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that C_i is a chorded cycle of length four for $1 \leq i \leq k - 1$.*

On the other hand, Kostochka and Yu (2009) considered a σ_2 condition for the existence of a spanning subgraph in which each component belongs to $\{K^1, K^2, C^3, C^{4+}, C^{5+}\}$. Here, C^{4+} and C^{5+} denote cycles of lengths four and five with a chord, respectively. (In Sect. 4.3, we will discuss results related to Theorem 3.4.30.)

Theorem 3.4.30 (Kostochka and Yu [160]) *Let G be a graph of order n and H be a graph of order n whose components are isomorphic to graphs in $\{K^1, K^2, C^3, C^{4+}, C^{5+}\}$. If $\sigma_2(G) \geq \frac{4n}{3} - 1$, then G contains H as a spanning subgraph.*

As a corollary of this, we can obtain the following El-Zahár-type result. (Apply Theorem 3.4.30 with $H = H_1 \cup \cdots \cup H_k$, where each H_i is a graph of order n_i such that $H_i \in \{K^1, K^2, C^3, C^{4+}, C^{5+}\}$.)

Corollary 3.4.31 *Let k be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $1 \leq n_i \leq 5$ for $1 \leq i \leq k$. If $\sigma_2(G) \geq \frac{4n}{3} - 1$, then G can be partitioned into k cycles and degenerate cycles $C_1, \ldots, C_k$ such that $|C_i| = n_i$ for $1 \leq i \leq k$, and each C_i of length 4 or 5 has a chord.*

4 Generalizations of Disjoint Cycles in Graphs

In this section, we consider generalizations of packing cycles and partitions into cycles, and we discuss results which are obtained from the generalizations.

4.1 Subgraphs with Degree Constraints

If a graph contains k disjoint subgraphs with minimum degree at least two, then by Proposition 3.1.1, each subgraph contains a cycle, that is, the graph contains k disjoint cycles. In this sense, packing and partition problems into subgraphs with degree constraints are one of generalizations of packing and partition problems into cycles. We will discuss such type of results in this section.

4.1.1 Subgraphs with Degree Constraints

In 1981, Györi has suggested the following problem in the Sixth Hungarian Colloquim on Combinatorics held at Eger: For positive integers s_1 and s_2, find a (smallest) natural number $f(s_1, s_2)$ such that every graph of minimum degree at least $f(s_1, s_2)$ can be partitioned into two subgraphs of minimum degree at least s_1 and s_2, respectively. In 1983, Thomassen [221] answered this question by showing that every graph of minimum degree at least $6(s_1+s_2)$ can be partitioned into two subgraphs of minimum degree at least s_1 and s_2, respectively. In the same year, Hajnal [123] improved the minimum degree condition "$6(s_1+s_2)$" into "$2s_1+s_2-3$" for $s_1 \geq 4$. Thomassen (1988) also conjectured the smallest $f(s_1, s_2)$ as follows.

Conjecture 4.1.1 (Thomassen [224, Conjecture 6.1]) *Let s_1 and s_2 be positive integers, and let G be a graph. If $\delta(G) \geq s_1+s_2+1$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i = 1, 2$.*

In 1996, Stiebitz settled this conjecture by showing the following stronger version. (In 2017, Ban gave a weighted graph version of this theorem, see [13].)

Theorem 4.1.2 (Stiebitz [216]) *Let G be a graph, and let $s_1, s_2 : V(G) \to \mathbb{N}$ be two functions. If $d_G(x) \geq s_1(x)+s_2(x)+1$ for every vertex $x \in V(G)$, then G can be partitioned into two subgraphs H_1 and H_2 such that $d_{H_1}(x) \geq s_1(x)$ for every vertex $x \in V(H_1)$, and $d_{H_2}(x) \geq s_2(x)$ for every vertex $x \in V(H_2)$.*

Kaneko (1998) improved the degree condition in Conjecture 4.1.1 for triangle-free graphs (i.e., graphs with girth at least 4).

Theorem 4.1.3 (Kaneko [138]) *Let s_1 and s_2 be positive integers, and let G be a graph. If $g(G) \geq 4$ and $\delta(G) \geq s_1+s_2$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i = 1, 2$.*

Diwan (2000) further improved this theorem for graphs with girth at least 5.

Theorem 4.1.4 (Diwan [65]) *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a graph. If $g(G) \geq 5$ and $\delta(G) \geq s_1+s_2-1$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i = 1, 2$.*

The proofs of these theorems consist of Step 1 (Packing) and Step 2 (Partitioning) as ones mentioned in Sect. 3.1.2. In particular, the following result by Stiebitz [216] has been used in Step 2. Hence, Kaneko and Diwan actually proved Step 1 for Theorems 4.1.3 and 4.1.4, respectively.

Theorem 4.1.5 (Stiebitz [216]) *Let G be a graph, and let $s_1, s_2 : V(G) \to \mathbb{N}$ be two functions. Suppose that G contains two disjoint subgraph F_1 and F_2 such that $d_{F_1}(x) \geq s_1(x)$ for every vertex $x \in V(F_1)$, and $d_{F_2}(x) \geq s_2(x)$ for every vertex $x \in V(F_2)$. If $d_G(x) \geq s_1(x)+s_2(x)-1$ for every vertex $x \in V(G)$, then the same conclusion as Theorem 4.1.2 holds.*

In 2007, Bazgan et al. [15] gave polynomial-time algorithms that find such partitions under the conditions of Theorems 4.1.2, 4.1.3 and 4.1.4, respectively.

In 2003, Kühn and Osthus considered a partition into two subgraphs with a stronger property.

Theorem 4.1.6 (Kühn and Osthus [163]) *For any positive integer s, there exists an integer $h(s)$ depending on only s such that, if G is a graph with $\delta(G) \geq h(s)$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\delta(H_i) \geq s$ for $i = 1, 2$ and $|N_G(x) \cap V(H_2)| \geq s$ for $x \in V(H_1)$.*

Theorem 4.1.7 (Kühn and Osthus [163]) *Let s be a positive integer, and let G be a graph. If $\delta(G) \geq 2^{32}s$, then G can be partitioned into two subgraphs H_1 and H_2 such that $|H_i| \geq |G|/2^{18}$ and $\delta(H_i) \geq s$ for $i = 1, 2$, and $G[V(H_1), V(H_2)]$ has average degree at least s.*

These theorems are related to Conjecture 4.2.4.

In 2018, Chiba and Lichiardopol considered σ_2 versions of Conjecture 4.1.1 and Theorem 4.1.3 and they proved the following results, which correspond to Step 1 for the problem.

Theorem 4.1.8 (Chiba and Lichiardopol [45]) *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then G contains two disjoint subgraphs H_1 and H_2 such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq s_i + 1$ for $i = 1, 2$.*

Theorem 4.1.9 (Chiba and Lichiardopol [45]) *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a graph of order at least 3. If $g(G) \geq 4$ and $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then G contains two disjoint subgraphs H_1 and H_2 such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq 2s_i$ for $i = 1, 2$.*

It is an open problem whether the subgraphs H_1 and H_2 in Theorems 4.1.8 and 4.1.9 can be transformed into a partition of the graph.

Problem 4.1.10 (Chiba and Lichiardopol [45]) *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq s_i + 1$ for $i = 1, 2$.*

Problem 4.1.11 (Chiba and Lichiardopol [45]) *Let s_1 and s_2 be integers with $s_1 \geq 2$ and $s_2 \geq 2$, and let G be a graph of order at least 3. If $g(G) \geq 4$ and $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then G can be partitioned into two subgraphs H_1 and H_2 such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq 2s_i$ for $i = 1, 2$.*

The degree conditions in Problems 4.1.10 and 4.1.11 are best possible if they are true.

4.1.2 Applications

The results in Sect. 4.1.1 are sometimes useful tools to get degree conditions for packing k cycles. In fact, in order to attack the problem, one may use the induction on k, and then the results can work effectively in the inductive step. As an immediate corollary of Theorems 4.1.2–4.1.4, we can obtain the following.

Corollary 4.1.12 (Stiebitz [216], Kaneko [138], Diwan [65]) *Let k be an integer with $k \geq 2$, $s_1, \ldots, s_k$ be positive integers, and G be a graph.*

(1) *If $\delta(G) \geq \sum_{i=1}^{k} s_i + (k-1)$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that $\delta(H_i) \geq s_i$ for $1 \leq i \leq k$.*
(2) *If $g(G) \geq 4$ and $\delta(G) \geq \sum_{i=1}^{k} s_i$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that $\delta(H_i) \geq s_i$ for $1 \leq i \leq k$.*
(3) *If $g(G) \geq 5$ and $\delta(G) \geq \sum_{i=1}^{k} s_i - (k-1)$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that $\delta(H_i) \geq s_i$ for $1 \leq i \leq k$.*

This implies that it is enough to consider only the basis step for packing cycles. For example, we know that every graph G with $\delta(G) \geq 2$ contains a cycle (Proposition 3.1.1). Hence by applying Corollary 4.1.12 with $s_i = 2$ $(1 \leq i \leq k)$, we can obtain the following:

(1) Every graph G with $\delta(G) \geq 3k - 1$ $(= 2k + (k-1))$ contains k disjoint cycles.
(2) Every graph G with $g(G) \geq 4$ and $\delta(G) \geq 2k$ contains k disjoint cycles.
(3) Every graph G with $g(G) \geq 5$ and $\delta(G) \geq k + 1$ $(= 2k - (k-1))$ contains k disjoint cycles.

Although the degree condition in (1) is not sharp, the degree condition in (2) is the same condition as the sharp condition in Corrádi and Hajnal's Theorem (Theorem 3.1.2). Recall Theorem 3.3.24. If we assume that the order is sufficiently large compared with k, then a much weaker degree condition than (3) guarantees the existence of k disjoint cycles. (Note that (1) and (2) can be easily shown without the use of Corollary 4.1.12.)

By using the fact that every graph G with $\delta(G) \geq 3$ contains a chorded cycle, we can also obtain degree conditions for the existence of k disjoint chorded cycles. As mentioned in Sect. 3.4.3, Gould, Horn and Magnant [113] conjectured that every graph G of order at least $(c+1)k$ and of minimum degree at least ck contains k disjoint $(c+1)(c-2)/2$-chorded cycles (Conjecture 3.4.14) and they showed that this conjecture is true for sufficiently large graphs (Theorem 3.4.15).

Combining Theorem 3.4.13 with Corollary 4.1.12, we can obtain the following.

Corollary 4.1.13 *Let k be a positive integer, and let G be a graph.*

(1) *If $\delta(G) \geq (c+1)k - 1$, then G contains k disjoint $\frac{(c+1)(c-2)}{2}$-chorded cycles.*
(2) *If $g(G) \geq 4$ and $\delta(G) \geq ck$, then G contains k disjoint $\frac{(c+1)(c-2)}{2}$-chorded cycles.*
(3) *If $g(G) \geq 5$ and $\delta(G) \geq (c-1)k + 1$, then G contains k disjoint $\frac{(c+1)(c-2)}{2}$-chorded cycles.*

We do not know whether the degree conditions in this corollary are sharp or not.

We introduce another example, which gives a sharp degree condition for the existence of k disjoint cycles with some additional condition.

Chiba and Lichiardopol gave the following σ_2 condition for the existence of a cycle of length 0 mod 3, which is a relaxed structure of a triangle. Here, a cycle C is called a *cycle of length* 0 *mod* 3 if $|C| \equiv 0 \pmod 3$. (In 1994, Chen and Saito [38] gave a minimum degree condition.)

Theorem 4.1.14 (Chiba and Lichiardopol [45]) *Let G be a graph of order at least* 3. *If $\sigma_2(G) \geq 5$, then G contains a cycle of length* 0 *mod* 3.

By using this result and Theorem 4.1.9, they proved the following result for packing k cycles of length 0 mod 3, which is a natural generalization of Theorem 4.1.14 (see also [45, Proposition 6]).

Theorem 4.1.15 (Chiba and Lichiardopol [45]) *Let k be a positive integer, and let G be a graph of order at least $3k$. If $\sigma_2(G) \geq 6k - 1$, then G contains k disjoint cycles of length 0 mod 3.*

The complete bipartite graph $K^{3k-1,n-3k+1}$ shows the sharpness of the lower bound on the degree condition.

The above arguments might also work effectively for partitions into cycles. In fact, if each subgraph H_i in Corollary 4.1.12 contains a hamiltonian cycle, then we can obtain a partition into k cycles. However, s_i will need to depend on the order of H_i in order to work this argument, because the sharp degree condition for the existence of a hamiltonian cycle depends on the order of a graph. Therefore, we also need to control the order of the subgraph H_i in Theorem 4.1.2. This problem concerns with the following conjecture of Bollobás and Scott (2002).

Conjecture 4.1.16 (Bollobás and Scott [24]) *Let G be a graph. Then G can be partitioned into two induced subgraphs H_1 and H_2 with $|H_1| \leq |H_2| \leq |H_1| + 1$ such that for $i = 1, 2$, the following holds: $|N_G(v_i) \cap V(H_{3-i})| \geq d_{H_i}(v_i) - 1$ for $v_i \in V(H_i)$.*

In 2015, Liu and Xu pointed out the relationship between this conjecture and a partition into subgraphs with degree constraints.

Proposition 4.1.17 (see [177]) *Conjecture 4.1.16 on graphs of even orders is equivalent to the following statement* (S).

(S) *Every graph G with even order can be partitioned into two induced subgraphs H_1 and H_2 with $|H_1| = |H_2|$ such that for $i = 1, 2$, the following holds: $d_{H_i}(v_i) \geq \left\lceil \frac{d_G(v_i)}{2} \right\rceil - 1$ for $v_i \in V(H_i)$.*

If statement (S) is true, then by using it and a well-known result of Dirac [61] ("every graph G of order n and of minimum degree at least $\frac{n}{2}$ (> 1) is hamiltonian"), we can obtain the following as a corollary: Every graph G of order $2n$ and of minimum degree at least $n + 2$ contains a 2-factor with two cycles. Therefore, in order to get the degree conditions for partitions into k cycles, this kind of results may be useful tools.

4.2 Subgraphs with Connectivity Constraints

In this section, we consider connectivity versions of problems in Sect. 4.1, i.e., packing and partition problems into subgraphs with connectivity constraints. Since a cycle is a 2-connected graph, these problems are also one of generalizations of packing and partition problems into cycles. Therefore, this might be helpful to consider degree conditions.

Györi considered the following problem (in the Sixth Hungarian Colloquim on Combinatorics 1981): For positive integers s_1 and s_2, find a (smallest) natural number $g(s_1, s_2)$ such that every graph of connectivity at least $g(s_1, s_2)$ can be partitioned into two subgraphs of connectivity at least s_1 and s_2, respectively. Thosmassen showed that there is such a natural number $g(s_1, s_2)$ (see [221, Theorem 2]). By using a result of Marder [184], Hajnal (1983) proved that $g(s_1, s_2) \leq 4s_1 + 4s_2 - 13$ for $s_1 \geq 3$ and $s_2 \geq 2$.

Theorem 4.2.1 (Hajnal [123]) *Let s_1 and s_2 be integers with $s_1 \geq 3$ and $s_2 \geq 2$, and let G be a graph. If G is $(s_1 + s_2 - 1)$-connected and $\delta(G) \geq 4s_1 + 4s_2 - 13$, then G can be partitioned into two subgraphs H_1 and H_2 such that H_i is s_i-connected for $i = 1, 2$. (Hence, if G is $(4s_1 + 4s_2 - 13)$-connected, then G has such a partition.)*

Thomassen (1988) conjectured that "minimum degree" in Conjecture 4.1.1 is replaced by "connectivity" and that the value is the smallest $g(s_1, s_2)$.

Conjecture 4.2.2 (Thomassen [224, Conjecture 6.2]) *Let s_1 and s_2 be positive integers, and let G be a graph. If G is $(s_1 + s_2 + 1)$-connected, then G can be partitioned into two subgraphs H_1 and H_2 such that H_i is s_i-connected for $i = 1, 2$.*

However, unlike the situation for Conjecture 4.1.1, this conjecture is still wide open in general. It is clearly true for $s_i = 1$ for some i, and it is also true for $s_i = 2$ for some i, as is shown by the following result of Thomassen (1981), which was conjectured by Lovász [179]. Hence, the remaining case is $s_i \geq 3$ for $i = 1, 2$.

Theorem 4.2.3 (Thomassen [218]) *Let s be a positive integer, and let G be a graph. If G is $(s + 3)$-connected, then G contains an induced cycle C such that $G - C$ is s-connected.*

Thomassen (1983) also considered a partition of a graph G into two subgraphs H_1 and H_2 such that each of the graphs H_1, H_2 and the bipartite graph consisting of all edges between H_1 and H_2 in some sense has large connectivity or edge density.

Conjecture 4.2.4 (Thomassen [222]) *For any positive integer s, there exists an integer $h(s)$ depending on only s such that, if G is an $h(s)$-connected graph and S is a vertex subset of order at most s in G, then G can be partitioned into two subgraphs H_1 and H_2 such that $S \subseteq V(H_1)$, both H_1 and H_2 are s-connected, and $|N_G(x) \cap V(H_2)| \geq s$ for $x \in V(H_1)$.*

This conjecture can be applied to show the existence of a non-separating structure in highly connected graphs, i.e., a subgraph in a graph of sufficiently high connectivity compared with a given integer s whose deletion results in an s-connected graph. Theorem 5.2.14 implies that every sufficiently highly connected graph contains a subdivision of a given graph H with prescribed branch vertices (see also [222, Corollay 1]). Conjecture 4.2.4 implies the existence of a non-separating such subdivision. As a special case of this, Conjecture 4.2.4 implies the following conjecture of Lovász (see also [222,224]), which is closely related to Theorem 4.2.3.

Conjecture 4.2.5 (Lovász [179]) *For any positive integer s, there exists an integer $h'(s)$ depending on only s such that, if G is an $h'(s)$-connected graph and, u and v are two vertices in G, then G contains a path P such that u and v are the end vertices, and $G - P$ is s-connected.*

Kühn and Osthus (2003) proved that if we drop the condition about the specified vertex subset S in Conjecture 4.2.4, then the conjecture is true.

Theorem 4.2.6 (Kühn and Osthus [163]) *For any positive integer s, there exists an integer $l(s)$ depending on only s such that, if G is an $l(s)$-connected graph, then G can be partitioned into two subgraphs H_1 and H_2 such that both H_1 and H_2 are s-connected, and $|N_G(x) \cap V(H_2)| \geq s$ for $x \in V(H_1)$.*

Since every sufficiently highly connected graph contains a subdivision of a given graph H ([166,184]), this theorem immediately yields the following.

Corollary 4.2.7 (Kühn and Osthus [163]) *For any positive integer s and any graph H, there exists an integer $l'(s, H)$ depending on only s and H such that, if G is an $l'(s, H)$-connected graph, then G contains a subdivision H' of H such that $G - H'$ is s-connected.*

4.3 The BEC-conjecture and the Pósa–Seymour's Conjecture

In this section, we will discuss the BEC-conjecture and the Pósa–Seymour's conjecture. These conjectures are related to the El-Zahár's conjecture in Sect. 3.3.1. We will focus on only results related to the El-Zahár's conjecture. If the readers want to know other results on two conjectures in more detail, refer a survey paper [152].

We first mention the BEC-conjecture and related results. Aigner and Brandt (1993) gave a minimum degree condition for the existence of a subgraph with maximum degree two. (This is a conjecture due to Sauer and Spencer [211] and, it has been proved by Alon and Fischer [6] for sufficiently large graphs.)

Theorem 4.3.1 (Aigner and Brandt [2]) *Let G be a graph of order n and H be a graph of order at most n with $\Delta(H) \leq 2$. If $\delta(G) \geq \frac{2n-1}{3}$, then G contains H as a subgraph.*

This theorem is the case $r = 2$ in the following famous BEC-conjecture, due to Bollobás and Eldridge (1978) and independently due to Catlin (1976).

Conjecture 4.3.2 (Bollobás and Eldridge [23], Catlin [31]) *Let G be a graph of order n and H be a graph of order at most n with $\Delta(H) \leq r$. If $\delta(G) \geq \frac{rn-1}{r+1}$, then G contains H as a subgraph.*

The case $r = 3$ is also settled for sufficiently large graphs by Csaba et al. [55].

By using Theorem 3.4.30, Kostochka and Yu (2012) improved Theorem 4.3.1 into a σ_2 version.

Theorem 4.3.3 (Kostochka and Yu [161]) *Let G be a graph of order n and H be a graph of order at most n with $\Delta(H) \leq 2$. If $\sigma_2(G) \geq \frac{4n-3}{3}$, then G contains H as a subgraph.*

This result shows that the case $r = 2$ in the following conjecture, due to Kostochka and Yu (2007), is true. (In [161], they actually gave a slightly stronger result concerning the case $r = 2$ in Conjecture 4.3.4.)

Conjecture 4.3.4 (Kostochka and Yu [157]) *Let G be a graph of order n and H be a graph of order at most n with $\Delta(H) \le r$. If $\sigma_2(G) \ge \frac{2(rn-1)}{r+1}$, then G contains H as a subgraph.*

Theorems 4.3.1 and 4.3.3 lead to an El-Zahár-type result as a corollary. In fact, by applying Theorem 4.3.1 with $H = C^{n_1} \cup C^{n_2} \cup \cdots \cup C^{n_k}$, $|G| = n = \sum_{i=1}^{k} n_i$, we can obtain the following. (By using Theorem 4.3.3, we can obtain a σ_2 version.)

Corollary 4.3.5 *Let k be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \ge 3$ for $1 \le i \le k$. If $\delta(G) \ge \frac{2n-1}{3}$, then G can be partitioned into k cycles of lengths $n_1, n_2, \ldots, n_k$.*

In this sense, the study along this line is one of generalizations of an El-Zahár-type problem. However, note that the degree condition in Corollary 4.3.5 is much stronger than the one in Conjecture 3.3.1. (It is caused by the chromatic number, see Sect. 4.5.) Therefore, it may not be an effective approach to get sharp degree conditions for an El-Zahár-type problem in general graphs. On the other hand, interestingly, the situation is quite different for bipartite graphs. In fact, a bipartite version of this type of result implies a bipartite version of the El-Zahár's conjecture (see Sect. 6.1.4).

We next mention the Pósa–Seymour's conjecture and related results. In 1996, Fan and Kierstead showed that the same degree condition as Theorem 4.3.1 guarantees the existence of the square of P^n. A *square* of a path P is the graph obtained from P by joining every pair of vertices with distance two in P.

Theorem 4.3.6 (Fan and Kierstead [89]) *Let G be a graph of order n. If $\delta(G) \ge \frac{2n-1}{3}$, then G contains the square of P^n.*

Theorem 4.3.1 is a corollary of this theorem, since any graph H of order at most n with $\Delta(H) \le 2$ is contained in the square of P^n.

Theorem 4.3.6 is related to a conjecture of Posá which says that every graph G of order $n \ge 3$ and of minimum degree at least $2n/3$ contains the square of C^n (see Erdős [83]). More generally, it is related to the following conjecture, due to Seymour (1974). Here, the rth *power* of a cycle C is the graph obtained from C by joining every pair of vertices with distance at most r in C.

Conjecture 4.3.7 (Seymour [214]) *Let r be a positive integer, and let G be a graph of order $n \ge 3$. If $\delta(G) \ge \frac{rn}{r+1}$, then G contains the rth power of C^n.*

This conjecture is called the Pósa–Seymour's conjecture, and is settled for sufficiently large graphs by Komlós et al. [155]. Note that if this conjecture is true, then every graph G of order n with $\delta(G) \ge \frac{rn-1}{r+1}$ contains the rth power of P^n (because, adding a new vertex v to such a graph G and joining v to all the vertices of G, the resulting graph satisfies the minimum degree condition in Conjecture 4.3.7). Therefore, Conjecture 4.3.7 is a generalization of Theorem 4.3.6, that is, it is a generalization of Theorem 4.3.1. So, the study on Conjecture 4.3.7 is also one of generalizations of an El-Zahár-type problem in a sense.

4.4 The Erdős–Pósa Property and the Corrádi–Hajnal Property

In this section, we discuss a relation between the Erdős–Pósa property and the Corrádi–Hajnal property.

A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property*, if for every positive integer k, there is an integer $f = f(k, \mathcal{F})$ depending on only k and $\mathcal{F}$ such that every graph G contains either k disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set X of at most f vertices such that $G - X$ has no subgraph isomorphic to a graph in $\mathcal{F}$. Erdős and Pósa (1965) proved that the family of cycles has the property.

Theorem 4.4.1 (Erdős and Pósa [87]) *For any positive integer k, there is an integer $f = f(k)$ depending on only k such that, if G is a graph, then G contains k disjoint cycles, or G contains a vertex set X with $|X| \leq f$ such that $G - X$ is a forest.*

The function f in this theorem is $O(k \log k)$. This theorem concerns with *the feedback vertex set problem*, i.e., the problem of finding a minimum vertex set of a given graph whose removal results in a graph that contains no cycle, which is a fundamental combinatorial optimization problem and has many applications (see a survey [98] for more details). It is also one of the well-known NP-complete problems in Karp's list [143]. Theorem 4.4.1 says that if a given graph contains no k disjoint cycles, then the size of a minimum feedback vertex set is at most $O(k \log k)$.

A family $\mathcal{F}$ of graphs is said to have the *Corrádi–Hajnal property*, if for every positive integer k, there are integers $n = n(k, \mathcal{F})$ and $\delta = \delta(k, \mathcal{F})$ depending on only k and $\mathcal{F}$ such that every graph G of order at least n and of minimum degree at least δ contains k disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$. By Corrádi–Hajnal's Theorem (Theorem 3.1.2), the family of cycles has the property. This fact is also obtained by Theorem 4.4.1 and Proposition 3.1.1 as follows.

Corollary 4.4.2 *For any positive integer k, there is an integer $h(k)$ depending on only k such that if G is a graph with $\delta(G) \geq h(k)$, then G contains k disjoint cycles.*

Proof Let f be the function in Theorem 4.4.1, and let $h(k) = f(k) + 2$. We show that every graph G with $\delta(G) \geq h(k)$ contains k disjoint cycles. Suppose that there is a graph G with $\delta(G) \geq h(k)$ which contains no k disjoint cycles. Then, by Theorem 4.4.1, G contains a vertex set X with $|X| \leq f$ such that $G - X$ is a forest. However, by the definition of h, we have $\delta(G - X) \geq 2$, and hence Proposition 3.1.1 guarantees the existence of a cycle in $G - X$, a contradiction. □

In general, we can see the following proposition.

Proposition 4.4.3 *Let $\mathcal{F}$ be a family of graphs, and suppose that there is an integer $\delta = \delta(\mathcal{F})$ such that if G is a graph with $\delta(G) \geq \delta$, then G contains a graph in $\mathcal{F}$. If $\mathcal{F}$ has the Erdős–Pósa property, then $\mathcal{F}$ also has the Corrádi–Hajnal property.*

From this proposition, we see that the study on the Erdős–Pósa property is important to consider degree conditions for packing cycles.

In the rest of this section, we discuss families of even cycles (i.e., cycles of even length), long cycles, cycles passing through a vertex in a pre-specified vertex set. These families have the Erdős–Pósa property.

We first consider a family of even cycles. Thomassen [223] proved that the family of even cycles has the Erdős–Pósa property. On the other hand, as mentioned in Sect. 3.4, we can see that $\delta(G) \geq 3$ guarantees the existence of a chorded cycle, and a chorded cycle contains an even cycle. Therefore the family of even cycles has the Corrádi–Hajnal property. In fact, Theorem 3.3.26 confirms the fact.

We next consider a family of long cycles. In [21], Birmelé et al. proved that the family of cycles of length at least l (≥ 3) has the Erdős-Pósa property, where l is a fixed positive integer. On the other hand, it is easy to see that every graph of minimum degree at least $l - 1$ (≥ 2) contains a cycle of length at least l. Therefore the family of cycles of length at least l (≥ 3) has the Corrádi-Hajnal property. (See also Theorem 3.1.38.)

We finally consider a family of cycles passing through a vertex in a pre-specified vertex set. An S-*cycle* is a cycle containing at least one vertex from a given vertex set S. In [202], Pontecorvi and Wollan showed that the family of S-cycles has the Erdős-Pósa property which is an improvement of the result in [137]. (It is also known that a family of S-cycles of long length also has the Erdős–Pósa property (see [29]).) On the other hand, it is known that the minimum degree condition that guarantees the existence of S-cycles depends on the order of the graph (see [22,215]). Hence, from only these facts, we cannot see whether a family of S-cycles has the Corrádi–Hajnal property. But, it follows from the sharpness of the degree condition in Proposition 3.2.15 that the family of S-cycles does not have the Corrádi–Hajnal property in general.

Finally, we propose the following problem.

Problem 4.4.4 *Which is true that, "for a family $\mathcal{C}$ of cycles (paths) with a property P, if $\mathcal{C}$ has the Corrádi–Hajnal property, then $\mathcal{C}$ has the Erdős–Pósa property" or "there is a family $\mathcal{C}$ of cycles (paths) with a property P such that $\mathcal{C}$ has the Corrádi–Hajnal property, but does not have the Erdős–Pósa property"?*

4.5 Partitions of a Graph into any Fixed Graphs

In this section, we discuss Alon and Yuster's result mentioned in Sect. 3.3.1 (see the paragraph following Conjecture 3.3.5). In 1996, Alon and Yuster proved the following theorem for partitions into any fixed graph H. Here, $\chi(H)$ denotes the chromatic number of a graph H.

Theorem 4.5.1 (Alon and Yuster [7]) *For any graph H and a real number $\varepsilon > 0$, there exists an integer $n_0 = n_0(H, \varepsilon)$ depending on only H and ε such that, if k is an integer with $k|H| \geq n_0$ and G is a graph of order $n = k|H|$ such that $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \varepsilon\right)n$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that $H_i \simeq H$ for $1 \leq i \leq k$.*

Note that the case where H is a complete graph corresponds to an asymptotic version of Theorem 3.4.11.

They also conjectured in the same paper that εn $(= \varepsilon k|H|)$ is not the best possible error term, and a constant depending on H would suffice. In 2001, Komlós et al. settled this conjecture as follows.

Theorem 4.5.2 (Komlós et al. [154]) *Let H be a graph with the chromatic number χ, and assume that H has a χ-coloring with color-class sizes $h_1 \leq h_2 \leq \cdots \leq h_\chi$.*

There exists an integer $n_0 = n_0(H)$ *depending on only* H *such that, if* k *is an integer with* $k|H| \geq n_0$ *and* G *is a graph of order* $n = k|H|$ *such that* $\delta(G) \geq \left(1 - \frac{1}{\chi}\right)n + h_\chi + h_{\chi-1} - 1$*, then* G *can be partitioned into* k *subgraphs* $H_1, \ldots, H_k$ *such that* $H_i \simeq H$ *for* $1 \leq i \leq k$*.*

Kühn and Osthus (2009) gave a minimum degree condition on the chromatic number or the critical chromatic number, which was introduced by Komlós [153]. Let H be a graph with the chromatic number $\chi(H) = l$. The *critical chromatic number* $\chi_{cr}(H)$ is defined as $\frac{(l-1)|H|}{|H|-\tau(H)}$, where $\tau(H)$ is the size of the smallest color-class over all l-colorings of H. We also define $\mathrm{hcf}_c(H)$ as the highest common factor of all the orders of components of H. Given an l-coloring f with $h_1 \leq h_2 \leq \cdots \leq h_l$ as the sizes of color-classes, let $D(f) = \{h_{i+1} - h_i : 1 \leq i \leq l-1\}$, and let $D(H)$ be the union of all the sets $D(f)$ taken over all l-colorings of H. Define $\mathrm{hcf}_\chi(H)$ as the highest common factor of $D(H)$ (we let $\mathrm{hcf}_\chi(H) = \infty$ if $D(H) = \{0\}$). Lastly, we say that

$$H \text{ is in } \textit{Class } 1 \text{ if } \begin{cases} \mathrm{hcf}_\chi(H) = 1 & \text{if } \chi(H) \neq 2, \\ \mathrm{hcf}_\chi(H) \leq 2 \text{ and } \mathrm{hcf}_c(H) = 1 & \text{if } \chi(H) = 2, \end{cases}$$

otherwise, H is in *Class* 2.

Theorem 4.5.3 (Kühn and Osthus [164]) *Let* H *be a graph. There exist integers* $C = C(H)$ *and* $n_0 = n_0(H)$ *depending on only* H *such that, if* k *is an integer with* $k|H| \geq n_0$ *and* G *is a graph of order* $n = k|H|$ *such that*

$$\delta(G) \geq \begin{cases} \left(1 - \frac{1}{\chi_{cr}(H)}\right)n + C & \textit{if } H \textit{ is in Class 1,} \\ \left(1 - \frac{1}{\chi(H)}\right)n + C & \textit{if } H \textit{ is in Class 2,} \end{cases}$$

then G *can be partitioned into* k *subgraphs* $H_1, \ldots, H_k$ *such that* $H_i \simeq H$ *for* $1 \leq i \leq k$*.*

This degree condition is best possible up to the constant C.

Kühn et al. (2009) gave a σ_2 version of Theorem 4.5.1 (they actually gave a σ_2 condition on the chromatic number or the critical chromatic number as Theorem 4.5.3, see [165] for more details).

Theorem 4.5.4 (Kühn et al. [165]) *For any graph* H *and a real number* $\varepsilon > 0$*, there exists an integer* $n_0 = n_0(H, \varepsilon)$ *depending on only* H *and* ε *such that, if* k *is an integer with* $k|H| \geq n_0$ *and* G *is a graph of order* $n = k|H|$ *such that* $\sigma_2(G) \geq 2\left(1 - \frac{1}{\chi(H)} + \varepsilon\right)n$*, then* G *can be partitioned into* k *subgraphs* $H_1, \ldots, H_k$ *such that* $H_i \simeq H$ *for* $1 \leq i \leq k$*.*

The degree condition in this theorem is not sharp. Similar to Theorems 4.5.2 and 4.5.3, the error term $2\varepsilon n$ could be improved to a constant depending on only H.

These results also lead to related results with an El-Zahár-type problem as in Sect. 3.3.1. In fact, applying Theorem 4.5.4 as $H = C^l$, we can obtain the following.

Corollary 4.5.5 *For any integer $l \geq 3$ and a real number $\varepsilon > 0$, there exists an integer $n_0 = n_0(l, \varepsilon)$ depending on only l and ε such that the following hold. Let k be an integer with $kl \geq n_0$, and let G be a graph of order $n = kl$. Suppose that if l is even, then $\sigma_2(G) \geq 2\left(\frac{1}{2} + \varepsilon\right)n$; otherwise, $\sigma_2(G) \geq 2\left(\frac{2}{3} + \varepsilon\right)n$. Then G can be partitioned into k cycles of length l.*

However, in this corollary, if l is odd and $l \geq 5$, then the σ_2 condition is much stronger than the one in Conjecture 3.3.1. Therefore, we need to consider the case directly.

5 Disjoint Paths in Graphs

5.1 Paths with Length Constraints

5.1.1 Partitions into Paths

We first consider a relationship between the independence number and the connectivity for partitions into paths with a pre-specified order. Theorem 2.1.4 implies the following corollary.

Corollary 5.1.1 *Let k and $n_1, \ldots, n_k$ be positive integers, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$. If $\alpha(G) \leq \kappa(G) + 1$, then G contains a hamiltonian path, that is, G can be partitioned into k paths of orders $n_1, n_2, \ldots, n_k$.*

From this corollary, we may consider the existence of k disjoint paths in a connected graph G with $\alpha(G) \geq 3$. The following proposition is obtained from Theorem 2.1.7 (refer to the paragraph following Theorem 5.1.3 for the proof).

Proposition 5.1.2 *Let k be a positive integer, and let G be a connected graph of order $n \geq k$. If $\sigma_2^3(G) \geq n - k$, then G can be partitioned into k paths.*

Li and Steiner (2005) considered a characterization of connected graphs with high degree sum that cannot be partitioned into k paths.

Theorem 5.1.3 (Li and Steiner [170]) *Let k be a positive integer, and let G be a connected graph of order $n \geq k$. If $\sigma_2(G) \geq n - k - 1$, then one of the following holds:*

(i) *G can be partitioned into k paths,*
(ii) $K_{\frac{n-k-1}{2}, \frac{n+k+1}{2}} \subseteq G \subseteq K_{\frac{n-k-1}{2}} + \frac{n+k+1}{2} K_1$.

This theorem is also obtained from Theorem 2.1.8 as follows.

Proof Let k be a positive integer, and let G be a connected graph of order $n \geq k$ such that $\sigma_2(G) \geq n - k - 1$. It suffices to consider only the case $n \geq k + 1$. Let G' be the graph obtained from G by adding a complete graph F of order k and joining each vertex of G to all vertices of F. Then it is easy to check that G' is 2-connected and $\sigma_2(G') = (n-k-1)+2k = (n+k)-1 = |G'|-1$. Hence, by applying Theorem 2.1.8 to G', Theorem 2.1.8 (i) or Theorem 2.1.8 (ii) holds. If Theorem 2.1.8 (i) holds, then

$G = G' - F$ contains at most k disjoint paths as a spanning subgraph, and thus G can be partitioned into exactly k paths. Thus we may assume that Theorem 2.1.8 (ii) holds. Then $G' \simeq H + (m+1)K^1$ for some graph H such that $mK^1 \subseteq H \subseteq K^m$, where $m = \frac{|G'|-1}{2}$. Hence, by the definition of G', either (a) $|V(F) \cap V(H)| = k-1$ or (b) $V(F) \subseteq V(H)$ holds. If (a) holds, then G can be partitioned into k paths; if (b) holds, then $K^{\frac{n-k-1}{2}, \frac{n+k+1}{2}} \subseteq G \subseteq K^{\frac{n-k-1}{2}} + \frac{n+k+1}{2} K^1$. □

We consider the case not admitting paths of order one.

Proposition 5.1.4 *Let k be a positive integer, and let G be a connected graph of order n. If $n \geq 3k-1$ and $\sigma_2^3(G) \geq n-k$, then G can be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| \geq 2$ for $1 \leq i \leq k$.*

Proof Since $|G| \geq 3k-1$, $\sigma_2^3(G) \geq n-k$ and G is connected, we can show that G contains k disjoint paths $P_1, \ldots, P_k$ of order at least 2. Choose such disjoint paths so that $\sum_{i=1}^k |P_i|$ is as large as possible, and suppose that $V(G) \setminus \bigcup_{i=1}^k V(P_i) \neq \emptyset$. Let $H = G - \bigcup_{i=1}^k V(P_i)$, and let $P_1 = x_1x_2 \ldots x_{l_1}$, where $l_1 = |P_1|$. Since G is connected, we may assume that there exists a vertex y in H such that $yx_m \in E(G)$ for some m with $1 \leq m \leq l_1$. Then, by the maximality of $\sum |P_i|$, we have $m \notin \{1, l_1\}$ and $X := \{x_1, x_{m+1}, y\}$ is an independent set of order 3 in G. We can also see that for each pair u, v of vertices in X,

$$|N_G(u) \cap V(H)| + |N_G(v) \cap V(H)| \leq |H| - 1.$$

By using a standard crossing argument, we can further show that

$$|N_G(u) \cap V(P_i)| + |N_G(v) \cap V(P_i)| \leq |P_i| - 1 \text{ for } 1 \leq i \leq k.$$

In fact, for the path $P_i = z_1z_2 \ldots z_{l_i}$ ($2 \leq i \leq k$ and $l_i = |P_i|$), if $x_1z_p, yz_{p+1} \in E(G)$, then by considering the two paths $z_1z_2 \ldots z_px_1x_2 \ldots x_{l_1}$ and $yz_{p+1}z_{p+2} \ldots z_{l_i}$ in $G[V(P_1 \cup P_i) \cup \{y\}]$, this contradicts the maximality of $\sum |P_i|$. Similarly, if $x_{m+1}z_p, yz_{p+1} \in E(G)$ or $x_1z_p, x_{m+1}z_{p+1} \in E(G)$, then we can get a contradiction. Note that every vertex of X are not adjacent to end vertices of P_i ($2 \leq i \leq k$), and so we get the above inequality for $2 \leq i \leq k$. On the other hand, for the path P_1, if $x_1x_p, yx_{p+1} \in E(G)$ ($p \geq m+1$), then $x_{m-1}x_{m-2} \ldots x_1x_px_{p-1} \ldots x_myx_{p+1}x_{p+2} \ldots x_{l_1}$ is a hamiltonian path of $G[V(P_1) \cup \{y\}]$, which contradicts the maximality of $\sum |P_i|$. Similarly, if $yx_p, x_{m+1}x_{p+1} \in E(G)$ or $x_1x_p, x_{m+1}x_{p+1} \in E(G)$ ($p \geq m+1$), then we can get a contradiction again. Moreover, if $x_1x_p, yx_{p-1} \in E(G)$, $x_{m+1}x_p, yx_{p-1} \in E(G)$ or $x_1x_p, x_{m+1}x_{p-1} \in E(G)$ ($p \leq m$), then we can find a hamiltonian path of $G[V(P_1) \cup \{y\}]$, a contradiction. Note that by the maximality of $\sum |P_i|$, $\{x_1, x_{l_1}, y\}$ is also an independent set, and so we get the above inequality for $i = 1$.

The above two inequalities imply that for each pair u, v of vertices in X,

$$d_G(u) + d_G(v) \leq |H| - 1 + \sum_{i=1}^{k} (|P_i| - 1) = n - k - 1,$$

which contradicts $\sigma_2^3(G) \geq n-k$. □

The degree condition in Proposition 5.1.4 is best possible (consider the graph in (ii) of Theorem 5.1.3). The complete bipartite graph $K^{k-1,2k-1}$ shows that the order condition is also best possible.

Considering the above situations, we can consider the following problem.

Problem 5.1.5 *Let k and l be positive integers. Determine a sharp $\sigma_2^3(G)$ condition for a sufficiently large connected graph G to be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| \geq l$ for $1 \leq i \leq k$. Moreover, characterize connected graphs with high degree sum that cannot be partitioned into k paths.*

5.1.2 The El-Zahár-type Problem

For partitions into k "connected subgraphs" with a pre-specified order, Enomoto (1995) proved the following, which was conjectured by Frank [104] in 1975.

Theorem 5.1.6 (Eonomoto [76]) *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a connected graph of order $n = \sum_{i=1}^{k} n_i$. If $\delta(G) \geq k$, then G can be partitioned into k connected subgraphs $H_1, \ldots, H_k$ such that $|H_i| = n_i$ for $1 \leq i \leq k$.*

The path version of this theorem was obtained by Johansson (1998) and it is a similar type of El-Zahár's Conjecture (Conjecture 3.3.1). (The case $n_i = 3$ $(1 \leq i \leq k)$ of Theorem 5.1.7 was proved by Enomoto et al. [79] in 1987.) Here, for integers $n_1, \ldots, n_k$, we let $\lambda(n_1, \ldots, n_k) = \big|\{1 \leq i \leq k : n_i \text{ is odd}\}\big|$. For convenience, we abbreviate $\lambda(n_1, \ldots, n_k)$ to λ. Note that $\sum_{i=1}^{k} \lfloor \frac{n_i}{2} \rfloor = \frac{1}{2}\big((\sum_{i=1}^{k} n_i) - \lambda\big)$.

Theorem 5.1.7 (Johansson [134]) *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a connected graph of order $n = \sum_{i=1}^{k} n_i$. If $\delta(G) \geq \frac{1}{2}(n - \lambda)$, then G can be partitioned into k paths of orders $n_1, n_2, \ldots, n_k$.*

Chen et al. (2001) showed that this degree condition can be weakened by excluding some sequences $(n_1, \ldots, n_k)$ of integers n_i.

Theorem 5.1.8 (Chen et al. [39]) *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a connected graph of order $n = \sum_{i=1}^{k} n_i$. If $n \geq 3\lambda + 4$ and $\sigma_3(G) \geq \frac{3}{2}(n - \lambda) - 2$, then G can be partitioned into k paths of orders $n_1, n_2, \ldots, n_k$.*

Note that the condition $n \geq 3\lambda + 4$ holds if and only if the integers $n, n_1, \ldots, n_k$ are not any of the following (a), (b) and (c): (a) $n = 3k + 2$, $n_1 = \cdots = n_{k-1} = 3$, $n_k = 5$; (b) $n = 3k$, $n_1 = \cdots = n_k = 3$; (c) $n = 3k - 1$, $n_1 = 2$, $n_2 = \cdots = n_k = 3$, where we assumed $n_1 \leq n_2 \leq \cdots \leq n_k$ (see also [39, Proposition 1]). They also considered σ_3 conditions for $n < 3\lambda + 4$, i.e., for each of (a), (b) and (c) (see [39, Theorems B and C]).

On the other hand, Egawa and Ota (2001) extended Theorem 5.1.7 so that it also corresponds to the packing problem. (In fact, they completely characterized the graphs in the exceptional cases.)

Theorem 5.1.9 (Egawa and Ota [74]) *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a connected graph of order at least $\sum_{i=1}^{k} n_i$. If $\sigma_3(G) \geq \frac{3}{2}\big((\sum_{i=1}^{k} n_i) - \lambda\big)$, then G contains k disjoint paths of orders $n_1, n_2, \ldots, n_k$, unless all $n_i = 3$, or $k = 2$ and n_1 and n_2 are odd.*

As in Sect. 5.1.1, we can pose the following problem.

Problem 5.1.10 *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a connected graph of order at least $\sum_{i=1}^{k} n_i$. Is it true that, if $\sigma_2^3(G) \geq (\sum_{i=1}^{k} n_i) - \lambda$, then G contains k disjoint paths of orders $n_1, n_2, \ldots, n_k$, unless some exceptions?*

Chiba et al. (2010) gave a σ_4 version of Theorem 5.1.9 for 2-connected graphs. By Corollary 5.1.1, we may consider a 2-connected graph G with $\alpha(G) \geq 4$.

Theorem 5.1.11 (Chiba et al. [46]) *Let k and $n_1, \ldots, n_k$ be integers with $k \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq k$, and let G be a 2-connected graph of order at least $\sum_{i=1}^{k} n_i$. If $\sigma_4(G) \geq 2\big((\sum_{i=1}^{k} n_i) - \lambda\big)$, then G contains k disjoint paths of orders $n_1, n_2, \ldots, n_k$, unless all $n_i = 3$ and $lK^2 + (k-1)K^1 \subseteq G \subseteq lK^2 + K^{k-1}$ $(l \geq k+1)$.*

In 2014, Chiba and Tsugaki [47] considered the case where the integers $n, n_1, \ldots, n_k$ are not any of the above (a), (b) and (c).

5.1.3 Specified Lengths and One End Vertex

Győri (1978) and Lovász (1977), independently, proved the following theorem for partitions into k "connected subgraphs" in which each subgraph has a pre-specified order and contains a vertex in pre-specified k vertices.

Theorem 5.1.12 (Györi [119], Lovász [180]) *Let k and $n_1, \ldots, n_k$ be positive integers, G be a graph of order $n = \sum_{i=1}^{k} n_i$, and $x_1, \ldots, x_k$ be k vertices in G. If G is k-connected, then G can be partitioned into k connected subgraphs $H_1, \ldots, H_k$ such that $|H_i| = n_i$ and $x_i \in V(H_i)$ for $1 \leq i \leq k$.*

As mentioned in Sect. 5.1.2, Johansson gave a minimum degree condition for a graph to be partitioned into k paths with a pre-specified length (Theorem 5.1.7). On the other hand, Ore's Theorem (Theorem 2.1.2) implies the following corollary concerning partitions into k paths with a pre-specified end vertex. The σ_2 condition is sharp by considering $K^{m,m+1}$.

Corollary 5.1.13 *Let k be a positive integer, G be a graph of order n, and $x_1, \ldots, x_k$ be k vertices in G. If $\sigma_2(G) \geq n$, then G can be partitioned into k paths $P_1, \ldots, P_k$ such that x_i is an end vertex of P_i for $1 \leq i \leq k$.*

Enomoto and Ota (2000) considered a path version of Theorem 5.1.12, i.e., a partition into "k paths" with a pre-specified length and a pre-specified end vertex, and they posed the following conjecture. (Kawarabayashi proposed the minimum degree version, see [145].)

Conjecture 5.1.14 (Enomoto and Ota [82]) *Let k and $n_1, \ldots, n_k$ be positive integers, G be a graph of order $n = \sum_{i=1}^{k} n_i$, and $x_1, \ldots, x_k$ be k vertices in G. If $\sigma_2(G) \geq n + k - 1$, then G can be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| = n_i$ and x_i is an end vertex of P_i for $1 \leq i \leq k$.*

In Sect. 5.2.3, we will discuss relations between this conjecture and other conjectures on k-linkedness (see Fig. 5 in Sect. 5.2.3).

For the case $k = 1$, Conjecture 5.1.14 is obvious from Corollary 5.1.13. For the case $k = 2$, it follows from Theorem 2.1.11. In [82], Enomoto and Ota proved the conjecture for the case $k = 3$ or $n_i \leq 5$ for $1 \leq i \leq k - 2$. Kawarabayashi (2002) proved the statement of the conjecture under the following stronger degree condition.

Theorem 5.1.15 (Kawarabayashi [145]) *Let k and $n_1, \ldots, n_k$ be positive integers, G be a graph of order $n = \sum_{i=1}^{k} n_i$, and $x_1, \ldots, x_k$ be k vertices in G. If $\sigma_2(G) \geq \left(\sum_{i=1}^{k} \max\{\lfloor \frac{4}{3} n_i \rfloor, n_i + 1\}\right) - 1$, then G can be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| = n_i$ and x_i is an end vertex of P_i for $1 \leq i \leq k$.*

Magnant and Martin (2010) proved an asymptotic version of Conjecture 5.1.14 as follows.

Theorem 5.1.16 (Magnant and Martin [186]) *Let k be a positive integer. For any set of k positive real numbers $r_1, \ldots, r_k$ with $\sum_{i=1}^{k} r_i = 1$ and for any positive real number ε, there exists an integer n_0 such that, if G is a graph of order $n \geq n_0$ and $\sigma_2(G) \geq n + k - 1$, then for any k vertices $x_1, \ldots, x_k$, G can be partitioned into k paths $P_1, \ldots, P_k$ such that $(r_i - \varepsilon)n < |P_i| < (r_i + \varepsilon)n$ and x_i is an end vertex of P_i for $1 \leq i \leq k$.*

By using the regularity lemma, Hall et al. (2014) proved that Conjecture 5.1.14 holds without the spanning assumption if the order of a graph G is sufficiently large compared with $n_1, \ldots, n_k$.

Theorem 5.1.17 (Hall et al. [124]) *Let k and $n_1, \ldots, n_k$ be positive integers. Then there exists an integer n_0 such that, if G is a graph of order $n \geq n_0$ and $\sigma_2(G) \geq n + k - 2$, then for any k vertices $x_1, \ldots, x_k$, G contains k disjoint paths $P_1, \ldots, P_k$ such that $|P_i| = n_i$ and x_i is an end vertex of P_i for $1 \leq i \leq k$.*

Moreover, in 2014, Coll et al. [52] announced that Conjecture 5.1.14 is settled for sufficiently large graphs by using the regularity lemma.

5.2 Paths Connecting Two Vertices in Pre-specified Vertex Sets

5.2.1 X-paths

For a vertex subset X of a graph G, a path in G is an *X-path* if it begins and ends in X, and none of its internal vertices are contained in X. (We do not admit a path of order one as an X-path.) Before mentioning results on degree conditions, we show a necessary and sufficient condition for the existence of $|X|/2$ disjoint X-paths.

Gallai (1961) gave a Tutte-Berge type formula for the existence of disjoint X-paths. Let $w(G)$ be the number of components of a graph G.

Theorem 5.2.1 (Gallai [105]) *Let G be a graph, and let $X \subseteq V(G)$. The maximum number of disjoint X-paths is equal to*

$$\min_{S \subseteq V(G)} \left\{ |S| + \sum_{D \in w(G-S)} \left\lfloor \frac{1}{2}|D \cap X| \right\rfloor \right\}.$$

This theorem implies the following necessary and sufficient condition for the existence of $|X|/2$ disjoint X-paths.

Corollary 5.2.2 *Let G be a graph, and let $X \subseteq V(G)$. Then, there exist $|X|/2$ disjoint X-paths if and only if for all $S \subseteq V(G)$,*

$$\frac{|X|}{2} \le |S| + \sum_{D \in w(G-S)} \left\lfloor \frac{1}{2}|D \cap X| \right\rfloor.$$

Berman (1983) obtained a σ_2 condition for the existence of a cycle passing through a given matching.

Theorem 5.2.3 (Berman [17]) *Let G be a graph of order $n \ge 3$, and let M be a matching. If $\sigma_2(G) \ge n+1$, then G contains a cycle passing through every edge of M.*

As a corollary of this theorem, we can obtain the following for packing $|X|/2$ X-paths (refer to the proof of Corollary 5.2.5).

Corollary 5.2.4 *Let k be a positive integer, G be a graph of order $n \ge 2k$, and X be a set of $2k$ vertices in G. If $\sigma_2(G) \ge n+1$, then G contains k disjoint X-paths.*

On the other hand, as a corollary of Theorem 2.1.13, we can also obtain the following σ_2 condition for partitions into $|X|/2$ X-paths.

Corollary 5.2.5 *Let k be a positive integer, G be a graph of order $n \ge 2k$ and X be a set of $2k$ vertices in G. If $\sigma_2(G) \ge n+k$, then G can be partitioned into k X-paths.*

Proof We construct a graph H from G by adding edges so that $H[X]$ has a perfect matching M. Note that $\sigma_2(H) \ge \sigma_2(G) \ge |G|+k \ge |H|+|M|$. By Theorem 2.1.13, H has a hamiltonian cycle C passing through M. Then $C - M$ is a union of k disjoint X-paths of G. □

The degree condition in this corollary is best possible. In addition, this degree condition guarantees the existence of a partition into k paths with a stronger property (see Theorem 5.2.9 in the next section).

5.2.2 (X, Y)-paths

For a graph G and $X, Y \subseteq V(G)$, a path in G is an (X, Y)-*path* if one end vertex of the path belongs to X, another end vertex belongs to Y, and the internal vertices do not belong to $X \cup Y$. Furthermore, we let $\kappa(X, Y; G)$ be the minimum number of vertices separating X from Y.

Before discussing results on degree conditions, we show a necessary and sufficient condition for the existence of disjoint (X, Y)-paths. The well-known Menger's theorem implies the existence of k disjoint (X, Y)-paths.

Theorem 5.2.6 (Menger [195]) *Let k be a positive integer, G be a graph, and X and Y be subsets of $V(G)$ with $|X| = |Y| = k$. Then, there exist k disjoint (X, Y)-paths if and only if $\kappa(X, Y; G) \geq k$.*

It is easy to show that a graph having high degree sum also has high connectivity.

Proposition 5.2.7 *Let k be a positive integer, let G be a graph of order $n \geq k+1$, and X and Y be subsets of $V(G)$ with $|X| = |Y| = k$. If $\sigma_2(G) \geq n+k-2$, then $\kappa(X, Y; G) \geq k$.*

Proof Suppose that $\kappa(X, Y; G) < k$. Then there exists a cut set $S \subseteq V(G)$ separating X and Y such that $|S| \leq k-1$. Since S is a cut set, $G-S$ has at least two components. Let x_1 and x_2 be two distinct vertices which belong to different components of $G-S$. Then note that $x_1x_2 \notin E(G)$ and $|N_G(x_1) \cap N_G(x_2)| \leq |S| \leq k-1$. Since $N_G(x_1) \cup N_G(x_2) \subseteq V(G) \setminus \{x_1, x_2\}$, we have $|N_G(x_1) \cup N_G(x_2)| \leq n-2$. By the degree condition and the above two inequalities, we obtain that $n+k-2 \leq \sigma_2(G) \leq |N_G(x_1)| + |N_G(x_2)| = |N_G(x_1) \cup N_G(x_2)| + |N_G(x_1) \cap N_G(x_2)| \leq n+k-3$, a contradiction. □

By Theorem 5.2.6 and Proposition 5.2.7, we can obtain the following corollary.

Corollary 5.2.8 *Let k be a positive integer, G be a graph of order n, and X and Y be subsets of $V(G)$ with $|X| = |Y| = k$. If $\sigma_2(G) \geq n+k-2$, then G contains k disjoint (X, Y)-paths.*

Lim et al. (2016) showed that the degree condition in Corollary 5.2.5 guarantees the existence of a partition into k (X, Y)-paths. (Gould and Whalen [115, Corollary 8] proved it for a graph G of order at least $3k$.)

Theorem 5.2.9 (Lim et al. [175]) *Let k be a positive integer, G be a graph of order $n \geq 2k$, and X and Y be disjoint subsets of $V(G)$ with $|X| = |Y| = k$. If $\sigma_2(G) \geq n+k$, then G can be partitioned into k disjoint (X, Y)-paths.*

Surprisingly, we can prove Theorem 5.2.9 by using the result on directed hamiltonian cycles in digraphs (Theorem 2.2.4).

Proof Let $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and $Z = V(G) - (X \cup Y)$. We construct a digraph D^* from a given graph G satisfying the conditions in Theorem 5.2.9 as follows (see Figure 3).

(1) Delete edges in $G[X]$ and $G[Y]$.
(2) Replace each edge incident to a vertex x of X with an arc whose tail is x.
(3) Replace each edge incident to a vertex y of Y with an arc whose head is y.
(4) Replace each edge joining vertices z_1 and z_2 of Z with two arcs z_1z_2 and z_2z_1.
(5) Delete an edge x_iy_i, if there exists, for $1 \leq i \leq k$.

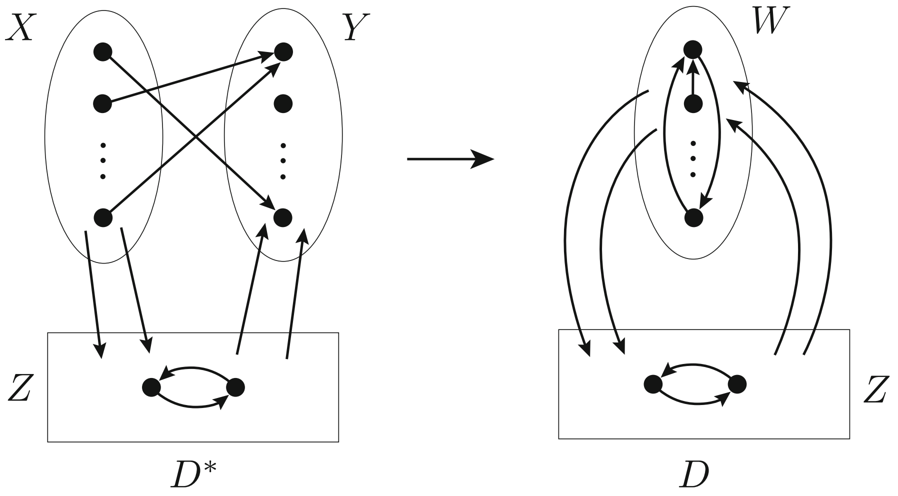

Fig. 3 The construction of a digraph D from D^*

We construct a digraph D from D^* by identifying x_i with y_i for $1 \le i \le k$.

Let w_i be the vertex obtained by identifying x_i with y_i for $1 \le i \le k$. Let $W = \{w_1, w_2, \ldots, w_k\}$. We now check the out-degree and in-degree of each vertex in D. For $z \in Z$,

$$d_D^+(z) = d_G(z) - |N_G(z) \cap X| \ge d_G(z) - k$$

and

$$d_D^-(z) = d_G(z) - |N_G(z) \cap Y| \ge d_G(z) - k.$$

For $w_i \in W$,

$$d_D^+(w_i) = d_G(x_i) - |N_G(x_i) \cap (X \cup \{y_i\})| \ge d_G(x_i) - k$$

and

$$d_D^-(w_i) = d_G(y_i) - |N_G(y_i) \cap (Y \cup \{x_i\})| \ge d_G(y_i) - k.$$

We next check the out-degree and in-degree sum condition in D. If $z_1z_2 \notin A(D)$ for $z_1, z_2 \in Z$, then $z_1z_2 \notin E(G)$, and hence it follows from the degree sum condition that

$$d_D^+(z_1) + d_D^-(z_2) \ge d_G(z_1) - k + d_G(z_2) - k \ge |G| + k - 2k = |D|.$$

If $zw_i \notin A(D)$ for $z \in Z$ and $w_i \in W$, then $zy_i \notin E(G)$, and hence

$$d_D^+(z) + d_D^-(w_i) \ge d_G(z) - k + d_G(y_i) - k \ge |G| + k - 2k = |D|.$$

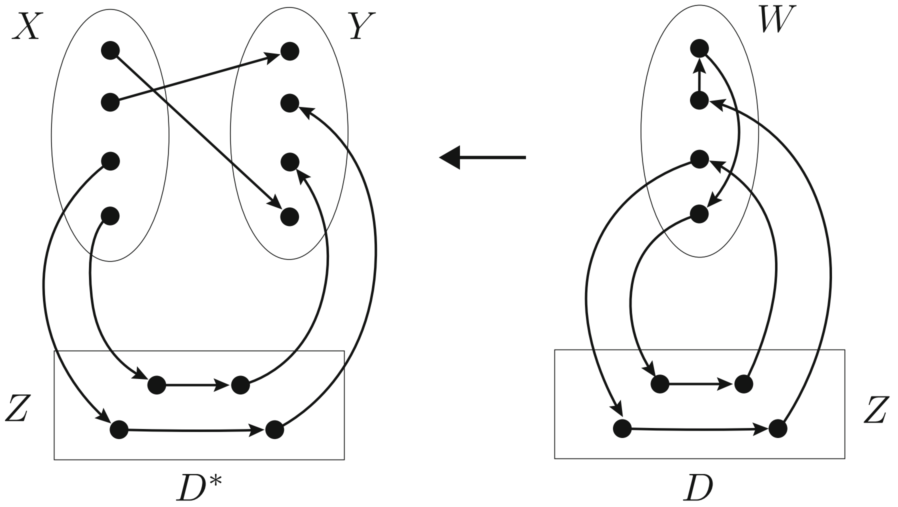

Fig. 4 The transform from a directed hamiltonian cycle in D to spanning k directed disjoint paths in D^*

If $w_i z \notin A(D)$ for $w_i \in W$ and $z \in Z$, then $x_i z \notin E(G)$, and so

$$d_D^+(w_i) + d_D^-(z) \geq d_G(x_i) - k + d_G(z) - k \geq |G| + k - 2k = |D|.$$

If $w_i w_j \notin A(D)$ for $w_i \in W$ and $w_j \in W$, then $x_i y_j \notin E(G)$, and

$$d_D^+(w_i) + d_D^-(w_j) \geq d_G(x_i) - k + d_G(y_j) - k \geq |G| + k - 2k = |D|.$$

Thus, D satisfies the degree sum condition of Theorem 2.2.4, and so D has a directed hamiltonian cycle. By putting w_i back to x_i and y_i, we can obtain spanning k disjoint directed paths from vertices of X to vertices of Y in D^* (see Fig. 4).

Furthermore, by putting the arcs of directed paths back to edges, we can obtain spanning k disjoint (X, Y)-paths in G. □

We finally mention Mader's result as a generalization of (X, Y)-paths. For a graph G and a collection $\mathcal{X}$ of disjoint subsets of $V(G)$, a path in G is an $\mathcal{X}$-*path* if it connects two different sets in $\mathcal{X}$ and has no internal vertex in any set in $\mathcal{X}$.

Theorem 5.2.10 (Mader [185]) *Let G be a graph, and let $\mathcal{X}$ be a collection of disjoint subsets of $V(G)$. The maximum number of disjoint $\mathcal{X}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \left\lfloor \frac{|B_i|}{2} \right\rfloor,$$

taken over all partitions $U_0, U_1, \ldots, U_n$ of $V(G)$ such that each $\mathcal{X}$-path disjoint from U_0 traverses some edge spanned by some U_i. Here B_i denotes the set of vertices in U_i that belong to $\bigcup_{X \in \mathcal{X}} X$ or have a neighbor in $V(G) \backslash (U_0 \cup U_i)$.

If $X = \{x_1, x_2, \cdots, x_k\}$ and $\mathcal{X} = \{\{x_1\}, \{x_2\}, \ldots, \{x_k\}\}$, then an $\mathcal{X}$-path is an X-path. If $\mathcal{X} = \{X, Y\}$, then an $\mathcal{X}$-path is an (X, Y)-path. Hence this is a common generalization of Gallai's Theorem (Theorem 5.2.1) and Menger's Theorem (Theorem 5.2.6).

Lovász [181] gave an alternative proof of Theorem 5.2.10 by deriving it from his matroid matching theorem, and Schrijver [213] also gave a short proof of it.

We propose the following problem on $\mathcal{X}$-paths.

Problem 5.2.11 *Let G be a graph, and let $\mathcal{X}$ be a collection of disjoint subsets of $V(G)$. Determine a sharp degree condition for the graph G to be partitioned into disjoint $\mathcal{X}$-paths.*

5.2.3 k-linked

A graph G of order at least $2k$ is *k-linked* if for any ordered subset of $2k$ distinct vertices $\{x_1, y_1, \ldots, x_k, y_k\}$, there exist k disjoint paths $P_1, \ldots, P_k$ such that x_i and y_i are end vertices of P_i for $1 \le i \le k$. In particular, if there exist such k disjoint paths that contain all vertices of G, we say that G is *fully k-linked*. Many researchers have investigated sufficient conditions for a graph to be k-linked. We refer the readers to [94] for other conditions than degree conditions.

Kawarabayashi et al. (2006) gave σ_2 conditions for a graph to be k-linked. (They also gave minimum degree conditions.)

Theorem 5.2.12 (Kawarabayashi et al. [147]) *Let k be an integer with $k \ge 2$, and let G be a graph of order $n \ge 2k$. If*

$$\sigma_2(G) \ge \begin{cases} 2n-3 & n \le 3k-1 \\ \left\lfloor \frac{2(n+5k)}{3} \right\rfloor - 3 & 3k \le n \le 4k-2 \\ n+2k-3 & n \ge 4k-1 \end{cases},$$

then G is k-linked.

In 2013, Li et al. [168] extended this theorem by giving a μ_2 condition for connected graphs G with $|G| \ge 232k$. In 2015, Dong and Li improved the order condition as follows. (It is unknown that the order condition in Theorem 5.2.13 is sharp.)

Theorem 5.2.13 (Dong and Li [67]) *Let k be a positive integer, and let G be a connected graph of order $n \ge 111k + 9$. If $\mu_2(G) \ge n + 2k - 3$, then G is k-linked.*

Thomas and Wollan (2005) gave an average degree condition for a graph with high connectivity to be k-linked. (The degree condition in Theorem 5.2.14 is not sharp. See also [217, Conjecture 5.1].)

Theorem 5.2.14 (Thomas and Wollan [217]) *Let k be a positive integer, and let G be a $2k$-connected graph. If $d(G) \ge 10k$, then G is k-linked (and hence, if G is $10k$-connected, then G is k-linked).*

The concept of k-linked concerns with packing and partition problems into k cycles in which each cycle contains an edge in a pre-specified k-matching. In fact, the result of Egawa et al. (Theorem 3.2.3) leads to the following. (For ordered subset of $2k$ vertices $\{x_1, y_1, \ldots, x_k, y_k\}$ in a graph G, join x_i and y_i if $x_i y_i \notin E(G)$, and then apply Theorem 3.2.3 as $M = \{x_1 y_1, \ldots, x_k y_k\}$.)

Corollary 5.2.15 (Egawa et al. [70]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k - 1$. If $\sigma_2(G) \geq n + 2k - 2$, then G is k-linked, in particular, each path has order at least three.*

Including the length constraint "each path has order at least three", the degree condition in this corollary is sharp.

On the other hand, Gould and Whalen (2006) gave a σ_2 condition for a graph to be fully k-linked. (In fact, they also consider the case $x_i = y_i$ for some integers i.)

Theorem 5.2.16 (Gould and Whalen [115, Corollary 9]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k$. If $\sigma_2(G) \geq n + 2k - 3$, then G is fully k-linked.*

Similar to Corollary 5.2.15, the result of Egawa et al. (Theorem 3.2.1) also leads to the following result on fully k-linked. (The degree condition in Corollary 5.2.17 is sharp, including the length constraint.)

Corollary 5.2.17 (Egawa et al. [70]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k - 1$. If $\sigma_2(G) \geq n + 2k - 2$, then G is fully k-linked, in particular, each path has order at least three.*

Magnant and Ozeki (2012) considered a stronger length constraint than the one of Corollary 5.2.17, and they conjectured the following.

Conjecture 5.2.18 (Magnant and Ozeki [187]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 5$ for $1 \leq i \leq k$. Further, let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be $2k$ vertices in G. If $\sigma_2(G) \geq n + 2k - 1$, then G can be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| = n_i$, and x_i, y_i are end vertices of P_i for $1 \leq i \leq k$.*

The degree condition is sharp in a sense if it is true (consider the graph G in the paragraph following Conjecture 3.3.19 and suppose that $x_1, \ldots, x_k, y_1, \ldots, y_k$ be $2k$ distinct vertices in some component of K^{2k}). Moreover, the condition $n_i \geq 5$ is necessary (consider the graph G in the second paragraph following Conjecture 3.3.19, and suppose that $x_1, \ldots, x_k, y_1, \ldots, y_k$ are $2k$ distinct vertices in some component of lK^{3k-2} and that $n_1 = \cdots = n_{k-1} = 4$ and $n_k = n - 4k + 4$).

Concerning Conjecture 5.2.18, Magnant and Ozeki gave the following σ_2 condition for a sufficiently large $(2k + 1)$-connected graph to be partitioned into k paths, with pre-specified end vertices, such that these k paths have approximately pre-specified lengths.

Theorem 5.2.19 (Magnant and Ozeki [187]) *Let k be an integer with $k \geq 2$. For any set of k positive real numbers $r_1, \ldots, r_k$ with $\sum_{i=1}^{k} r_i = 1$ and for any positive real number ε with $\varepsilon < \min\{\frac{1}{18^2 t^2}, \frac{r_1}{2}, \frac{r_2}{2}, \ldots, \frac{r_k}{2}\}$, there exists an integer n_0 such that, if G is a $(2k + 1)$-connected graph of order $n \geq n_0$ and $\sigma_2(G) \geq n + 2k - 2$, then for any $2k$ vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$, G can be partitioned into k paths $P_1, \ldots, P_k$ such that $(r_i - \varepsilon)n \leq |P_i| \leq (r_i + \varepsilon)n$, and x_i, y_i are end vertices of P_i for $1 \leq i \leq k$.*

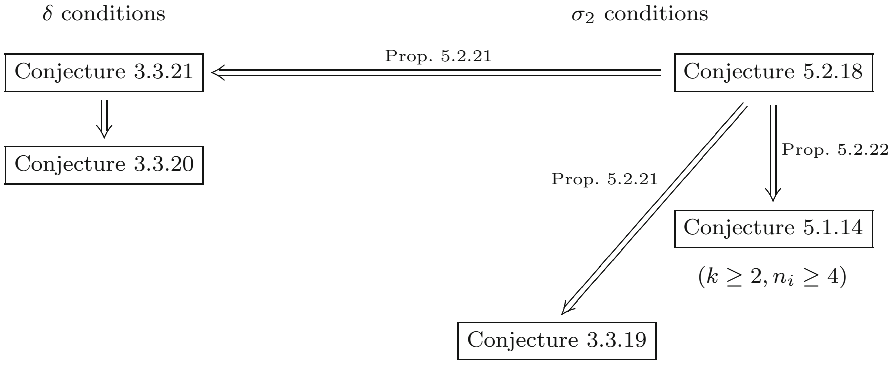

Fig. 5 The relations between conjectures

Since $\sigma_2(G) \geq n+2k-1$ implies that $\kappa(G) \geq 2k+1$, we can omit the connectivity condition in this theorem by replacing "$\sigma_2(G) \geq n+2k-2$" with $\sigma_2(G) \geq n+2k-1$.

In 2013, Faudree and Gould gave a minimum degree condition for packing k paths in which each path has a pre-specified length and end vertices. Moreover, they asked whether the condition that n is sufficiently large can be removed.

Theorem 5.2.20 (Faudree and Gould [91]) *Let k be an integer with $k \geq 2$ and $n_1, \ldots, n_k$ be integers with $n_i \geq 3$ for $1 \leq i \leq k$. Let G be a graph of order n with $\delta(G) \geq \frac{n+3k-1}{2}$ and $x_1, \ldots, x_k, y_1, \ldots, y_k$ be $2k$ vertices in G. Then there exists an integer $n_0 = n_0(k, n_1, n_2, \cdots, n_k)$ such that, if $n \geq n_0$, then G contains k disjoint paths $P_1, \ldots, P_k$ such that $|P_i| = n_i$, and x_i, y_i are end vertices of P_i for $1 \leq i \leq k$.*

One might wonder why the minimum degree condition is stronger than the one of Conjecture 5.2.18. This degree condition is sharp when $n_i \geq 3$.

In the last of this section, we will discuss relations between Magnant–Ozeki's conjecture (Conjecture 5.2.18), Enomoto–Ota's Conjecture (Conjecture 5.1.14) and Conjectures 3.3.19–3.3.21 including Wang's conjecture in Sect. 3.3.1. Figure 5 summarizes the relations between these conjectures.

We can easily obtain the following relation for Conjectures 5.2.18, 3.3.19 and 3.3.21, such as the relation between Theorem 3.2.1 and Corollary 5.2.17.

Proposition 5.2.21 *Conjecture 5.2.18 implies Conjectures 3.3.19 and 3.3.21.*

Proof Suppose that Conjecture 5.2.18 is true. Let G be a graph satisfying the conditions in Conjecture 3.3.21. For a given k-matching $\{e_1, \ldots, e_k\}$ in G, let $e_i = x_i y_i$ for $1 \leq i \leq k$, and apply Conjecture 5.2.18 (note that by Proposition 1(1), $\sigma_2(G) \geq 2\delta(G) \geq n + 2k - 1$). Then the partition in Conjecture 5.2.18 and the edges $e_1, \ldots, e_k$ form the partition in Conjecture 3.3.21. Thus, Conjecture 3.3.21 is also true. Since the degree condition in Conjecture 3.3.19 is the same as the one of Conjecture 5.2.18, and since the conclusion in Conjecture 3.3.19 is weaker than the one of Conjecture 3.3.21, we can also show that Conjecture 3.3.19 is true. □

On the other hand, we can obtain the following relation for Conjectures 5.2.18 and 5.1.14.

Proposition 5.2.22 *Conjecture 5.2.18 implies the case $k \geq 2$ and $n_i \geq 4$ $(1 \leq i \leq k)$ in Conjecture 5.1.14.*

Proof Suppose that Conjecture 5.2.18 is true, and we show that Conjecture 5.1.14 is also true for $k \geq 2$ and $n_i \geq 4$ $(1 \leq i \leq k)$. Let G be a graph satisfying the conditions in Conjecture 5.1.14, and let $x_1, \ldots, x_k$ be k vertices in G, where $k \geq 2$ and $n_i \geq 4$ $(1 \leq i \leq k)$. Let G' be the graph obtained from G by adding a complete graph K^k with vertices $y_1, \ldots, y_k$ and by joining each y_i to all vertices of G. Put $n'_i = n_i + 1$ for $1 \leq i \leq k$. The by the definition of G', we have

$$\sigma_2(G') = \sigma_2(G) + 2k \geq (n + k - 1) + 2k = |G'| + 2k - 1 \left(= \sum_{1 \leq i \leq k} n'_i + 2k - 1 \right).$$

Hence, we can apply Conjecture 5.2.18 for (G', n'_i, x_i, y_i), that is, G' can be partitioned into k paths $P_1, \ldots, P_k$ such that $|P_i| = n'_i$ $(= n_i + 1)$, and x_i, y_i are end vertices of P_i for $1 \leq i \leq k$. Then $P_1 - y_1, \ldots, P_k - y_k$ forms the desired partition in G. □

5.2.4 H-linked

Let $\mathcal{P}(G)$ be the set of paths in a graph G, and let H be a fixed multigraph (possibly containing loops). An *H-subdivision* in a graph G is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H) \to \mathcal{P}(G)$ such that (a) $f(u) \neq f(v)$ for $u, v \in V(H)$ with $u \neq v$, and (b) for every edge $uv \in E(H)$, $g(uv)$ is a path connecting $f(u)$ and $f(v)$, and distinct edges map into internally disjoint paths in G. An H-subdivision (f, g) is *spanning* if it further satisfies (c) $\bigcup_{e \in E(H)} V(g(e)) = V(G)$. A graph G is *H-linked* (resp., *fully H-linked*) if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision (resp., a spanning H-subdivision) in G. This concept is a generalization of k-connected and k-linked. In fact, G is k-connected if and only if it is $\left(K^2 \cup (k-1)K^1\right)$-linked, and G is k-linked if and only if it is $\left(kK^2\right)$-linked.

Kostochka and Yu (2005) gave the following minimum degree condition for a graph to be H-linked when H is a simple graph with $\delta(H) \geq 2$.

Theorem 5.2.23 (Kostochka and Yu [156]) *Let H be a simple graph with l edges and $\delta(H) \geq 2$, and let G be a graph of order $n \geq 5l + 6$. If $\delta(G) \geq \frac{n+l-2}{2}$, then G is H-linked.*

The degree condition is best possible for all bipartite graphs H (see [156]).

In order to introduce a sharp minimum degree condition for all multigraphs H, we prepare the following notation. Let H be a multigraph (possibly containing loops) with at least one non-loop edge. For two disjoint vertex subsets A and B of H, let $E_H(A, B)$ denote the set of edges of H between A and B, and we define

$$b(H) = \max_{\substack{A \cup B \cup C = V(H) \\ |E_H(A,B)| \geq 1}} \left\{ |E_H(A, B)| + |C| \right\},$$

where A, B and C are disjoint vertex subsets of H. Note that if H is connected, then $b(H)$ is the maximum number of edges in a spanning bipartite subgraph of H.

Gould et al. (2006) proved that the following minimum degree condition including the invariant $b(H)$ implies H-linkedness, and they also showed that the degree condition is sharp for all H (see also [96,159] for the sharpness of the lower bound). Here, let $u(H)$ be the number of components of a multigraph H which does not contain cycles of even length.

Theorem 5.2.24 (Gould et al. [114]) *Let H be a multigraph (possibly containing loops) with l edges including at least one non-loop edge, and let G be a graph of order $n \geq \frac{19}{2}(l + u(H) + 1)$. If $\delta(G) \geq \frac{n+b(H)-2}{2}$, then G is H-linked.*

In [114], they actually considered another invariant $b'(H)$ as follows: For a multigraph H of order at least two (possibly containing loops), let

$$b'(H) = \begin{cases} |H| - 1 & \text{if } H \text{ contains no cycles of even length,} \\ \max\limits_{A \cup B = V(H)} \{|E_H(A, B)|\} + u(H) & \text{otherwise,} \end{cases}$$

where A and B are disjoint non-empty vertex subsets of H. Note that, if H contains at least one non-loop edge, then $b(H) = b'(H)$. Therefore, $b'(H)$ is a generalization of $b(H)$.

Recall that if H is connected and $|H| \geq 2$, then $b(H)$ is the maximum number of edges in a spanning bipartite subgraph of H, that is, $b(H) \leq |E(H)|$ holds for a connected graph H. This implies that Theorem 5.2.24 is a generalization of Theorem 5.2.23. In addition, if $H = kK^2$, then we can check that $b(H) = 2k - 1$, and hence it follows that Theorem 5.2.24 is also a generalization of a minimum degree version of Theorem 5.2.12.

In 2008, Kostochka and Yu [159] improved the order condition for the case where H satisfies loopless, connected and $\delta(H) \geq 2$.

On the other hand, Kostochka and Yu (2008) gave a sharp σ_2 condition for H-linkedness when H is a simple graph with $\delta(H) \geq 2$. (In fact, they also gave it for $|E(H)| \leq |G| \leq (5|E(H)| - 11)/2$.)

Theorem 5.2.25 (Kostochka and Yu [158]) *Let H be a simple graph with l edges and $\delta(H) \geq 2$, and let G be a graph of order $n > \frac{5l-11}{2}$. If $\sigma_2(G) \geq n + \frac{3l-9}{2}$, then G is H-linked.*

Note that the lower bound of the σ_2 condition in this theorem is not twice the minimum degree given in Theorem 5.2.23. Ferrara et al. (2012) showed that if we add a mild minimum degree condition, then the σ_2 condition which is twice the minimum degree given in Theorem 5.2.24 implies H-linkedness. Here, $h_0(H)$ is the number of vertices of degree zero in a multigraph H.

Theorem 5.2.26 (Ferrara et al. [95]) *Let H be a loopless multigraph with l (≥ 1) edges and let G be a graph of order $n \geq 20l + h_0(H)$. If*

$$\delta(G) \geq 4l + h_0(H) \text{ and } \sigma_2(G) \geq n + b(H) - 2,$$

then G is H-linked.

In the same paper, they also obtained a sharp σ_2 condition with another invariant than $b(H)$, not adding any minimum degree condition. For a simple graph H with $|E(H)| \geq 1$, let

$$a(H) = \max_{\substack{A \cup B = V(H) \\ |E_H(A,B)| \geq 1}} \left\{ |E_H(A, B)| + |B| - \Delta_B(A)) \right\},$$

where A and B are disjoint vertex subsets of H, and $\Delta_B(A) = \max\{|N_G(v) \cap B| : v \in A\}$. It is known that $a(H) \geq b(H)$ for arbitrary graph H with $|E(H)| \geq 1$, and there are many graphs H for which $a(H) > b(H)$.

Theorem 5.2.27 (Ferrara et al. [95]) *Let H be a simple graph with l (≥ 1) edges, and let G be a graph of order $n > 20l$. If $\sigma_2(G) \geq n + a(H) - 2$, then G is H-linked.*

We next introduce results on fully H-linkedness. The following lemma elaborates some ideas of a result of Egawa et al. [70] (see [70, Theorem 1] and Theorem 3.2.4).

Lemma 5.2.28 (Kostochka and Yu [156]) *Let H be a simple graph with l edges and $\delta(H) \geq 2$, and let G be a graph of order n. Suppose that G is H-linked. If $\sigma_2(G) \geq n + l - 2$, then G is fully H-linked.*

The results preceding Lemma 5.2.28 corresponds to Step 1 mentioned in Section 3.1.2, i.e., to show the existence of an H-subdivision. On the other hand, Lemma 5.2.28 corresponds to Step 2, i.e., to show that the H-subdivision can be transformed into a spanning H-subdivision. Therefore, Theorem 5.2.23 and Lemma 5.2.28 together immediately imply the following.

Corollary 5.2.29 (Kostochka and Yu [156]) *Let H be a simple graph with l edges and $\delta(H) \geq 2$, and let G be a graph of order $n \geq 5l + 6$. If $\delta(G) \geq \frac{n+l-2}{2}$, then G is fully H-linked.*

We introduce another result related to Lemma 5.2.28. For a multigraph H (possibly containing loops), we say that a graph G is *H-extendable* if whenever there exists an H-subdivision which is not spanning, then there exists a spanning H-subdivision with the same set of vertices playing the role of $V(H)$ in G. In 2007, Gould and Whalen gave the following σ_2 condition for H-extendablity. Here, for a multigraph H, $\beta(H)$ is the maximum order of a matching of H, and $h_0(H)$ and $h_1(H)$ are the numbers of vertices of degree zero and one, respectively.

Theorem 5.2.30 (Gould and Whalen [116]) *Let H be a multigraph (possibly containing loops) with l edges, and let G be a $\left(\max\{\alpha(H), \beta(H)\} + 1\right)$-connected graph of order $n > 11l + 7(|H| - h_1(H))$. If*

$$\sigma_2(G) \geq n + l - |H| + 2h_0(H) + h_1(H),$$

then G is H-extendable.

This theorem implies many well-known results as corollaries, e.g., Theorem 2.1.2 for graphs with $n > 18$ and Theorem 3.1.17 for graphs with order $n > 18k$ and Theorem 5.2.9 for graphs with order $n > 11k$.

Since $h_0(H) = h_1(H) = 0$ for a simple graph H with $\delta(H) \geq 2$, Theorems 5.2.25 and 5.2.30 together immediately imply the following (note that if a graph G is H-linked and H-extendable, then G is fully H-linked).

Corollary 5.2.31 *Let H be a simple graph with l edges and $\delta(H) \geq 2$, and let G be a $\big(\max\{\alpha(H), \beta(H)\} + 1\big)$-connected graph of order $n > 11l + 7|H|$. If $\sigma_2(G) \geq n + \frac{3l-9}{2}$, then G is fully H-linked.*

Considering the above situations, we can consider the following problem.

Problem 5.2.32 *Let H be a simple graph (or a loopless multigraph) with l edges, and let G be a graph of sufficiently large order n. Is it true that, the degree conditions in Theorems 5.2.24, 5.2.26 and 5.2.27 imply that G is fully H-linked, respectively?*

We refer the readers to [94] for other results, including the above results.

6 Disjoint Cycles and Paths in Bipartite Graphs

6.1 Cycles in Bipartite Graphs

6.1.1 Packing Cycles

In 1996, Wang gave a bipartite version of Crrádi and Hajnal's Theorem (Theorem 3.1.2) as follows.

Theorem 6.1.1 (Wang [230]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$. If $\delta(G) \geq k + 1$, then G contains k disjoint cycles.*

Wang also conjectured that the same holds for the case $n = 2k$ and gave a weaker result than the conjecture (see also the paragraph following Conjecture 6.1.13).

Conjecture 6.1.2 (Wang [230]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = 2k$. If $\delta(G) \geq k + 1$, then G contains k disjoint cycles* (*i.e., G can be partitioned into k cycles of length* 4).

Li et al. (2004) improved Theorem 6.1.1 into a $\sigma_{1,1}$ condition.

Theorem 6.1.3 (Li et al. [173]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$. If $\sigma_{1,1}(G) \geq 2k + 2$, then G contains k disjoint cycles.*

We do not know whether this degree condition is sharp or not.

For unbalanced bipartite graphs, we can consider the same problem as above.

Problem 6.1.4 *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|B| > |A| \geq 2k$. Determine a sharp $\sigma_{1,1}(G)$ condition for the existence of k disjoint cycles in G.*

As a positive result for this problem, in 2009, Yan and Gao [250] showed that Theorem 6.1.3 holds for the case where $|A| \geq 2k + 1$ and $|B| - |A| = 1$.

6.1.2 Partitions into Cycles

In 2000, Chen et al. gave a $\delta_{1,1}$ condition for partitions of balanced bipartite graphs into k cycles.

Theorem 6.1.5 (Chen et al. [34]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq \max\{51, \frac{k^2}{2} + 1\}$. If $\delta_{1,1}(G) \geq n + 1$, then G can be partitioned into k cycles.*

As a related result, Li et al. (2001) gave the following σ_2 condition. (In 1999, Wang [234] proved a minimum degree version.)

Theorem 6.1.6 (Li et al. [172]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$. If $\sigma_2(G) \geq n + 2$, then G can be partitioned into k cycles.*

The degree condition in this theorem is sharp when $n = 2k+1$. Chiba and Yamashita (2017) showed that the $\sigma_{1,1}$ condition in Theorem 2.2.1 also guarantees the existence of a partition into k cycles. Note that this theorem is also a generalization of Theorems 6.1.5 and 6.1.6.

Theorem 6.1.7 (Chiba and Yamashita [48]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 12k + 2$. If $\sigma_{1,1}(G) \geq n + 1$, then G can be partitioned into k cycles.*

Considering the results in Sect. 3.1.4, we can expect that a weaker degree condition than the one of Theorem 6.1.7 is sufficient for partitions of balanced bipartite graphs into cycles and degenerate cycles.

Problem 6.1.8 *Let k be a positive integer, and let G be a balanced bipartite graph. Determine a sharp $\sigma_{1,1}(G)$ condition for partitions of G into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 or K_2 for $1 \leq i \leq k$.*

6.1.3 Disjoint Cycles Covers

In this section, we discuss bipartite versions of the results in Sect. 3.1.5, that is, we dicuss results concerning a vertex cover by disjoint cycles.

Kaneko and Yoshimoto (unpublished) gave a $\sigma_{1,1}$ condition for circumference of balanced bipartite graphs, which is a bipartite version of Theorem 3.1.36.

Theorem 6.1.9 (Kaneko and Yoshimoto [141]) *Let d be a positive integer, and let G be a 2-connected balanced bipartite graph of order $2n$. If $\sigma_{1,1}(G) \geq d + 1$, then $c(G) \geq \min\{2d, 2n\}$.*

We can consider a generalization of this theorem in terms of vertex cover by k disjoint cycles as in Sect. 3.1.5. In fact, Wang (2005) gave a minimum degree condition as follows, which corresponds to Theorem 3.1.37. (Note that $\sigma_{1,1}(G) \geq 2\delta(G)$.)

Theorem 6.1.10 (Wang [239]) *Let k and d be integers with $d \geq k \geq 2$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$. If $\delta(G) \geq d + 1$, then G contains k disjoint cycles covering at least $\min\{4d, 2n\}$ vertices of G, i.e., G contains k disjoint cycles $C_1, \ldots, C_k$ such that $\left|\bigcup_{1 \leq i \leq k} C_i\right| \geq \min\{4d, 2n\}$.*

Considering Theorems 6.1.3 and 6.1.9, we can consider the following problem.

Problem 6.1.11 *Let k and d be integers with $d \geq 2k + 1 \geq 5$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$. Is it true that, if $\sigma_{1,1}(G) \geq d + 1$, then G contains k disjoint cycles covering at least $\min\{2d, 2n\}$ vertices of G?*

On the other hand, Wang (2000) gave a minimum degree condition for the existence of disjoint cycles each of which has long length, which is a bipartite version of Theorem 3.1.38.

Theorem 6.1.12 (Wang [237]) *Let k and l be integers with $k \geq 1$ and $l \geq 2$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| \geq (l + 1)k$. If $\delta(G) \geq lk + 1$, then G contains k disjoint cycles of length at least $2(l + 1)$.*

We do not know whether this degree condition is sharp or not.

6.1.4 The El-Zahár-type Problem

Amar (1986) posed the following conjecture which is a bipartite version of the El-Zahár's conjecture (Conjecture 3.3.1), and proved the case $k = 2$.

Conjecture 6.1.13 (Amar [8]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n = \sum_{i=1}^{k} n_i$, where $n_i \geq 2$ for $1 \leq i \leq k$. If $\delta_{1,1}(G) \geq n + k$, then G can be partitioned into k cycles of lengths $2n_1, 2n_2, \ldots, 2n_k$.*

As a related result for the case $n_i = 2$ $(1 \leq i \leq k)$ in this conjecture, Wang [230] proved that every bipartite graph $G[A, B]$ with $|A| = |B| = 2k$ and $\delta(G) \geq k + 1$ can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i $(1 \leq i \leq k - 1)$ is a cycle and H_k is a path of order 4, see [230]. Later, Li et al. (2004) improved this into a $\sigma_{1,1}$ condition, see [173]. Zou et al. (2011) obtained a similar one to these results for the case $n_i = 3$ $(1 \leq i \leq k)$, see [261]. Another related result can be found in [169].

There are no known examples to show the sharpness of the lower bound on the degree condition in Conjecture 6.1.13 when $k \geq 3$. In fact, the following stronger conjecture was posed by Wang in 1999 (see also [58, Conjecture 1.10]).

Conjecture 6.1.14 (Wang [234]) *Let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2$ and $H[A', B']$ be a bipartite graph with $|A'| = |B'| \leq n$ and $\Delta(H) \leq 2$. If $\delta_{1,1}(G) \geq n + 2$, then G contains H as a subgraph.*

This conjecture is a bipartite version of Theorem 4.3.1. By applying Conjecture 6.1.14 with $H = C^{2n_1} \cup C^{2n_2} \cup \cdots \cup C^{2n_k}$, it follows that Conjecture 6.1.14 implies Conjecture 6.1.13 for $k \geq 2$.

Czygrinow et al. (2010) settled Conjecture 6.1.14 for sufficiently large graphs. Here, $w(H)$ denotes the number of components of a graph H.

Theorem 6.1.15 (Czygrinow et al. [58]) *For any positive integer k, there exists an integer $n_0 = n_0(k)$ depending on only k such that, if $G[A, B]$ is a bipartite graph with $|A| = |B| = n \geq n_0$ and $\delta_{1,1}(G) \geq n + 2$, and $H[A', B']$ is a bipartite graph with $|A'| = |B'| \leq n$, $\Delta(H) = 2$ and $w(H) = k$, then G contains H as a subgraph.*

Therefore, we can obtain a stronger result than Conjecture 6.1.13 for sufficiently large graphs as follows.

Corollary 6.1.16 (Czygrinow et al. [58]) *For any positive integer k, there exists an integer $n_0 = n_0(k)$ depending on only k such that, if $G[A, B]$ is a bipartite graph with $|A| = |B| = n = \sum_{i=1}^{k} n_i \geq n_0$, where $n_i \geq 2$ for $1 \leq i \leq k$, and $\delta_{1,1}(G) \geq n + 2$, then G can be partitioned into k cycles of lengths $2n_1, 2n_2, \ldots, 2n_k$.*

Unlike the situation for general graphs, this type of result, such as Theorem 6.1.15, directly leads to the solution of an El-Zahár-type problem in bipartite graphs (see Sect. 4.3 for the general graph case).

On the other hand, as a special case of Conjecture 6.1.13, Wang (2001) proved the case $n_i = 2$ $(1 \leq i \leq k-1)$, which corresponds to Theorem 3.3.8.

Theorem 6.1.17 (Wang [238]) *Let q be a positive integer, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2q + 3$. If $\delta(G) \geq \frac{n}{2} + 1$, then G can be partitioned into $q + 1$ cycles $C_1, C_2, \cdots, C_{q+1}$ such that $|C_i| = 4$ for $1 \leq i \leq q$.*

6.2 Cycles Passing Through Pre-specified Elements

6.2.1 Specified Edges

In this section, we focus on degree conditions for bipartite graphs to be partitioned into k cycles in which each cycle contains an edge in a pre-specified k-matching.

Chen et al. (2001) and Wang (1999), independently, gave the following $\sigma_{1,1}$ condition in Step 1 (Packing) and Step 2 (Partitioning) for the above mentioned problem. (Chen et al. actually gave δ and $\sigma_{1,1}$ conditions for $n \geq 2k$.)

Theorem 6.2.1 (Chen et al. [32], Wang [235]) *Let k be an integer with $k \geq 2$, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 3k$ and M be a k-matching in G. If $\sigma_{1,1}(G) \geq n + k$, then*

(1) *G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ and $|C_i| \leq 6$ for $1 \leq i \leq k$, and*
(2) *G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$.*

This theorem corresponds to Theorems 3.2.1 and 3.2.3.

Matsumura (2005) proved that a stronger degree condition than the one in this theorem guarantees the existence of a prescribed number of disjoint cycles of length 4. (He also gave a minimum degree condition.)

Theorem 6.2.2 (Matsumura [193]) *Let k and s be integers with $k \geq s \geq 1$, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k$ and M be a k-matching in G. If*

$$\sigma_{1,1}(G) \geq \max\left\{\left\lceil \frac{4n + 2s - 1}{3} \right\rceil, \left\lceil \frac{2n - 1}{3} \right\rceil + 2k\right\},$$

then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$, $|C_i| = 4$ for $1 \leq i \leq s$, and $|C_i| \leq 6$ for $s + 1 \leq i \leq k$.

It is known that the coefficient of n in this degree condition cannot be improved to 1 in the following sense. Let c be any constant, and let k and s be integers with $k \geq s \geq 1$, and let G be a graph of order $n \geq 3s+3c+2$ such that $V(G) = \bigcup_{i=1}^{8} W_i$, where $W_1, \ldots, W_8$ are pairwise disjoint, $|W_1| = |W_2| = s-1$, $|W_3| = |W_4| = k-s+1$, $|W_5| = |W_8| = (n-s-c)/2$, $|W_6| = |W_7| = (n-2k+s+c)/2$, and $E(G) = \bigcup_{i=1}^{7} E_{i,i+1} \cup E_{8,1} \cup E_{1,4} \cup E_{1,6} \cup E_{2,5} \cup E_{2,7} \cup E_{3,8}$, where $E_{i,j}$ is the set of all possible edges between W_i and W_j. Let M_1 be any perfect matching in $G[W_1 \cup W_2]$ and M_2 be any perfect matching in $G[W_3 \cup W_4]$. Then there is no cycle C in $G-(W_1 \cup W_2)$ such that $|E(C) \cap M_2| = 1$ and $|C| = 4$, and hence for $M = M_1 \cup M_2$, G does not satisfy the conclusion of Theorem 6.2.2. Moreover, for $w_5 \in W_5$ and $w_8 \in W_8$, we have $\sigma_{1,1}(G) = d_G(w_5) + d_G(w_8) = n+s+c$.

On the other hand, Yan and Liu (2006) showed that the $\sigma_{1,1}$ condition for the case $s = k$ in Theorem 6.2.2 guarantees the existence of a partition into k cycles of length 4, except at most one cycle. Note that $\frac{4n+2k-1}{3} \geq \frac{2n-1}{3} + 2k$ for $n \geq 2k$. (Another related result to this theorem can be found in [253].)

Theorem 6.2.3 (Yan and Liu [251]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k+1$ and M be a k-matching in G. If $\sigma_{1,1}(G) \geq \left\lceil \frac{4n+2k-1}{3} \right\rceil$, then G can be partitioned into k cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|E(C_i) \cap M| = 1$ for $1 \leq i \leq k$ and $|C_i| = 4$ for $1 \leq i \leq k-1$.*

6.2.2 Specified Perfect Matchings

As mentioned in Sect. 2.2 (see Remark 2.2.6), cycles passing through every edge of a pre-specified perfect matching in bipartite graphs correspond to cycles in digraphs.

In 1999, Chen et al. gave a minimum degree condition for partitions into k cycles passing through every edge of a pre-specified perfect matching. Here, for a graph with a matching M, a cycle in the graph is called an *M-alternating cycle* if the edges belong to M and not to M, alternately.

Theorem 6.2.4 (Chen et al. [36]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 9k$ and M be a perfect matching in G. If $\delta(G) \geq \frac{n+2}{2}$, then G can be partitioned into k M-alternating cycles.*

The condition $n \geq 9k$ comes from the proof techniques and they gave an example showing that $n \geq 3k+1$ is necessary. Recently, Chiba and Yamashita improved the degree condition and the conclusion of this theorem as follows.

Theorem 6.2.5 (Chiba and Yamashita [49]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 12k+3$ and M be a perfect matching in G. If $\sigma_{1,1}(G) \geq n+2$, then G can be partitioned into k M-alternating cycles of length at least* 6.

The order condition comes from the proof techniques. It is known that there exists an example showing that $n \geq 4k-1$ is necessary.

Theorem 6.2.5 implies the theorem of Las Vergnas (Theorem 2.2.5) for graphs of large order. By Remark 2.2.6, we also see that Theorem 6.2.5 is equivalent to the following theorem.

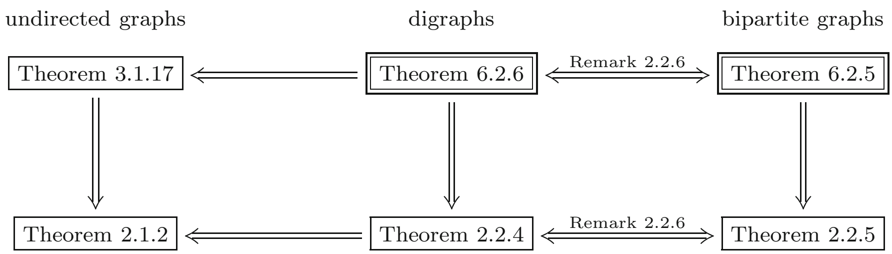

Fig. 6 The relations between theorems

Theorem 6.2.6 *Let k be a positive integer, and let D be a digraph of order $n \geq 12k+3$. If $\sigma_{1^+,1^-}(D) \geq n$, then D can be partitioned into k directed cycles of length at least* 3.

Thus, Theorem 6.2.5 also implies the theorem of Woodall (Theorem 2.2.4) for graphs of large order. Moreover, similar to the relation between Theorems 2.1.2 and 2.2.4, Theorem 6.2.5 implies the theorem of Brandt et al. (Theorem 3.1.17) for graphs of large order. Figure 6 summarizes the relations between theorems.

The problem on packing in digraphs seems to be difficult. For example, Bermond and Thomassen (1981) conjectured the following for packing k directed cycles. Here, for a digraph D, we define $\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}$.

Conjecture 6.2.7 (Bermond and Thomassen [19]) *Let k be a positive integer, and let D be a digraph of order at least $2k$. If $\delta^+(D) \geq 2k-1$, then D contain k disjoint directed cycles.*

The case $k = 1$ of this conjecture is an easy problem, and the cases $k = 2$ and $k = 3$ are proved in [219] and [174], respectively. Alon [5] proved that the conclusion holds if every vertex has out-degree at least $64k$, but the conjecture is still open in general.

6.2.3 Specified Vertices

In this section, we consider a vertex version of Sect. 6.2.1.

We first consider $\sigma_{1,1}$ conditions for balanced bipartite graphs to be partitioned into k cycles in which each cycle contains a vertex in pre-specified k vertices. As a corollary of Theorems 2.2.1 and 6.2.1, we can obtain the following corollary (note that, for any set S of k vertices in a balanced bipartite graph G with $\sigma_{1,1}(G) \geq |G|/2 + k$, we can take a k-matching M in G such that each edge of M contains a vertex of S, refer the proof of Proposition 3.2.12).

Corollary 6.2.8 (Chen et al. [32], Moon and Moser [197], Wang [235]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 3k$, and S be set of k vertices in G. If $\sigma_{1,1}(G) \geq n + k$, then*

(1) *G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \leq 6$ for $1 \leq i \leq k$, and*
(2) *G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$.*

It is known that the degree condition in this corollary is best possible. Therefore, in a sense, there is no difference between "specified k-matchings" and "specified k vertices" in terms of $\sigma_{1,1}$ conditions as in the case of general graphs (see Theorem 3.2.1 and Corollary 3.2.13).

Gao et al. (2009) considered a $\sigma_{1,1}$ condition for packing k cycles of length 4 in which each cycle contains a vertex in pre-specified k vertices. The following result corresponds to Theorem 6.2.2. (It is unknown that the degree condition is sharp for $k \geq 2$ or $n \geq 5$.)

Theorem 6.2.9 (Gao et al. [109]) *Let k and s be integers with $k \geq s \geq 1$, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 3k - s$, and S be a set of k vertices in G. If $\sigma_{1,1}(G) \geq \lceil \frac{4n+s}{3} \rceil$, then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$, $|C_i| = 4$ for $1 \leq i \leq s$, and $|C_i| \leq 6$ for $s+1 \leq i \leq k$.*

In [109], they also considered partitions into $k + 1$ cycles $C_1, \ldots, C_{k+1}$ such that each cycle C_i ($1 \leq i \leq k$) has length four and contains a pre-specified k vertices. (Before this result, in 2007, Yan and Liu proved the case $s = k$, see [252].)

On the other hand, for the case $s = k - 1$ in Theorem 6.2.9, if we do not specify the length of the cycle C_k, then the degree condition can be replaced with the one of Corollary 6.2.8.

Theorem 6.2.10 (Zhang et al. [257]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k+1$, and S be a set of k vertices in G. If $\sigma_{1,1}(G) \geq n+k$, then G contains k disjoint cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$ and $|C_i| = 4$ for $1 \leq i \leq k - 1$.*

In [257], they also showed the existence of the following partition.

Theorem 6.2.11 (Zhang et al. [257]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k+1$, and S be a set of k vertices in G. If $\sigma_{1,1}(G) \geq n+k$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that $|V(H_i) \cap S| = 1$ for $1 \leq i \leq k$, H_i is a cycle with $|H_i| = 4$ for $1 \leq i \leq k - 1$ and H_k is a path.*

Considering Corollary 6.2.8 and Theorem 6.2.10, we conjecture that "H_k is a path" in Theorem 6.2.11 can be replaced with "H_k is also a cycle".

Conjecture 6.2.12 *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 2k + 1$, and S be a set of k vertices in G. If $\sigma_{1,1}(G) \geq n + k$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$ and $|C_i| = 4$ for $1 \leq i \leq k - 1$.*

We next consider minimum degree conditions for the existence of such disjoint cycles. Chen et al. (2004) gave a bipartite version of Theorem 3.2.23 as follows.

Theorem 6.2.13 (Chen et al. [33]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 4k - 2$, and S be a set of k vertices in G. If $\delta(G) \geq (n + 1)/2$, then*

(1) *G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ and $|C_i| \leq 6$ for $1 \leq i \leq k$, and*

(2) *either (i) G can be partitioned into k cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \le i \le k$, or (ii) $k = 2$ and $G - S \simeq 2K^{(n-1)/2,(n-1)/2}$.*

The condition $n \ge 4k - 2$ is sharp, and we cannot replace $\delta(G) \ge (n+1)/2$ with $\delta_{1,1}(G) \ge n + 1$.

To prove (2) of this theorem, they further applied the following result as another Step 1 (Packing), that is, they actually prepared two results (Theorem 6.2.13 (1) and Theorem 6.2.14) in order to get Theorem 6.2.13 (2). Here, for a graph G and a set S of k vertices in G, a set of k disjoint cycles $\{C_1, \dots, C_k\}$ is *minimal system* (*with respect to* S) if $|V(C_i) \cap S| = 1$ for $1 \le i \le k$ and G does not contain another set of k disjoint cycles $\{D_1, \dots, D_k\}$ such that $|V(D_i) \cap S| = 1$ for $1 \le i \le k$ and $\sum_{i=1}^{k} |D_i| < \sum_{i=1}^{k} |C_i|$.

Theorem 6.2.14 (Chen et al. [33]) *Let k be a positive integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \ge 4k - 2$, and S be a set of k vertices in G. Suppose that G contains k disjoint cycles $D_1, \dots, D_k$ such that $|V(D_i) \cap S| = 1$ for $1 \le i \le k$. If $\delta(G) \ge (n+1)/2$, then G contains a minimal system $\{C_1, \dots, C_k\}$ such that $G - \bigcup_{i=1}^{k} V(C_i)$ contains a perfect matching.*

We finally consider packing k cycles in which each cycle contains at least two vertices in pre-specified vertices. Jiang and Yan (2017) gave the following bipartite analogy of Theorem 3.2.27.

Theorem 6.2.15 (Jiang and Yan [132]) *Let k be a positive integer and $G[A, B]$ be a bipartite graph with $|A| = |B| = n$. Let $S \subseteq A$ with $|S| \ge 2k$. If $d_G(x) + d_G(y) \ge n + k$ for every $x \in S$ and $y \in B$ with $xy \notin E(G)$, then G contains k disjoint cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| \ge 2$ for $1 \le i \le k$.*

In the same paper, they remarked that this degree condition is sharp when $k = 1$, and the degree condition may be not sharp for $k \ge 2$. Then they proposed the following problem.

Problem 6.2.16 (Jiang and Yan [132]) *Determine a sharp degree condition to guarantee that G contains k disjoint cycles such that each of them contains at least two vertices of S.*

On the other hand, in 2009, Amar et al. showed a result on cyclability of balanced bipartite graphs.

Theorem 6.2.17 (Amar et al. [9]) *Let $G[A, B]$ be a 2-connected balanced bipartite graph of order $2n$, and let $S \subseteq V(G)$. If $d_G(x) + d_G(y) \ge n + 1$ for every $x \in A \cap S$ and $y \in B \setminus S$ with $xy \notin E(G)$ and $d_G(x) + d_G(y) \ge n + 1$ for every $x \in A \setminus S$ and $y \in B \cap S$ with $xy \notin E(G)$, then G contains a cycle passing through all vertices in S.*

By considering this theorem, we can also propose the following problem in order to improve Theorem 6.2.15.

Problem 6.2.18 *Can we replace the condition "$S \subseteq A$" into "$S \subseteq V(G)$" in Theorem 6.2.15? In particular, does the following hold? Let k be a positive integer and $G[A, B]$ be a bipartite graph with $|A| = |B| = n$. Let $S \subseteq V(G)$ with $|S| \ge 2k$. If $d_G(x) + d_G(y) \ge n + k$ for every $x \in A \cap S$ and $y \in B \setminus S$ with $xy \notin E(G)$ and $d_G(x) + d_G(y) \ge n + k$ for every $x \in A \setminus S$ and $y \in B \cap S$ with $xy \notin E(G)$, then G contains k disjoint cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| \ge 2$ for $1 \le i \le k$.*

6.2.4 Specified Paths

Matsubara and Matsumura (2005) considered a $\sigma_{1,1}$ condition for disjoint cycles in which each cycle contains a path in a pre-specified disjoint paths of order 3.

Theorem 6.2.19 (Matsubara and Matsumura [189]) *Let k be an integer, $G[A, B]$ be a bipartite graph with $|A| = |B| = n \geq 3k$, and $P_1, \ldots, P_k$ be k disjoint paths of order 3 in G.*

(1) *If $k \geq 2$ and $\sigma_{1,1}(G) \geq n + 2k - 1$, then G contains k disjoint cycles $C_1, \ldots, C_k$ such that C_i contains P_i as a subpath and $|C_i| \leq 6$ for $1 \leq i \leq k$.*
(2) *If $k \geq 3$ and $\sigma_{1,1}(G) \geq n + 2k - 1$, then G can be partitioned into k cycles $C_1, \ldots, C_k$ such that C_i contains P_i as a subpath for $1 \leq i \leq k$.*

6.3 Chorded Cycles

As mentioned in Sect. 3.4.3, the study on packing chorded cycles is related to the one on packing complete subgraphs. In 1998, Wang conjectured the following for partitions into k balanced complete bipartite subgraphs $K^{c,c}$, which is a bipartite version of Theorem 3.4.11.

Conjecture 6.3.1 (Wang [232]) *Let k and c be integers with $k \geq 1$ and $c \geq 2$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n = ck$. If $\delta(G) \geq \frac{c-1}{c}n + 1$, then G can be partitioned into k subgraphs isomorphic to $K^{c,c}$.*

The cases $1 \leq k \leq 3$ and $k = 4$ are proved in [232] and [236], respectively. On the other hand, for the case $c = 2$, this conjecture is stronger than a minimum degree version of Amar's Conjecture (Conjecture 6.1.13) and Wang (1996) gave a partial answer to this case in [230] (see also the paragraph following Conjecture 6.1.13). For the case $c = 3$, Wang (1999) also gave the following partial answer (note that a hamiltonian cycle in $K^{3,3}$ is a 3-chorded cycle of length 6).

Theorem 6.3.2 (Wang [232]) *Let k be an integer with $k \geq 1$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n = 3k$. If $\delta(G) \geq \frac{2}{3}n+1$, then G can be partitioned into k 2-chorded cycles of length 6.*

For the case $c = 4$, Zou et al. (2013) gave the following partial answer (note that a hamiltonian cycle in $K^{4,4}$ is an 8-chorded cycle of length 8).

Theorem 6.3.3 (Zou et al. [260]) *Let k be an integer with $k \geq 1$, and let $G[A, B]$ be a bipartite graph with $|A| = |B| = n = 4k$. If $\delta(G) \geq \frac{3}{4}n + 1$, then G can be partitioned into k 2-chorded cycles of length 8.*

We do not know whether the degree conditions in Conjecture 6.3.1, Theorems 6.3.2 and 6.3.3 are sharp or not.

For a general integer $c \geq 2$, Zhao (2009) proved a stronger result than Conjecture 6.3.1 for sufficiently large balanced bipartite graphs by using the regularity lemma. (The degree conditions in Theorem 6.3.4 are sharp.)

Theorem 6.3.4 (Zhao [259]) *For an integer c with $c \geq 2$, there exists an integer k_0 such that, if k is an integer with $k \geq k_0$ and $G[A, B]$ is a bipartite graph with $|A| = |B| = n = ck$ such that*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + c - 1 & k \text{ is even,} \\ \frac{n+3c}{2} - 2 & k \text{ is odd,} \end{cases}$$

then G can be partitioned into k subgraphs isomorphic to $K^{c,c}$.

In [259], Zhao also asked about minimum degree conditions for a sufficiently large balanced bipartite graph to be partitioned into any fixed bipartite graph H and suggested using the critical chromatic number of H as the result due to Kühn and Osthus [164] in Sect. 4.5. Bush and Zhao answered this problem affirmatively (see [30, Theorem 1.4]).

Martin and Skokan considered a multipartite version of Theorem 3.4.11 and Conjecture 6.3.1, and they gave a degree condition for a sufficiently large balanced multipartite graph with l partite sets to be partitioned into balanced complete multipartite graph with l partite sets (see [188, Theorem 4]).

6.4 Paths in Bipartite Graphs

6.4.1 Partitions into Paths

In 2006, Li and Steiner characterized bipartite graphs with high degree sum that cannot be partitioned into k paths. The following result is a bipartite version of Theorem 5.1.3 (note that the condition $\big||A| - |B|\big| \leq k$ is a necessary condition for partitions into k paths).

Theorem 6.4.1 (Li and Steiner [171]) *Let k be a positive integer, and let $G[A, B]$ be a bipartite graph of order $n \geq k$ such that $\big||A| - |B|\big| \leq k$. If $2\sigma_{1,1}(G) \geq n - k + 1$, then one of the following holds:*

(i) *G can be partitioned into k paths,*
(ii) *$k = 1$, $|A| = |B|$ and $G \in \{K^{s,s} \cup K^{\frac{n}{2}-s, \frac{n}{2}-s} : 1 \leq s \leq \frac{n}{2} - 1\}$.*

6.4.2 The El-Zahár-type Problem

By applying Theorem 6.4.1 with $k = 1$, it follows that every connected balanced bipartite graph G with $2\sigma_{1,1}(G) \geq |G|$ contains a hamiltonian path (note that the graph in Theorem 6.4.1 (ii) is disconnected). Therefore, we can obtain the following path version of Conjecture 6.1.13. (We can obtain this corollary also from Theorem 2.2.2.)

Corollary 6.4.2 (Li and Steiner [171]) *Let k be a positive integer, and let G be a connected balanced bipartite graph of order $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 1$ for $1 \leq i \leq k$. If $2\sigma_{1,1}(G) \geq n$, then G can be partitioned into k paths of orders $n_1, n_2, \ldots, n_k$.*

The degree condition is best possible in the following sense. Let G be a graph obtained from two disjoint complete bipartite graphs $H_1 \simeq K^{l+1,l}$ and $H_2 \simeq K^{l,l+1}$ by joining each vertex in the partite set of size l in H_1 to all vertices in the partite set of size l in H_2, and let $n = 4l + 2$. Then it is easy to see that G is a connected balanced bipartite graph of order n with $2\sigma_{1,1}(G) = n - 2$. Moreover, for any k positive even integers $n_1, \ldots, n_k$ such that $\sum_{i=1}^{k} n_i = n$, we can check that G cannot be partitioned into k paths of orders $n_1, n_2, \ldots, n_k$.

For unbalanced bipartite graphs, Li and Steiner (2006) gave the following result on partitions into two paths with a pre-specified length. (Note that Theorem 6.4.3 also implies Corollary 6.4.2 for the case $k = 2$.)

Theorem 6.4.3 (Li and Steiner [171]) *Let $G[A, B]$ be a bipartite graph of order $n = n_1 + n_2$ such that $\big||A| - |B|\big| \le 2$, where $n_1 \ge 1$ and $n_2 \ge 1$. If $2\sigma_{1,1}(G) \ge n - 1$, then one of the following holds:*

(i) *G can be partitioned into two paths of order n_1 and n_2, respectively,*
(ii) *G is a graph obtained from $K^{1,3}$ by replacing every edge in it by a path of length 2,*
(iii) *$G \in \{K^{s,s} \cup K^{\lfloor \frac{n}{2} \rfloor - s, \lceil \frac{n}{2} \rceil - s} : 1 \le s \le \lfloor \frac{n}{2} \rfloor - 1,\ 2s \ne n_1,\ 2s \ne n_2\}$,*
(iv) *$G \in \{K^{s,s+1} \cup K^{\lfloor \frac{n-1}{2} \rfloor - s, \lceil \frac{n-1}{2} \rceil - s} : 1 \le s \le \lfloor \frac{n-1}{2} \rfloor - 1,\ 2s+1 \ne n_1,\ 2s+1 \ne n_2\}$,*
(v) *$G \subseteq K^{\frac{n-2}{2}, \frac{n+2}{2}}$ and n_i is even for $i \in \{1, 2\}$,*
(vi) *G is a graph obtained from $K^{m,m+1} \cup K^{m',m'}$ by adding at least one edge between the part of m vertices and the opposing part of m' vertices.*

6.4.3 X-paths

In 2017, Matsubara et al. proved the following theorem.

Theorem 6.4.4 (Matsubara et al. [191]) *Let k be a positive integer and $G[A, B]$ be a bipartite graph of order $n \ge 2k$ such that $|B| \ge |A|$. Let X be a set of $2k$ vertices in G such that $|A \setminus X| - |B \setminus X| = |B| - |A|$. If*

$$2\sigma_{1,1}(G) \ge \begin{cases} n + 4\ (= n + k + 3) & |X \cap A| = |X \cap B| = 1, \\ n + k + 2 & |X \cap A| = 0,\ \textit{or}\ |X \cap A| = |X \cap B| = 2\ \textit{and}\ n = 10, \\ n + k & \textit{otherwise}, \end{cases}$$

then G can be partitioned into k X-paths.

The degree condition is best possible for each case. The condition $|A \setminus X| - |B \setminus X| = |B| - |A|$ is a necessary condition for the existence of the partition in this theorem.

As mentioned in Sect. 5.2.1, the results on degree conditions for the existence of a hamiltonian cycle passing through every component of a pre-specified linear forest, are useful tools to get degree condition for partitions into X-paths for a pre-specified vertex set X. In fact, Matsubara et al. [191] pointed out that a weaker version of Theorem 6.4.4 can be obtained by using Theorem 2.2.3 (see [191, Theorem 4]).

6.4.4 (X, Y)-paths

Gould and Whalen considered a $\sigma_{1,1}$ condition for bipartite graphs to be k-connected for $k \geq 2$. (In fact, they also considered the case $k = 1$.)

Theorem 6.4.5 (Gould and Whalen, preprint [117]) *Let k be an integer with $k \geq 2$, and let $G[A, B]$ be a bipartite graph of order n such that $|B| \geq |A| \geq k$. If $\sigma_{1,1}(G) \geq |B| + k - 1$, then G is k-connected.*

As mentioned in Sect. 5.2.2, by combining this with Menger's Theorem (Theorem 5.2.6), we can obtain the following corollary.

Corollary 6.4.6 *Let k be an integer with $k \geq 2$ and $G[A, B]$ be a bipartite graph of order n such that $|B| \geq |A| \geq k$. Let X and Y be subsets of $V(G)$ with $|X| = |Y| = k$. If $\sigma_{1,1}(G) \geq |B| + k - 1$, then G contains k disjoint (X, Y)-paths.*

6.4.5 k-linked

Gould and Whalen gave the following $\sigma_{1,1}$ condition for bipartite graphs to be k-linked for $k \geq 2$, which is a bipartite version Theorem 5.2.12 and Corollary 5.2.15.

Theorem 6.4.7 (Gould and Whalen, preprint [117]) *Let k be an integer with $k \geq 2$, and let $G[A, B]$ be a bipartite graph of order n such that $|B| \geq |A| \geq 3k$ and $\delta(G) \geq 2k - 1$. If $2\sigma_{1,1}(G) \geq n + 4k - 4$, then G is k-linked.*

On the other hand, considering a bipartite version of the results on fully k-linked (Theorem 5.2.16 and Corollary 5.2.17, is more difficult since it depends on the respective cardinalities of the partite sets A and B of a bipartite graph G, and the pre-specified $2k$ distinct vertices. For example, there is no hamiltonian path between a vertex of A and a vertex of B if $|A| \neq |B|$. Considering this situation, in [117], Gould and Whalen also considered the following concept "(k, k_0)-extendible": Let k be a positive integer, and let G be a graph. We define $\mathcal{W}_k(G)$ be the family of all sets $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ of k pairs of vertices of G, where $x_1, \ldots, x_k, y_1, \ldots, y_k$ are all distinct. For $W = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\} \in \mathcal{W}_k(G)$, a set of k disjoint paths $P_1, \ldots, P_k$ such that x_i and y_i are end vertices of P_i for $1 \leq i \leq k$, is called a *W-linkage*; thus, a graph G is k-linked if and only if there exists a W-linkage for every $W \in \mathcal{W}_k(G)$. Let now $G[A, B]$ be a bipartite graph. For $W \in \mathcal{W}_k(G)$, we denote by W^A (resp., W^B) the set of pairs of W whose two vertices are in A (resp., in B). For $W \in \mathcal{W}_k(G)$, a W-linkage $\mathcal{P}$ *veneers* the graph G if $A \subseteq \bigcup_{P \in \mathcal{P}} V(P)$ or $B \subseteq \bigcup_{P \in \mathcal{P}} V(P)$. Note that a W-linkage $\mathcal{P}$ veneering G is spanning (i.e., $\bigcup_{P \in \mathcal{P}} V(P) = V(G)$) if and only if $|A| - |B| - (|W^A| - |W^B|) = 0$. A bipartite graph $G[A, B]$ is said to be *(k, k_0)-extendible* if for any $W \in \mathcal{W}_k(G)$ with $|W^A| + |W^B| = k_0$, whenever there exists a W-linkage, there exists a W-linkage veneering the graph G.

Theorem 6.4.8 (Gould and Whalen, preprint [117]) *Let k and k_0 be positive integers with $k \geq k_0$, and let $G[A, B]$ be a bipartite graph of order n such that $|B| \geq |A| \geq$*

$2k + k_0$ and $\delta(G) \geq k$. If

$$2\sigma_{1,1}(G) \geq \begin{cases} n + k_0 + 4 & k = 1, \\ n + k_0 + 2k & k \geq 2, \end{cases}$$

then G is (k, k_0)-extendible.

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