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On α-Domination in Graphs

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Abstract Let $G = (V, E)$ be an isolate-free graph. For some α with $0 < \alpha \leq 1$, a subset S of V is said to be an *α-dominating set* if for all $v \in V \backslash S, |N(v) \cap S| \geq \alpha|N(v)|$. The size of a smallest such S is called the *α-domination number* and is denoted by $\gamma_\alpha(G)$. A set $S \subseteq V$ is said to be an *α-rate dominating set* of G if for any vertex $v \in V$, $|N[v] \cap X| \geq \alpha|N(v)|$. The minimum cardinality of an α-rate dominating set of G is called the *α-rate domination number $\gamma_{\times\alpha}(G)$*. The set of distinct values of $\gamma_\alpha(G)$ as α runs over $(0, 1]$ is called the α-domination spectrum of a graph G, i.e., $\mathsf{Sp}_\alpha(G) = \{\gamma_\alpha(G) : \alpha \in (0, 1]\}$. In this paper, we study some properties of $\mathsf{Sp}_\alpha(G)$ and show that $\gamma_\alpha(G)$ changes its value only at rational points as α runs over $(0, 1]$. Using this result, we characterize some values of α such that $\gamma_\alpha(G) \leq n\alpha$, where n is the number of vertices in G, holds. Finally, we present some improved probabilistic upper bounds of α-domination number and α-rate domination number of a graph G.

Keywords α-Domination spectrum · Probabilistic methods

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1 Introduction

The domination in graphs has been an active area of research from the time of its inception. Two domination books [8,9] provide a comprehensive report of vastness of the area of the domination and its relation to other graph parameters. Many variations of the domination problem can be found in literature most of which are motivated by many real-life scenarios. One such variation is α-domination in graphs, introduced by Dunbar et al. [6]. The concept of α-domination also models some problems in the spread of diseases or influence in social or virtual networks (see [5,10]). Subsequent works on α-domination can be found in [3,4,7,11–13]. In the current paper, we introduce the notion of α-domination spectrum of a graph G and using it, we show that $\gamma_\alpha(G)$, the α-domination number changes its value only at rational points as α runs over $(0, 1]$ and demonstrate some extremal graphs with respect to critical values.

In [4], the authors proved that for all $\alpha \in (0, 1)$ and for all $\epsilon > 0$, $\gamma_\alpha(G) \leq n\alpha + \epsilon$, where n is the number of vertices in G, holds for all graphs G with sufficiently large minimum degree. On the other hand, authors in [13] characterized the values of α for a particular class of graphs such that $\gamma_\alpha(G) \leq n\alpha$ holds. We show that for certain values of α, a tighter upper bound $\gamma_\alpha(G) \leq n\alpha$ holds. It is to be noted here, in contrast with [13], that our results hold good for all graphs.

Finally, we present some improved probabilistic upper bounds of $\gamma_\alpha(G)$ and $\gamma_{\times\alpha}(G)$. Throughout the paper, we assume that G is an isolate-free graph. For standard graph-theoretic terminologies, please refer to [16], and for the probabilistic methods terminologies, please refer to [1].

1.1 Preliminaries

Let $G = (V, E)$ be an isolate-free simple undirected graph.[1] For any vertex $v \in V$, $N(v)$ denotes the set of all vertices adjacent to v, $N[v] = N(v) \cup \{v\}$ and $\deg(v) = |N(v)|$. For some α with $0 < \alpha \leq 1$, a subset S of V is said to be an α-*dominating set* of G if for all $v \in V \backslash S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The size of a smallest such set S is called the α-*domination number* and is denoted by $\gamma_\alpha(G)$. An α-dominating set of size $\gamma_\alpha(G)$ is called a $\gamma_\alpha(G)$-set or γ_α-set. A set $S \subseteq V$ is said to be an α-*rate dominating set* of G if for any vertex $v \in V$, $|N[v] \cap X| \geq \alpha|N(v)|$. The minimum cardinality of an α-rate dominating set of G is called the α-*rate domination number* and is denoted by $\gamma_{\times\alpha}(G)$. Assume that $V = \{v_1, v_2, \ldots, v_n\}$ and $d_i = \deg(v_i)$ for $i = 1, 2, \ldots, n$. For $0 < \alpha \leq 1$, the α-*degree* of a graph G is defined as follows:

$$\widehat{d_\alpha} = \widehat{d_\alpha}(G) = \frac{1}{n} \sum_{i=1}^{n} \binom{d_i}{\lceil \alpha d_i \rceil - 1}.$$

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[1] The original definition of α-domination does not require the graph to be isolate-free. But this condition is imposed to ensure $\gamma(G) \leq \gamma_\alpha(G)$. See Concluding Remarks in [13].

The *closed α-degree* of a graph G is defined as follows:

$$\widetilde{d}_\alpha = \widetilde{d}_\alpha(G) = \frac{1}{n}\sum_{i=1}^{n}\binom{d_i+1}{\lceil\alpha d_i\rceil-1}.$$

It is known that for any isolate-free graph G of order n, $\gamma(G) \le \gamma_\alpha(G) \le \beta(G)$, where $\beta(G)$ is the vertex covering number of G. Define α-domination spectrum of a graph G, denoted by $\mathsf{Sp}_\alpha(G)$, to be the set of distinct values of $\gamma_\alpha(G)$ as α runs over $(0, 1]$, i.e., $\mathsf{Sp}_\alpha(G) = \{\gamma_\alpha(G) : \alpha \in (0, 1]\}$. Now, two cases may arise: either $\mathsf{Sp}_\alpha(G)$ is singleton or not. It is known that if for a graph G, $\gamma(G) = \beta(G)$, then $\gamma(G) = \gamma_\alpha(G) = \beta(G)$, i.e., $|\mathsf{Sp}_\alpha(G)| = 1$. Examples of graphs with singleton α-domination spectrum are star graphs $K_{1,n-1}$.

On the other hand, if $\gamma(G) < \beta(G)$, then for $0 < \alpha \le 1/\Delta$, $\gamma_\alpha(G) = \gamma(G)$ and for $1 - 1/\Delta < \alpha \le 1$, $\gamma_\alpha(G) = \beta(G)$. Thus for $\Delta \ge 2$, we have $\gamma(G), \beta(G) \in \mathsf{Sp}_\alpha(G)$, i.e., $|\mathsf{Sp}_\alpha(G)| \ge 2$. The complete graph K_n has a α-domination spectrum of size $n-1$, i.e., $\mathsf{Sp}_\alpha(K_n) \subseteq \{1, 2, \ldots, n-1\}$. Thus, the only case left is when $\Delta = 1$. However, if G is an isolate-free graph and $\Delta = 1$, then G is a disjoint union of a copies of P_2. Thus in this case we have $\gamma(G) = \beta(G)$. Thus for an isolate-free graph G, $|\mathsf{Sp}_\alpha(G)| > 1$ implies $\gamma(G) \ne \beta(G)$ and $\Delta \ge 2$.

2 α-Domination Spectrum of a Graph and its Consequences

In this section, we focus on the case when $\mathsf{Sp}_\alpha(G)$ is not singleton, i.e., $|\mathsf{Sp}_\alpha(G)| > 1$. For the sake of clarity, we assume that the elements in $\mathsf{Sp}_\alpha(G)$ are arranged in ascending order. Now, we move towards proving our main result that the α-domination number changes its value only at rational points as α runs over $(0, 1]$. However before doing that, we prove a lemma which we will use later.

Lemma 2.1 *Let G be a graph such that $|\mathsf{Sp}_\alpha(G)| > 1$. Let $q \in \mathsf{Sp}_\alpha(G)$ such that $q \ne \gamma(G)$. Let $A = \{\alpha \in (0, 1] : \gamma_\alpha(G) < q\}$ and $B = \{\alpha \in (0, 1] : \gamma_\alpha(G) \ge q\}$. Then there exists a rational number $\alpha^* \in (0, 1)$ such that $A = (0, \alpha^*]$ and $B = (\alpha^*, 1]$.*

Proof Since $q \ne \gamma(G)$, q is not the least element of $\mathsf{Sp}_\alpha(G)$. Observe that both A and B are non-empty, because $(0, 1/\Delta] \subseteq A$ and $(1 - 1/\Delta, 1] \subseteq B$. In fact, both A and B are intervals. Because if $a, b \in A$ with $a < b$, then for any c with $a < c < b$, we have $c \in A$. It follows from the fact that $\alpha' < \alpha''$ implies $\gamma_{\alpha'}(G) \le \gamma_{\alpha''}(G)$. Moreover, from the definition, it follows that $A \cup B = (0, 1]$ and $A \cap B = \emptyset$. Thus there exists $\alpha^* \in (0, 1]$ such that either $A = (0, \alpha^*)$, $B = [\alpha^*, 1]$ or $A = (0, \alpha^*]$, $B = (\alpha^*, 1]$.

Claim 1 Both A and B are left-open, right-closed intervals, i.e., $A = (0, \alpha^*]$ and $B = (\alpha^*, 1]$.

Proof of Claim 1 If possible, let $A = (0, \alpha^*)$ and $B = [\alpha^*, 1]$. Let $p \in \mathsf{Sp}_\alpha(G)$ be the largest element in $\mathsf{Sp}_\alpha(G)$ less than q. Thus there exists $\alpha' \in A$ such that $\gamma_{\alpha'}(G) = p$. This imply that for all $\alpha \in [\alpha', \alpha^*)$, $\gamma_\alpha(G) = p$. Let (α_n) be a strictly monotonically increasing sequence in $[\alpha', \alpha^*)$ such that (α_n) converges to α^*. Now as

$\alpha_n \in [\alpha', \alpha^*)$, we have $\gamma_{\alpha_n} = p$, i.e., for each α_n, there exists $S_n \subseteq V$ with $|S_n| = p$ such that

$$\frac{|N(v) \cap S_n|}{|N(v)|} \geq \alpha_n, \quad \forall v \in V \backslash S_n.$$

Moreover as $|S_n| = p < q$, S_n is not a γ_{α^*}-set, i.e., there exists at least one $v_n \in V \backslash S_n$ such that

$$\frac{|N(v_n) \cap S_n|}{|N(v_n)|} < \alpha^*.$$

Thus we get a sequence of S_n of subsets of V and a sequence of vertices $v_n \in V \backslash S_n$ such that

$$\alpha_n \leq \frac{|N(v_n) \cap S_n|}{|N(v_n)|} < \alpha^*, \quad \forall n \in \mathbb{N} \tag{1}$$

As G is a finite graph, the number of choices for subsets S_n of size p and the number of choices for $v_n \in V \backslash S_n$ is finite. Thus, the sequence $\left(\frac{|N(v_n) \cap S_n|}{|N(v_n)|} \right)$ assumes finitely many values. Now, since (α_n) converges to α^*, by Sandwich Theorem, the sequence $\left(\frac{|N(v_n) \cap S_n|}{|N(v_n)|} \right)$ converges to α^*. As any convergent sequence taking finitely many values is eventually constant, we have $\left(\frac{|N(v_n) \cap S_n|}{|N(v_n)|} \right)$ to be eventually a constant sequence. Thus there exists $k \in \mathbb{N}$ such that $\frac{|N(v_n) \cap S_n|}{|N(v_n)|} = \alpha^*$ for all $n \geq k$. This is a contradiction to Eq. (1). Thus our claim is justified and hence $A = (0, \alpha^*]$ and $B = (\alpha^*, 1]$.

Claim 2 $\alpha^* \in \mathbb{Q}$, where $\mathbb{Q}$ denote the set of all rationals.

Proof of Claim 2 If possible, let $\alpha^* \in (0, 1] \backslash \mathbb{Q}$. We observe that $\gamma_{\alpha^*}(G) = p$, because if $\gamma_{\alpha^*}(G) < p$, then for all $\alpha \in A$, $\gamma_\alpha(G) < p$ which contradicts the fact that $p \in \mathsf{Sp}_\alpha(G)$. Since, α^* is an irrational number, $\alpha^*|N(v)|$ is not an integer for all $v \in V$. Now as $(0, 1] \cap (\mathbb{R} \backslash \mathbb{Q})$ is dense in $(0, 1]$, there exists an irrational number $\overline{\alpha} \in (0, 1] \cap (\mathbb{R} \backslash \mathbb{Q})$ with $\overline{\alpha} > \alpha^*$ such that $\lceil \alpha^*|N(v)| \rceil = \lceil \overline{\alpha}|N(v)| \rceil$ for all $v \in V$ (we omit the details of the proof).

Now let S be a $\gamma_{\overline{\alpha}}$-set of G. Then for all $v \in V \backslash S$, $|N(v) \cap S| \geq \overline{\alpha}|N(v)|$. As $\overline{\alpha} \in \mathbb{R} \backslash \mathbb{Q}$, we have $v \in V \backslash S$, $|N(v) \cap S| \geq \lceil \overline{\alpha}|N(v)| \rceil = \lceil \alpha^*|N(v)| \rceil \geq \alpha^*|N(v)|$. Thus S is a γ_{α^*}-set in G and hence $|S| = p$.

On the other hand, as $\overline{\alpha} > \alpha^*$, $\overline{\alpha} \in B$. But S being a $\gamma_{\overline{\alpha}}(G)$-set of G, we get $|S| \geq q$ (by definition of B). This is a contradiction. Hence $\alpha^* \in \mathbb{Q}$. $\square$

Now we are in a position to prove the following theorem.

Theorem 2.1 *Let G be a graph such that $\mathsf{Sp}_\alpha(G) = \{a_1, a_2, \ldots, a_t\}$ with $\gamma(G) = a_1 < a_2 < \cdots < a_t = \beta(G)$ and $t > 1$. Then there exists $t - 1$ rational numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{t-1}$ in $(0, 1) \cap \mathbb{Q}$ such that*
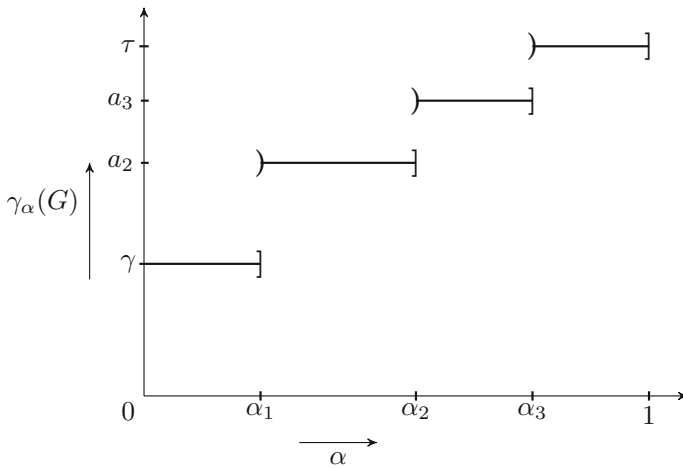
Fig. 1 A typical α-vs-γ_α plot

1. *for all $\alpha \in (0, \alpha_1]$, $\gamma_\alpha(G) = a_1$.*
2. *for all $i \in \{1, 2, \ldots, t-2\}$, for all $\alpha \in (\alpha_i, \alpha_{i+1}]$, $\gamma_\alpha(G) = a_{i+1}$.*
3. *for all $\alpha \in (\alpha_{t-1}, 1]$, $\gamma_\alpha(G) = a_t$.*

Proof Substituting $q = a_t$ in Lemma 2.1, we get a rational number α_{t-1} such that $A_1 = \{\alpha \in (0, 1] : \gamma_\alpha(G) < a_t\} = (0, \alpha_{t-1}]$ and $B_1 = \{\alpha \in (0, 1] : \gamma_\alpha(G) \geq a_t\} = (\alpha_{t-1}, 1]$. However, as a_t is the largest element in $\mathsf{Sp}_\alpha(G)$, we have $B_1 = \{\alpha \in (0, 1] : \gamma_\alpha(G) = a_t\} = (\alpha_{t-1}, 1]$.

Again, substituting $q = a_{t-1}$ in Lemma 2.1, we get a rational number α_{t-2} such that $A_2 = \{\alpha \in (0, 1] : \gamma_\alpha(G) < a_{t-1}\} = (0, \alpha_{t-2}]$ and $B_2 = \{\alpha \in (0, 1] : \gamma_\alpha(G) \geq a_{t-1}\} = (\alpha_{t-2}, 1]$. However, as a_t and a_{t-1} are the only two elements in $\mathsf{Sp}_\alpha(G)$ which are greater or equal to a_{t-1} and $B_1 = \{\alpha \in (0, 1] : \gamma_\alpha(G) = a_t\} = (\alpha_{t-1}, 1]$, we have $\{\alpha \in (0, 1] : \gamma_\alpha(G) = a_{t-1}\} = (\alpha_{t-2}, \alpha_{t-1}]$.

Continuing in this way, at one stage we substitute $q = a_2$ in Lemma 2.1 to get a rational number α_1 such that $A_{t-1} = \{\alpha \in (0, 1] : \gamma_\alpha(G) < a_2\} = (0, \alpha_1]$ and $B_{t-1} = \{\alpha \in (0, 1] : \gamma_\alpha(G) \geq a_2\} = (\alpha_1, 1]$. By similar argument as that of above, we get $\{\alpha \in (0, 1] : \gamma_\alpha(G) = a_2\} = (\alpha_1, \alpha_2]$. Moreover, as $\gamma(G) = a_1$ is the only value in $\mathsf{Sp}_\alpha(G)$ which is less than a_2, we have $A_{t-1} = \{\alpha \in (0, 1] : \gamma_\alpha(G) = a_1\} = (0, \alpha_1]$. Hence the theorem follows.   □

We call the α_i's obtained in Theorem 2.1 as *critical values* of α. A typical α vs $\gamma_\alpha(G)$-plot, as demonstrated in Theorem 2.1, is shown in Fig. 1. It is shown in Theorem 2.4 that for any finite set of rational numbers in $(0, 1)$, there exists a graph G which has that set as the set of its all critical values. Moreover, Theorem 2.1 has an immediate corollary.

Corollary 2.2 *Let G be a graph and $\overline{\alpha}$ be a irrational number in $(0, 1)$. Then there exists $\epsilon > 0$, such that for all $\alpha \in (\overline{\alpha} - \epsilon, \overline{\alpha} + \epsilon)$, $\gamma_\alpha(G)$ is constant.*

Proof The corollary follows from Theorem 2.1 and denseness of rationals and irrationals in $\mathbb{R}$. □

Our next goal is to find an upper bound on the size of the α-domination spectrum of a graph. Before that we prove a lemma and recall the definition of totient summatory function Φ.

Lemma 2.3 *Let G be a graph such that $\mathsf{Sp}_\alpha(G) = \{a_1, a_2, \ldots, a_t\}$ with $a_1 < a_2 < \ldots < a_t$ and let α_i's be as in Theorem 2.1. Then for each α_i, there exists a $\gamma_{\alpha_i}(G)$-set $S_i \subseteq V$ and a vertex $v_i \in V \backslash S_i$ such that*

$$\alpha_i = \frac{|N(v_i) \cap S_i|}{|N(v_i)|}.$$

Proof Since S_i is a $\gamma_{\alpha_i}(G)$-set of G, we have $|S_i| = a_i$ and

$$\frac{|N(v) \cap S_i|}{|N(v)|} \geq \alpha_i, \quad \forall v \in V \backslash S_i. \tag{2}$$

If possible, for all $v \in V \backslash S_i$, the inequality in Eq. (2) is strict. But in that case, by denseness of real numbers, we can find $\alpha' > \alpha_i$ such that $\frac{|N(v) \cap S_i|}{|N(v)|} \geq \alpha' > \alpha_i$, for all $v \in V \backslash S_i$. Thus S_i is a $\gamma_{\alpha'}(G)$-set of G and hence $\gamma_{\alpha'}(G) \leq |S_i| = a_i < a_{i+1}$. However as $\alpha' > \alpha_i$, we have $\gamma_{\alpha'}(G) \geq a_{i+1}$. This is a contradiction. Thus, there exists at least one vertex $v_i \in V \backslash S_i$ such that Eq. (2) holds with equality. Hence the theorem follows. □

Definition 2.1 The totient summatory function Φ is defined as

$$\Phi(n) = \sum_{k=1}^{n} \phi(k),$$

where ϕ is the Euler's totient function, i.e., $\phi(k)$ is the number of positive integers less than and relatively prime to k.

Theorem 2.2 *For any isolate-free graph G, the critical values belong to the set $\{\frac{p}{q} : gcd(p,q) = 1; 1 \leq p < q \leq \Delta\}$ and $|\mathsf{Sp}_\alpha(G)| \leq \Phi(\Delta)$, where Φ is the totient summatory function.*

Proof By Lemma 2.3, for every critical value α_i, there exists a $\gamma_{\alpha_i}(G)$-set $S_i \subseteq V$ and a vertex $v_i \in V \backslash S_i$ such that

$$\alpha_i = \frac{|N(v_i) \cap S_i|}{|N(v_i)|}.$$

Thus the first part of the theorem follows from the observation that $|N(v_i) \cap S_i| < |N(v_i)| \leq \Delta$.

For the second part, observe that $|\{\frac{p}{q} : gcd(p,q) = 1; 1 \leq p < q \leq \Delta\}| = \phi(2) + \phi(3) + \cdots + \phi(\Delta)$. Also note that $|\mathsf{Sp}_\alpha(G)|$ is one more than the number of critical values. Thus, $|\mathsf{Sp}_\alpha(G)| \leq 1 + \phi(2) + \phi(3) + \cdots + \phi(\Delta) = \Phi(\Delta)$. □

Corollary 2.4 *For a k-regular graph, the critical values belong to the set $\{\frac{p}{k} : 1 \leq p < k\}$ and $|\mathsf{Sp}_\alpha(G)| \leq k$.* □

Remark 2.1 As $\gamma(G) \leq \gamma_\alpha(G) \leq \beta(G)$, we also have $|\mathsf{Sp}_\alpha(G)| \leq \beta(G) - \gamma(G) + 1$. Theorem 2.3 shows that this bound is tight. Also this upper bound provides an idea if both $\beta(G)$ and $\gamma(G)$ are known. Moreover, comparing this upper bound with the upper bound given in Theorem 2.2, it is observed that this upper bound is tighter if the difference between $\beta(G)$ and $\gamma(G)$ is smaller than $\Phi(\Delta)$. However, if Δ is relatively small compared to the difference between $\beta(G)$ and $\gamma(G)$, then the upper bound in Theorem 2.2 is tighter.

Theorem 2.3 *Given any two natural numbers $r < s$, there exists an isolate-free graph G with $\gamma(G) = r$, $\beta(G) = s$ and $|\mathsf{Sp}_\alpha(G)| = s - r + 1$, i.e., all the integer values between r and s (including both) are the α-domination number $\gamma_\alpha(G)$ of G for some $\alpha \in (0, 1]$.*

Proof Let $p = s - r + 2$ and G be the disjoint union of $r - 1$ copies of K_2 and one copy of K_p. Then $\gamma(G) = r$ and $\beta(G) = (p-1) + (r-1) = (s-r+1) + (r-1) = s$. Now, the degree of all the vertices in $r - 1$ copies of K_2 are 1 and the degree of all the vertices in K_p are $p - 1$. Thus for any integer between r and s, say k, consider the set S_k consisting of one vertex from each K_2 and $k - r + 1$ vertices from K_p. Then $|N(v) \cap S_k| = 1$ or $k - r + 1$ according as v is a vertex in K_2's outside S_k or K_p outside S_k. Let $\alpha = \frac{k-r+1}{p-1} < 1$. Then $|N(v) \cap S_k| \geq \alpha |N(v)|$ for all $v \in V \setminus S_k$. Thus $\gamma_\alpha(G) \leq |S_k| = k$. If possible, let $\gamma_\alpha(G) < k$ and let S be a $\gamma_\alpha(G)$-set in G. Since $\gamma(G) \leq \gamma_\alpha(G)$ and all minimum dominating sets of G must contain exactly one vertex from each copy of K_2, S contains less than $k - r + 1$ vertices from K_p. Thus, for all vertices v in K_p outside S, we have $\alpha|N(v)| = \frac{k-r+1}{p-1}(p-1) = k-r+1 > |N(v) \cap S|$. Thus S is not a $\gamma_\alpha(G)$-set of G, a contradiction. Therefore $\gamma_\alpha(G) = k$ and hence the theorem follows. □

Theorem 2.4 *Given any finite set of rational numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{t-1}$ in $(0, 1)$ and any positive integer r, there exists an isolate-free graph G such that $\gamma(G) = r$ and α_i's are critical values for G.*

Proof Without loss of generality, let the α_i's have a common denominator, i.e., let $\alpha_i = \frac{s_i}{q}$, where $s_1 < s_2 < \cdots < s_{t-1}$. Now in the construction of G in the proof of Theorem 2.3, set $r = r$ and $s = q + r - 1$. Therefore $p = q + 1$. Let $k_i = s_i + r - 1$. Then as in Theorem 2.3, S_{k_i} is a α_i-dominating set, where $\alpha_i = \frac{k_i - r + 1}{p - 1} = \frac{s_i}{q}$. Moreover, each α_i is a critical value in the α-domination spectrum of G and $\gamma(G) = r$. □

Proposition 2.5 *Let G be a connected regular graph. Then $|\mathsf{Sp}_\alpha(G)| \geq 2$ if and only if G is not isomorphic to K_2 or C_4.*

Proof The proposition follows from Theorem 2.5 in [14], which states that for a connected graph G, $\gamma(G) = \beta(G)$ if and only if G is isomorphic to K_2 or C_4. □

Corollary 2.6 *For a connected k-regular graph which is not isomorphic to K_2 or C_4, $2 \leq |\mathsf{Sp}_\alpha(G)| \leq k$.*

Proof The proof follows from Corollary 2.4 and Proposition 2.5. □

2.1 A Tighter Upper Bound

In [4], the authors proved that if G is a graph of order n and $\alpha \in (0, 1)$, then $\gamma_\alpha(G) \leq \left(1 - \frac{1}{\lceil \frac{1}{1-\alpha} \rceil}\right) n$. They also proved (Theorem 2.6 in [4]) that for all $\alpha \in (0, 1)$ and for all $\epsilon > 0$, $\gamma_\alpha(G) \leq n\alpha + \epsilon$ holds for all graphs G with sufficiently large minimum degree (depending upon α and ϵ). On the other hand, the authors in [13] [see Concluding Remarks (1)] characterized the values of α for a particular class of graphs such that $\gamma_\alpha(G) \leq n\alpha$ holds. In this section, using the idea of critical values, we show that for certain values of α, a tighter upper bound $\gamma_\alpha(G) \leq n\alpha$ holds. It is to be noted here, in contrast with [13], that our results hold good for all graphs. We start by observing that if $\alpha = \frac{k}{k+1}$ for some $k \in \mathbb{N}$, then the upper bound in [4] reduces to $\gamma_\alpha(G) \leq n\alpha$.

Theorem 2.5 *Let G be a graph of order n, $k \in \mathbb{N}$ and α_i's be the critical values of α as defined in Theorem 2.1. If $\frac{k}{k+1} \in (\alpha_i, \alpha_{i+1}]$, then either $\gamma_\alpha(G) \leq n\alpha$, for all $\alpha \in (\alpha_i, \alpha_{i+1}]$ or*

$$\gamma_\alpha(G) = n\alpha \ \text{ for some } \ \alpha \in \left(\alpha_i, \frac{k}{k+1}\right) \ \text{ and}$$

$$\gamma_\alpha(G) \leq n\alpha, \quad \forall \alpha \in \left[\frac{k}{k+1}, \alpha_{i+1}\right].$$

Proof Since α_i's are the critical values, $\gamma_\alpha(G)$ remains constant in the interval $(\alpha_i, \alpha_{i+1}]$. Also $\gamma_{k/(k+1)}(G) \leq \frac{nk}{k+1}$. Let $\alpha \in \left(\frac{k}{k+1}, \alpha_{i+1}\right]$. Then $\gamma_\alpha(G) = \gamma_{k/(k+1)}(G) \leq \frac{nk}{k+1} \leq n\alpha$. If $\gamma_\alpha(G) \leq n\alpha$, for all $\alpha \in (\alpha_i, \alpha_{i+1}]$, then the theorem holds. If not, then there exists $\alpha' \in (\alpha_i, \frac{k}{k+1})$ such that $\gamma_{\alpha'}(G) > n\alpha'$. As $\gamma_\alpha(G)$ remains constant in the interval $(\alpha_i, \alpha_{i+1}]$ (i.e., $\gamma_{\alpha'}(G) = \gamma_{k/(k+1)}(G)$), $\gamma_{k/(k+1)}(G) \leq \frac{nk}{k+1}$ and $\gamma_{\alpha'}(G) > n\alpha'$, there exists $\alpha \in \left(\alpha', \frac{k}{k+1}\right)$ such that $\gamma_\alpha(G) = n\alpha$. Hence the theorem follows.   □

3 New Probabilistic Upper Bounds

In this section, we prove some improved probabilistic upper bounds of α-domination number and α-rate domination number of a graph G. Gagarin, Poghosyan and Zverovich obtained the following probabilistic upper bounds for the α-domination number of a graph.

Theorem 3.1 (Gagarin, Poghosyan, Zverovich [7]) *For any graph G,*

$$\gamma_\alpha(G) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1 + 1/\widehat{\delta}} \widehat{d_\alpha}^{1/\widehat{\delta}}}\right) n,$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1$.

Corollary 3.1 (Gagarin, Poghosyan, Zverovich [7]) *For any graph G,*

$$\gamma_\alpha(G) \leq \left\lceil \frac{\ln(\widehat{\delta} + 1) + \ln(\widehat{d_\alpha}) + 1}{\widehat{\delta} + 1} \right\rceil n,$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1.$

Gagarin, Poghosyan and Zverovich obtained the following probabilistic upper bounds for the α-rate domination number of a graph.

Theorem 3.2 (Gagarin, Poghosyan, Zverovich [7]) *For any graph G,*

$$\gamma_{\times\alpha}(G) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widehat{d_\alpha}^{1/\widehat{\delta}}} \right) n,$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1.$

Corollary 3.2 (Gagarin, Poghosyan, Zverovich [7]) *For any graph G,*

$$\gamma_{\times\alpha}(G) \leq \left(\frac{\ln(\widehat{\delta} + 1) + \ln \widetilde{d_\alpha} + 1}{\widehat{\delta} + 1} \right) n,$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1.$

We obtain new probabilistic upper bounds for the α-domination number of a graph, and improve Theorem 3.1 and Corollary 3.1. We also obtain new probabilistic upper bounds for the α-rate domination number of a claw-free graph, and improve Theorem 3.2 and Corollary 3.2 for claw-free graphs. The following is useful.

Theorem 3.3 (Caro [2] and Wei [15]) *For any graph G,* $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\deg(v)}.$

Theorem 3.4 *For any graph G,*

$$\gamma_\alpha(G) \leq \left[1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widehat{d_\alpha}^{1/\widehat{\delta}}} \right] n - \frac{n}{1 + \Delta} \left[1 - \left(\frac{1}{(1 + \widehat{\delta})\widehat{d_\alpha}} \right)^{\frac{1}{\widehat{\delta}}} \right]^{1+\Delta},$$

where $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1.$

Proof We follow the proof of Theorem 3.1 given in [7]. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{(1 + \widehat{\delta})\widehat{d_\alpha}} \right)^{\frac{1}{\widehat{\delta}}}.$$

Let

$$B = \{ v \in V(G) - A : |N(v_i) \cap A| \leq \lceil \alpha d_i \rceil - 1 \}.$$

Let $A' = \{v \in V(G) : N[v] \subseteq A\}$, and I be a maximum independent set in $G[A']$. Clearly $(A - I) \cup B$ is an α-dominating set for G. Thus $\gamma_\alpha(G) \leq E(|(A - I) \cup B|) \leq E(|A|) + E(|B|) - E(|I|)$. As it was shown in [7] (in the proof of Theorem 3.1),

$$E(|A|) + E(|B|) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1 + 1/\widehat{\delta}}\widehat{d}_\alpha^{1/\widehat{\delta}}}\right)n.$$

We compute the expectation of $|I|$. By Theorem 3.3,

$$\begin{aligned}
E(|I|) &\geq E\left(\sum_{v \in A'} \frac{1}{1 + \deg_{G[A']}(v)}\right) \\
&\geq \sum_{v \in V} \frac{1}{1 + \deg_G(v)} Pr(v \in A') \\
&= \sum_{v \in V} \frac{1}{1 + \deg_G(v)} p^{1 + \deg_G(v)} \\
&\geq \sum_{v \in V} \frac{1}{1 + \deg_G(v)} p^{1 + \Delta} \\
&\geq \frac{n}{1 + \Delta} p^{1 + \Delta} = \frac{n}{1 + \Delta}\left[1 - \left(\frac{1}{(1 + \widehat{\delta})\widehat{d}_\alpha}\right)^{\frac{1}{\widehat{\delta}}}\right]^{1 + \Delta}.
\end{aligned}$$

Thus the result follows immediately.   □

Following the proofs of Theorem 3.4 and Corollary 3.1 with $p = \frac{\ln(\widehat{\delta} + 1) + \ln\widehat{d}_\alpha}{\widehat{\delta} + 1}$, we obtain the following.

Corollary 3.3 *For any graph G,*

$$\gamma_\alpha(G) \leq \left[\frac{\ln(\widehat{\delta} + 1) + \ln\widehat{d}_\alpha + 1}{\widehat{\delta} + 1}\right]n - \frac{n}{1 + \Delta}\left(\frac{\ln(\widehat{\delta} + 1) + \ln\widehat{d}_\alpha}{\widehat{\delta} + 1}\right)^{1 + \Delta},$$

where $\widehat{\delta} = \lfloor\delta(1 - \alpha)\rfloor + 1$.

We next present new probabilistic upper bounds for the α-rate domination number in claw-free graphs.

Theorem 3.5 *For any claw-free graph G,*

$$\gamma_{\times\alpha}(G) \leq \left[1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1 + 1/\widehat{\delta}}\widetilde{d}_\alpha^{1/\widehat{\delta}}}\right]n - \frac{n}{1 + \Delta}\left[1 - \left(\frac{1}{(1 + \widehat{\delta})\widetilde{d}_\alpha}\right)^{\frac{1}{\widehat{\delta}}}\right]^{1 + \Delta^2},$$

where $\widehat{\delta} = \lfloor\delta(1 - \alpha)\rfloor + 1$.

Proof We follow the proof of Theorem 3.2. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{(1 + \widehat{\delta})\widetilde{d_\alpha}} \right)^{\frac{1}{\delta}}.$$

Let

$$B = \{v \in V(G) : |N(v_i) \cap A| \leq \lceil \alpha d_i \rceil - 1\}.$$

Let $A' = \{v \in V(G) : N[v] \subseteq A\}$, $A'' = \{v : N[v] \subseteq A'\}$, and let I be a maximum independent set in $G[A'']$. Clearly, $\deg_{G[A']}(v) = \deg(v)$ for every vertex $v \in A''$. Since G is claw-free, any vertex of $A' - A''$ is adjacent to at most one vertex of I. Then $(A - I) \cup B$ is an α-rate dominating set for G. Thus $\gamma_{\times\alpha}(G) \leq E(|(A - I) \cup B'|) = E(|A| + |B'| - |I|) = E(|A|) + E(|B'|) - E(|I|)$. As it was shown in [7] (in the proof of Theorem 3.2),

$$E(|A|) + E(|B|) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}}\widetilde{d_\alpha}^{1/\widehat{\delta}}} \right)n.$$

For a vertex v, if $N(v) = \{v_1, \ldots, v_d\}$, then

$$Pr(v \in A'') = pp^{\deg(v_1)} \cdots p^{\deg(v_d)} \geq p^{1+d\Delta} \geq p^{1+\Delta^2}.$$

Thus,

$$\begin{aligned}
E(|I|) &\geq E\left(\sum_{v \in A''} \frac{1}{1 + \deg_{G[A'']}(v)} \right) \\
&\geq \sum_{v \in V} \frac{1}{1 + \deg_{G[A']}(v)} Pr(v \in A'') \\
&\geq \sum_{v \in V} \frac{1}{1 + \deg_G(v)} Pr(v \in A'') \\
&= \sum_{v \in V} \frac{1}{1 + \deg_G(v)} p^{1+\Delta^2} \\
&\geq \sum_{v \in V} \frac{1}{1 + \deg_G(v)} p^{1+\Delta^2} \\
&\geq \frac{n}{1 + \Delta} \left[1 - \left(\frac{1}{(1 + \widehat{\delta})\widetilde{d_\alpha}} \right)^{\frac{1}{\delta}} \right]^{1+\Delta^2}.
\end{aligned}$$

□

Following the proofs of Theorem 3.5 and Corollary 3.2 with $p = \frac{\ln(\widehat{\delta}+1)+\ln\widetilde{d_\alpha}}{\delta+1}$, we obtain the following.

Corollary 3.4 *For any claw-free graph G,*

$$\gamma_{\times\alpha}(G) \leq \left(\frac{\ln(\widehat{\delta}+1) + \ln(\widetilde{d}_\alpha) + 1}{\widehat{\delta}+1}\right) n - \frac{n}{1+\Delta}\left(\frac{\ln(\widehat{\delta}+1) + \ln\widetilde{d}_\alpha}{\widehat{\delta}+1}\right)^{1+\Delta^2},$$

where $\widehat{\delta} = \lfloor \delta(1-\alpha)\rfloor + 1$.

4 Conclusion and Open Problems

In this paper, we introduced the notion of α-domination spectrum and critical values of α to prove a few bounds about $\gamma_\alpha(G)$ and the $\gamma_\alpha(G)$-spectrum of a graph G. Finally, we prove some improved probabilistic upper bounds of α-domination number and α-rate domination number of a graph G. We close with the following open problems.

- It was shown in [6], that α-domination spectrum of cycles and paths have exactly two values, namely the domination number and vertex cover number of the respective graphs. It can be an interesting problem to characterize graphs with exactly two values in its α-domination spectrum.
- In Theorem 2.5, certain values of α's were characterized for which $\gamma_\alpha(G) \leq n\alpha$ holds. To characterize graphs G on n vertices for which $\gamma_\alpha(G) \leq n\alpha$ holds for all $\alpha \in [1/n, 1]$ can be another interesting problem.

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