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ORIGINAL PAPER

Independence Number and k-Trees of Graphs

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Abstract A tree T is called a k-tree if the maximum degree of T is at most k. In this paper, we give a sufficient condition for a graph to have a k-tree containing specified vertices as following: let G be a connected graph and let S be a subset of $V(G)$. If $\alpha_G(S) \leq (k-1)\kappa_G(S)+1$, then G has a k-tree containing S. Moreover, this condition is sharp.

Keywords Independence number · k-tree

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. Let $\alpha(G)$ and $\kappa(G)$ denote the independence number of G and the connectivity of G, respectively. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a vertex set S of G, let $\langle S\rangle_G$ denote the subgraph induced by S in G. We define $\alpha_G(S)$ the maximum cardinality of the independent set of S in G, which is called the *independence number of* S in G. For two vertices x, y of G, the *local connectivity* $\kappa_G(x, y)$ is defined to be the maximum number of internally disjoint paths connecting x and y in G. We define $\kappa_G(S) := \min\{\kappa_G(x, y) : x, y \in S, x \neq y\}$. Moreover, if $|S| = 1$, $\kappa_G(S)$ is defined to be $+\infty$. For $X, Y \subset V(G)$, we denote the set of edges of

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G joining X to Y by $E_G(X, Y)$, and the number of edges in $E_G(X, Y)$ by $e_G(X, Y)$. For a tree T, a vertex of T with degree one is called a *leaf* of T and the set of leaves of T is denoted by $Leaf(T)$.

In 1972, Chvátal and Erdös gave an independence number condition for a graph to have a Hamiltonian cycle (path) as following:

Theorem 1 (Chvátal and Erdös [1]) *Let G be a connected graph.*

(1) *If* $\alpha(G) \leq \kappa(G)$, *then G has a Hamiltonian cycle unless* $G \cong K_1$ *or* K_2.
(2) *If* $\alpha(G) \leq \kappa(G) + 1$, *then G has a Hamiltonian path.*

A Hamiltonian cycle (path) is a cycle (path) which passes through all vertices of a graph. In this sense, we can consider a cycle (path) containing specified vertices as a generalization of a Hamiltonian cycle (path).

Theorem 2 (Fournier [2]) *Let G be a 2-connected graph, and let* $S \subset V(G)$. *If* $\alpha_G(S) \leq \kappa(G)$, *then G has a cycle covering S.*

Theorem 3 (Ozeki and Yamashita [3]) *Let G be a 2-connected graph and let* $S \subset V(G)$. *If* $\alpha_G(S) \leq \kappa_G(S)$, *then G has a cycle covering S.*

Let $k \geq 2$ be an integer. A tree T is called a k-*tree* if $\deg_T(x) \leq k$ for all $x \in V(T)$, that is, the maximum degree of a k-tree is at most k. Since a Hamiltonian path is nothing but a spanning 2-tree, we can consider the existence of a spanning k-tree as an generalization of a Hamiltonian path. Similarly to considering a cycle passing through specified vertices, in [4], the authors focus on the existence of a k-tree containing specified vertices and obtained the following result:

Theorem 4 (Chiba et al. [4]) *Let G be a graph and let k be an integer with* $k \geq 3$. *Let* $S \subset V(G)$ *with* $\kappa_G(S) \geq m \geq 1$. *If*

$$\alpha_G(S) \leq (k-1)m + 1 - \left\lfloor \frac{m-1}{k} \right\rfloor,$$

then G has a k-tree containing S.

In [4], the authors said that they did not know whether the condition of above theorem was best possible condition or not. Here, we give the best condition as the following result:

Theorem 5 *Let G be a connected graph and let k be an integer with* $k \geq 2$. *Let S be a vertex set of G. If*

$$\alpha_G(S) \leq (k-1)\kappa_G(S) + 1, \tag{1}$$

then G has a k-tree containing S.

We show that the condition in Theorem 5 is sharp. Let $m \geq 1$ and $k \geq 2$ be integers. Let K_m be a complete graph of order m and let $S = \{s_1, s_2, \ldots, s_N\}$ where $N = (k-1)m+2$. Join $s_i\,(1 \leq i \leq N)$ to all the vertices of K_m. Let G be the resulting graph. Then $\kappa_G(S) = m$ and $\alpha_G(S) = N = (k-1)\kappa_G(S) + 2$. Let T ba a tree of G containing S and $V(T) = S \cup L$, where $L \subseteq V(K_m)$. Then $E(T) = E_T(S, L) \cup E_T(L, L)$, $|E(T)| = |V(T)| - 1 = |S| + |L| - 1$ and $E_T(S, L) \cap E_T(L, L) = \emptyset$. We have $\sum_{v \in L} \deg_T(v) = e_T(S, L) + 2e_T(L, L) = |S| + |L| - 1 + e_T(L, L) = (k-1)\kappa_G(S)+1+|L|+e_T(L, L) = |L|k+1+(k-1)(\kappa_G(S)-|L|)+e_T(L, L) \geq |L|k+1$, that is, there exists at least one vertex of L with degree at least $k+1$. So G has no k-tree containing S. Therefore the condition (1) is sharp.

In Theorem 5, given $S = V(G)$, by Menger's theorem, we obtain the following result which is an extension of Theorem 1:

Theorem 6 (Neumann-Lara and Rivera-Campo [5]) *Let k be an integer with $k \geq 2$ and let G be a connected graph. If $\alpha(G) \leq (k-1)\kappa(G) + 1$, then G has a spanning k-tree.*

In [4], the authors gave not only an independence number condition but also a degree sum condition for the existence of a k-tree containing specified vertices. In this paper, we give a sharp independence number condition for the existence of a k-tree containing specified vertices. It is natural to ask the following problem.

Problem 1 Find a sharp degree sum condition for a graph to have a k-tree containing specified vertices.

In order to prove Theorem 5, we prove the following result which implies Theorem 5.

Theorem 7 *Let G be a connected graph and let k be an integer with $k \geq 2$. Let S be a vertex set of G. Then either G has a k-tree covering S, or there exists a k-tree T in G such that*

$$\alpha_G(S - V(T)) \leq \alpha_G(S) - (k-1)\kappa_G(S) - 1. \quad (2)$$

2 Proof of the Results

In order to prove Theorem 7, we need the following Lemmas.

Lemma 1 *Let G be a connected graph. Let T be a tree of G with $|T| \geq 2$ and C be a cycle of G. Then there exists a tree T^* of G such that $V(T) \cup V(C) \subseteq V(T^*)$ and $\Delta(T^*) \leq \Delta(T) + 1$.*

Proof We consider two cases: first, $V(T) \cap V(C) = \emptyset$. In this case, since G is a connected graph, there exists a path P connecting T and C. Let $V(P) \cap V(C) = \{u\}$ and $e = ux$ be an edge of C. Then $T^* = T + P + C - e$ is the desired tree of G.

Next we consider $V(T) \cap V(C) \neq \emptyset$. In this case, $T + C$ is a connected subgraph of G and $\Delta(T + C) \leq \Delta(T) + 2$. There exists l vertices $v_1, \ldots, v_l$ in C with $\deg_{T+C}(v_i) = \deg_T(v_i) + 2 (0 \leq i \leq l)$. We assign an orientation in C, and for every vertex v_i its successor v_i^+ is well-defined. Let $T_1 = T + C - \{v_i v_i^+ : 1 \leq i \leq l\}$. Then $\Delta(T_1) \leq \Delta(T) + 1$ and $V(T_1) = V(T) \cup V(C)$. Therefore, T_1 contains a desired tree. □

Lemma 2 *Let G be a connected graph and $S \subset V(G)$ with $|S| \geq 2$ and $\kappa_G(S) \geq 2$. Then either the vertices of S can be covered by a cycle of G, or there exists a cycle C of G such that $\alpha_G(S - V(C)) \leq \alpha_G(S) - \kappa_G(S)$.*

The connectivity conditions are only used to find a fan of certain width, which can be found even by the local connectivity condition. Therefore Lemma 2 can be shown by the same way as the proof of [6, Theorem 2].

By Lemma 2, we can obtain the following corollary.

Corollary 1 *Let G be a connected graph and $S \subset V(G)$. Then either the vertices of S can be covered by one path of G, or there exists a path P of G such that $\alpha_G(S - V(P)) \leq \alpha_G(S) - (\kappa_G(S) + 1)$.*

Proof Let w be a new vertex not contained in G, joining w to every vertex of G. Let G^* be the resulting graph and $S \subset V(G^*)$. Obviously, $\alpha_{G^*}(S) = \alpha_G(S)$ and $\kappa_{G^*}(S) = \kappa_G(S) + 1$. By Lemma 2, S is covered by a cycle D of G^* or there exists a cycle C in G^* such that $\alpha_{G^*}(S - V(C)) \leq \alpha_{G^*}(S) - \kappa_{G^*}(S)$.

If the former holds and D passes through w, then S is covered by the path $D - w$ of G; otherwise, S is covered by a path obtained from D by removing an edge of D. If C passes through w, then the path $C - w$ of G satisfies the condition since $S - V(C - w) = S - V(C)$; otherwise, the path $C - e$ obtained from C by removing an edge e of C satisfies the condition since $S - V(C - e) = S - V(C)$. And $\alpha_G(S - V(C)) = \alpha_{G^*}(S - V(C)) \leq \alpha_{G^*}(S) - \kappa_{G^*}(S) = \alpha_G(S) - (\kappa_G(S) + 1)$. Hence the corollary holds. □

Proof of Theorem 7. If G has a k-tree covering S, then the theorem holds. Hence we assume that G has no k-tree containing all vertices of S.

First, we consider $\kappa_G(S) = 1$. Choose a k-tree T of G so that

(T1) $|T \cap S|$ is as large as possible,

(T2) $Leaf(T) \subset S$ and $Leaf(T)$ is an independent set of G subject to (T1).

We say T is the desired k-tree of G, that is

$$\alpha_G(S - V(T)) \leq \alpha_G(S) - (k - 1)\kappa_G(S) - 1 = \alpha_G(S) - k.$$

Otherwise, $\alpha_G(S - V(T)) \geq \alpha_G(S) - k + 1$. Let $S_0 \subset S - V(T)$ be an independent set of G with $|S_0| = \alpha_G(S - V(T))$. By the choice of T, $|Leaf(T)| = l \geq k$. Suppose that $l \leq k - 1$. Then $\Delta(T) \leq |Leaf(T)| \leq k - 1$. By assumption, there exists a vertex $v \in S - V(T)$. Since G is a connected graph, there exists a path connecting v and T. We add this path to T and obtain a k-tree which contains more vertices of S than T, which contradicts the condition (T1). Therefore, the claim holds. We shall show that $X = Leaf(T) \cup S_0$ is an independent set of S in G. Let x and y are the vertices of X.

By the choice of S_0 (or the condition (T2)), if $x, y \in S_0$ (or $x, y \in Leaf(T)$), then x and y are not adjacent in G. Hence we assume that $x \in Leaf(T)$ and $y \in S_0$. If x and y are adjacent in G, then $T + xy$ is a k-tree of G which contains more vertices of S than T, which contradicts the condition (T1). Therefore, X is an independent set of S in G and $|X| = |Leaf(T)| + |S_0| = l + \alpha_G(S - V(T)) \geq \alpha_G(S) + 1$, contradiction. Therefore, the theorem holds for $\kappa_G(S) = 1$.

Next we consider $\kappa_G(S) \geq 2$. We prove this case by induction on k. For $k = 2$, by Corollary 1, the theorem holds. Assume the theorem holds for some $k = t \geq 2$, that is, there exists a t-tree T in G such that

$$\alpha_G(S - V(T)) \leq \alpha_G(S) - (t-1)\kappa_G(S) - 1.$$

Let $S_1 = S - V(T)$. By Lemma 2, there exists a cycle C such that $\alpha_G(S_1 - V(C)) \leq \alpha_G(S_1) - \kappa_G(S_1)$ or C covers S_1. By Lemma 1, there exists a tree T_1 such that

$$V(T) \cup V(C) \subseteq V(T_1), \quad \text{and} \quad \Delta(T_1) \leq \Delta(T) + 1 \leq t + 1.$$

If C covers S_1, then T_1 is a $(t+1)$-tree covering S, which contradicts the assumption. Hence, the cycle C satisfies $\alpha_G(S_1 - V(C)) \leq \alpha_G(S_1) - \kappa_G(S_1)$. Obviously, $\alpha_G(S-V(T_1)) \leq \alpha_G(S-(V(T) \cup V(C))) = \alpha_G(S-V(T)-V(C)) = \alpha_G(S_1-V(C))$. Hence

$$\begin{aligned}\alpha_G(S - V(T_1)) &\leq \alpha_G(S_1 - V(C)) \\ &\leq \alpha_G(S_1) - \kappa_G(S_1) \\ &\leq \alpha_G(S) - (t-1)\kappa_G(S) - 1 - \kappa_G(S_1) \\ &\leq \alpha_G(S) - t \cdot \kappa_G(S) - 1.\end{aligned}$$

Hence, the theorem holds for $k = t + 1$. Therefore, the theorem holds for all $k \geq 2$ by the principle of mathematical induction. □

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