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ORIGINAL PAPER

# Partial Degree Conditions and Cycle Coverings in Bipartite Graphs

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**Abstract** Let $k$ be a positive integer. Let $G$ be a graph of order $n \geq 3$ and $W$ a subset of $V(G)$ with $|W| \geq 3k$. Wang (J Graph Theory 78:295–304, 2015) proved that if $d(x) \geq 2n/3$ for each $x \in W$, then $G$ contains $k$ vertex-disjoint cycles such that each of them contains at least three vertices of $W$. In this paper, we obtain an analogue result of Wang's Theorem in bipartite graph with the partial degree condition. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n$, and let $W$ be a subset of $V_1$ with $|W| \geq 2k$, where $k$ is a positive integer. We show that if $d(x) + d(y) \geq n + k$ for every pair of nonadjacent vertices $x \in W$, $y \in V_2$, then $G$ contains $k$ vertex-disjoint cycles such that each of them contains at least two vertices of $W$.

**Keywords** Bipartite graph · Disjoint cycles · Partial degree

## 1 Introduction

We discuss only finite simple graphs and use standard terminology and notation from [3] except as indicated. Let $G = (V(G), E(G))$ be a graph. We use $E$ to denote the edge set of $G$ if there is no confusion. For a subgraph $H$ of $G$ and a vertex $x \in V(G)$, $N(x, H)$ stands for the set of neighbors of $x$ in $H$ and let $d(x, H) = |N(x, H)|$. The degree of $x$ in $G$ is briefly denoted by $d(x)$. For a subset $U$ of $V(G)$, $G[U]$ denotes the subgraph of $G$ induced by $U$. For disjoint vertex-sets $A$ and $B$, $G[A, B]$ is the bipartite subgraph on $A$ and $B$ with all the edges of $G$ between $A$ and $B$. A set of graphs is said

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to be disjoint if no two of them have any vertex in common. The minimum degree of $G$ is denoted by $\delta(G)$, and

$$\sigma_2(G) = \min\{d(x) + d(y) | x, y \in V(G), x \neq y, xy \notin E(G)\}$$

is the minimum degree sum of nonadjacent vertices (When $G$ is a complete graph, we define $\sigma_2(G) = \infty$). For a bipartite graph $G = (V_1, V_2; E)$, we define

$$\sigma_{1,1}(G) = \min\{d(x) + d(y) | x \in V_1, y \in V_2, xy \notin E(G)\}.$$

When $G$ is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$.

In 1952, Dirac [7] obtained the following classical result on hamiltonian graphs using a minimum degree condition: if $G$ is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then $G$ is hamiltonian. Ore [12] generalized the above result by using degree sum condition (Ore type condition) in 1960. He proved that if $G$ is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then $G$ is hamiltonian. Later, Moon and Moser [11] made the natural transition to bipartite graphs: if $G = (V_1, V_2; E)$ is a balanced bipartite graph of order $2n$ and $\sigma_{1,1}(G) \geq n + 1$, then $G$ is hamiltonian.

Let $W$ be a subset of $V(G)$, the set $W$ is called cyclable in $G$ if all vertices of $W$ belong to a common cycle in $G$. Similarly, we define $\delta(W)$ to be the minimum degree of $W$ in $G$ and define

$$\sigma_2(W, G) = \min\{d(x) + d(y) | x, y \in W, x \neq y, xy \notin E(G)\}$$

to be the minimum degree sum of nonadjacent vertices in $W$ (When $G[W]$ is a complete graph, we define $\sigma_2(W, G) = \infty$). For a bipartite graph $G = (V_1, V_2; E)$, let $W$ be a subset of $V_1$, we define

$$\sigma_{1,1}(W, G) = \min\{d(x) + d(y) | x \in W, y \in V_2, xy \notin E(G)\}.$$

When $G[W \cup V_2]$ is a complete bipartite graph, we define $\sigma_{1,1}(W, G) = \infty$.

Bollobás and Brightwell [2] considered partial degree condition for cyclable in graphs. They proved that if $G$ is a graph on $n$ vertices and $W$ is a subset of $V(G)$ with $|W| \geq 3$ and $\delta(W) \geq d$, then there is a cycle through at least $\lfloor \frac{|W|}{n/d-1} \rfloor$ vertices of $W$. When $d = n/2$, we have the following result, which is a generalization of Dirac's [7] result.

**Theorem 1.1** (Bollobás and Brightwell [2]) *Let $G$ be a graph of order $n$ and $W$ a subset of $V(G)$ with $|W| \geq 3$. If $\delta(W) \geq n/2$, then $W$ is cyclable.*

Analogously, Shi [13] generalized Ore's [12] result.

**Theorem 1.2** (Shi [13]) *Let $G$ be a 2-connected graph of order $n$ and $W$ a subset of $V(G)$ with $|W| \geq 3$. If $\sigma_2(W, G) \geq n$, then $W$ is cyclable in $G$.*

Later, Amar et al. [1] obtained a similar result for bipartite graphs:

**Theorem 1.3** (Amar et al. [1]) *Let $G = (V_1, V_2; E)$ be a 2-connected balanced bipartite graph of order $2n$ and $W$ a subset of $V_1$. If $\sigma_{1,1}(W, G) \geq n + 1$, then $W$ is cyclable in $G$.*

It is natural to ask that what is the degree condition and partial degree condition for disjoint cycles in graphs. In 1963, Corrádi and Hajnal [6] proved that every graph $G$ with $|V(G)| \geq 3k$ and $\delta(G) \geq 2k$ contains $k$ disjoint cycles. Later, Enomoto [8] and Wang [15] gave an Ore-type version, they proved that every graph $G$ with $|V(G)| \geq 3k$ and $\sigma_2(G) \geq 4k - 1$ contains $k$ disjoint cycles. In 1996, Wang [14] considered the bipartite graph, he proved that every bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = n > 2k$ and $\delta(G) \geq k + 1$ contains $k$ disjoint cycles. Recently, Wang [16] considered the partial degree condition for disjoint cycles.

**Theorem 1.4** (Wang [16]) *Let $G$ be a graph of order $n \geq 3$. Let $W$ be a subset of $V(G)$ with $|W| \geq 3k$, where $k$ is a positive integer. Suppose that $\delta(W) \geq 2n/3$. Then $G$ contains $k$ disjoint cycles such that each of the $k$ cycles contains at least three vertices of $W$.*

Naturally, can we consider the analogous problem on balanced bipartite graphs? We answer the question by proving the following theorem.

**Theorem 1.5** *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n$, and let $W$ be a subset of $V_1$ with $|W| \geq 2k$, where $k$ is a positive integer. If $\sigma_{1,1}(W, G) \geq n+k$, then $G$ contains $k$ disjoint cycles such that each of them contains at least two vertices of $W$.*

For other results on this topic, see [4,5,9,10].

*Remark 1* The following example shows that the degree condition in Theorem 1.5 is sharp when $k = 1$. Let $G = (V_1, V_2; E)$ be a balanced bipartite graph with $V_1 = \{u_1, \ldots, u_n\}$, $V_2 = \{v_1, \ldots, v_n\}$ and $E = \{u_1v_1\} \cup \{u_iv_j | i, j \geq 2\}$, and suppose $u_1, u_2 \in W$. Clearly, $G$ does not contain a desired cycle and $\sigma_{1,1}(W, G) = n$. For $k > 1$, the degree condition may be not sharp. But we can give an example to show that $\sigma_{1,1}(W, G) > n + \frac{\sqrt{16k+1}-1}{4}$ is necessary for the problem. Let $G = (V_1, V_2; E)$ be a balanced bipartite graph and let $W$ be a subset of $V_1$ with the following properties:

- $|V_1| = |V_2| = n = 2k + x$, $|W| = 2k$, where $k$ and $x$ are positive integers and $2k - 1$ is divisible by $x + 1$.
- Let $W = W_0 \cup W_1 \cup \cdots \cup W_{x+1}$, $|W_0| = 1$ and $|W_i| = \frac{2k-1}{x+1}$ for $1 \leq i \leq x + 1$.
- Let $U$ be a subset of $V_2$, and $U = U_1 \cup U_2 \cup \cdots \cup U_{x+1}$, $|U_i| = 1$ for $1 \leq i \leq x+1$.
- Each of $G[W_0, U]$, $G[W_i, U_i]$, $G[V_1 - W_0, V_2 - U]$ is a complete bipartite subgraph of $G$, where $1 \leq i \leq x + 1$.

Clearly, $\sigma_{1,1}(W, G) = \min\{n + x, n - x + \frac{2k-1}{x+1} + 1\}$. When $x = \frac{\sqrt{16k+1}-1}{4}$, we have $n + x = n - x + \frac{2k-1}{x+1} + 1$, and so $\sigma_{1,1}(W, G) = n + \frac{\sqrt{16k+1}-1}{4}$. From the construction of $G$, we observe that any cycle containing the special vertex in $W_0$ must contain at least three vertices of $W$. Note that $|W| = 2k$, $G$ does not contain $k$ disjoint cycles such that each of the $k$ cycles contains at least two vertices of $W$.

We propose the following problem:

**Problem 1.6** What is the best lower bound of $\sigma_{1,1}(W, G)$ to guarantee that $G$ contains $k$ disjoint cycles such that each of them contains at least two vertices of $W$?

Following [3], for a subgraph $H$ of $G$, define $G - H = G[V(G) - V(H)]$. Let $G_1$ and $G_2$ be subgraphs of $G$. The union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We denote by $E[G_1, G_2]$ the set of edges of $G$ with one end in $V(G_1)$ and the other end in $V(G_2)$, and by $e(G_1, G_2)$ their number. Clearly, $e(G_1, G_2) = \sum_{v \in G_i} d_{G_{3-i}}(v)$ for each $i = 1, 2$. If $H$ is a subgraph of $G$, written as $G \supseteq H$.

We use the following notation in this paper. The length of a cycle $C$ is denoted by $l(C)$. If $W$ is a subset of $V_1$, then the $W$-length of $C$ is the number of vertices of $C$ contained in $W$. We denote the $W$-length of $C$ by $l_W(C)$. Similarly, for a path $P$, we define $l(P)$ and $l_W(P)$ as above. If we write $C = x_1x_2 \cdots x_m x_1$, we assume that an orientation of $C$ is given such that $x_2$ is the successor of $x_1$ and operations in the subscripts of $x_i$'s will be taken modulo $m$ in $\{1, 2, \ldots, m\}$. Moreover, we use $x_i^+$ and $x_i^-$ to denote the successor and predecessor of $x_i$, respectively. We use $C[x_i, x_j]$ to represent the path of $C$ from $x_i$ to $x_j$ along the orientation of $C$. We adopt the notation $C(x_i, x_j] = C[x_i, x_j] - x_i$, $C[x_i, x_j) = C[x_i, x_j] - x_j$ and $C(x_i, x_j) = C[x_i, x_j] - x_i - x_j$, respectively. Moreover, we define $\overleftarrow{C}[x_i, x_j] = x_j x_{j-1} \cdots x_i$. Similarly, we define $P[x_i, x_j]$, $P(x_i, x_j]$, $P[x_i, x_j)$, $P(x_i, x_j)$ and $\overleftarrow{P}[x_i, x_j]$ as above.

The rest of the paper is organized as follows: we first present some useful lemmas in Sect. 2, and then prove the main theorem in Sect. 3.

## 2 Lemmas

In the following, $G = (V_1, V_2; E)$ is a balanced bipartite graph of order $2n$ and $W$ is a subset of $V_1$.

**Lemma 2.1** *Let $C$ be a cycle of $W$-length at least 2 and $l(C) \geq 6$. Let $x$ and $y$ be two distinct vertices of $G$ not on $C$. Then the following three statements hold:*

(1) *If $x \in W$ and $d(x, C) \geq 3$, then $G[V(C) \cup \{x\}]$ contains a cycle $C'$ such that $l(C') < l(C)$ and $l_W(C') \geq 2$.*
(2) *If $y \notin W$ and $d(y, C) \geq 5$, then $G[V(C) \cup \{y\}]$ contains a cycle $C'$ such that $l(C') < l(C)$ and $l_W(C') \geq 2$.*
(3) *If $x \in W$, $y \in V_2$, $xy \in E$ and $d(x, C) + d(y, C) \geq 5$, then $G[V(C) \cup \{x, y\}]$ contains a cycle $C'$ such that $l(C') < l(C)$ and $l_W(C') \geq 2$.*

*Proof* Let $C = x_1y_1x_2y_2 \cdots y_t x_1$ with $x_1 \in V_1$ and $t = l(C)/2$. First, we prove (1). We may assume $\{y_{i_1}, y_{i_2}, y_{i_3}\} \subseteq N(x, C)$ with $1 \leq i_1 < i_2 < i_3 \leq t$. As $l_W(C) \neq 0$, it follows that $V(C[y_{i_j}, y_{i_{j+1}}]) \cap W \neq \emptyset$ for some $j \in \{1, 2, 3\}$, without loss of generality, we say $j = 1$. Then the cycle $C' = xC[y_{i_1}, y_{i_2}]x$ satisfies the requirement.

Next, we prove (2). We may assume $\{z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}, z_{i_5}\} \subseteq N(y, C)$ with $i_j < i_{j+1}$ for each $1 \leq j \leq 4$, where $z_{i_j} = y_{i_j}$ if $y \in V_1$ and $z_{i_j} = x_{i_j}$ if $y \in V_2$. If $|V(C[z_{i_j}, z_{i_{j+3}}]) \cap W| \geq 2$ for some $j \in \{1, \ldots, 5\}$, then $C' = yC[z_{i_j}, z_{i_{j+3}}]y$ satisfies the requirement. Hence we may assume $|V(C[z_{i_j}, z_{i_{j+3}}]) \cap W| \leq 1$ for all $j \in \{1, \ldots, 5\}$. Since $l_W(C) \geq 2$ and $|V(C[z_{i_j}, z_{i_{j+3}}] \cup C[z_{i_{j+3}}, z_{i_{j+1}}]) \cap W| \leq 2$ for all $j \in \{1, \ldots, 5\}$, we have $V(C[z_{i_j}, z_{i_{j+1}}]) \cap W = \emptyset$ for all $j \in \{1, \ldots, 5\}$, it follows that $V(C) \cap W = \emptyset$, this is contrary to $l_W(C) \geq 2$.

Finally, we prove (3). By (1) and (2), we know if $d(x, C) \geq 3$ or $d(y, C) \geq 5$, we are done. So suppose that $d(x, C) \leq 2$ and $d(y, C) \leq 4$. Clearly, $1 \leq d(x, C) \leq 2$ as $d(x, C)+d(y, C) \geq 5$. Now we show that $G[V(C) \cup \{x, y\}]$ contains a cycle $C'$ satisfying the requirement. First we suppose that $d(x, C) = 1$. Thus $d(y, C) = 4$. We may assume $N(y, C) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$, where $i_j$ with ascending order, and $N(x, C) = \{y_j\}$ with $1 \leq j \leq t$. Without loss of generality, we say $y_j \in V(C[x_{i_1}, x_{i_2}])$. If $|V(C[y_j, x_{i_3}]) \cap W| \geq 1$ or $|V(C[x_{i_4}, y_j]) \cap W| \geq 1$, then $C' = xC[y_j, x_{i_3}]yx$ or $C' = yC[x_{i_4}, y_j]xy$. Otherwise, $|V(C[y_j, x_{i_3}]) \cap W| = |V(C[x_{i_4}, y_j]) \cap W| = 0$, then $|V(C[x_{i_3}, x_{i_4}]) \cap W| \geq 2$ as $l_W(C) \geq 2$, thus $C' = yC[x_{i_3}, x_{i_4}]y$.

Then suppose $d(x, C) = 2$. Thus $d(y, C) \geq 3$. We may assume $N(y, C) \supseteq \{x_{i_1}, x_{i_2}, x_{i_3}\}$ with $1 \leq i_1 < i_2 < i_3 \leq t$ and $N(x, C) = \{y_{j_1}, y_{j_2}\}$ with $1 \leq j_1 < j_2 \leq t$. Without loss of generality, we say $y_{j_1} \in V(C[x_{i_1}, x_{i_2}])$. First we show the case that $y_{j_2} \in V(C[x_{i_1}, x_{i_2}])$. If $|V(C[y_{j_1}, x_{i_2}]) \cap W| \geq 1$ or $|V(C[x_{i_3}, y_{j_1}]) \cap W| \geq 1$, then $C' = xC[y_{j_1}, x_{i_2}]yx$ or $C' = yC[x_{i_3}, y_{j_1}]xy$. Otherwise, $|V(C[y_{j_1}, x_{i_2}]) \cap W| = |V(C[x_{i_3}, y_{j_1}]) \cap W| = 0$, then $|V(C[x_{i_2}, x_{i_3}]) \cap W| \geq 2$ as $l_W(C) \geq 2$, thus $C' = yC[x_{i_2}, x_{i_3}]y$. Then we show the case that $y_{j_2} \notin V(C[x_{i_1}, x_{i_2}])$, by symmetry, say $y_{j_2} \in V(C[x_{i_2}, x_{i_3}])$. If one of $|V(C[x_{i_1}, y_{j_1}]) \cap W|$, $|V(C[y_{j_1}, x_{i_2}]) \cap W|$, $|V(C[x_{i_2}, y_{j_2}]) \cap W|$ and $|V(C[y_{j_2}, x_{i_3}]) \cap W|$ is at least 1, then one of the cycles $C' = yC[x_{i_1}, y_{j_1}]xy$, $C' = xC[y_{j_1}, x_{i_2}]yx$, $C' = yC[x_{i_2}, y_{j_2}]xy$ and $C' = xC[y_{j_2}, x_{i_3}]yx$ satisfies the requirement. Otherwise, $|V(C[x_{i_3}, x_{i_1}]) \cap W| \geq 2$ as $l_W(C) \geq 2$, thus $C' = yC[x_{i_3}, x_{i_1}]y$. □

**Lemma 2.2** [14] *Let $C$ be a quadrilateral and $P$ a path of order 4 in $G$ such that $P$ is disjoint from $C$ and $\sum_{x \in V(P)} d(x, C) \geq 6$. Then either $G[V(P \cup C)]$ contains two disjoint quadrilateral, or $P$ has an endvertex, say $z$, such that $d(z, C) = 0$.*

**Lemma 2.3** *Let $C$ be a cycle of $W$-length at least 2 with $l(C) \geq 4$. Let $x \in W$ and $y \in V_2$ be two distinct vertices of $G$ not on $C$ and $xy \notin E$. If $d(x, C) + d(y, C) \geq l(C)/2 + 2$, then $G[V(C) \cup \{x, y\}]$ contains a cycle $C'$ such that $l(C') < l(C)$ and $l_W(C') \geq 2$ or $l(C) = 4$ and $d(x, C) = d(y, C) = 2$.*

*Proof* By Lemma 2.1 (1), (2), if $d(x, C) \geq 3$ or $d(y, C) \geq 5$, we are done. Thus $d(x, C) \leq 2$ and $d(y, C) \leq 4$. Note that $d(x, C) + d(y, C) \geq l(C)/2 + 2$, we have $l(C) \leq 8$, i.e., $l(C) = 4, 6, 8$. Clearly, $d(x, C) = d(y, C) = 2$ if $l(C) = 4$. Now we consider the case $l(C) \neq 4$. Note that $d(y, C) \geq l(C)/2 + 2 - 2 = l(C)/2$. It is easy to see that $G[V(C) \cup \{y\}]$ contains a cycle $C'$ such that $l(C') < l(C)$ and $l_W(C') \geq 2$. □

**Lemma 2.4** [14] *Let $t$ and $s$ be two integers such that $t \geq s \geq 2$ and $t \geq 3$. Let $C_1$ and $C_2$ be two disjoint cycles of $G$ with lengths $2t$ and $2s$, respectively. Suppose that $\sum_{x \in V(C_1)} d(x, C_2) \geq 2t + 1$. Then $G[V(C_1 \cup C_2)]$ contains two disjoint cycles $C_1'$ and $C_2'$ such that $l(C_1') + l(C_2') < 2s + 2t$.*

**Lemma 2.5** *Let $t$ and $s$ be two integers such that $t \geq s \geq 2$ and $t \geq 3$. Let $C_1$ and $C_2$ be two disjoint cycles of $G$ such that $l_W(C_1) = t$, $l_W(C_2) = s$. Suppose* *that* $\sum_{x\in V(C_1)\cap W}(d(x, C_2) + d(x^+, C_2)) \geq tl(C_2)/2 + 1$. *Then* $G[V(C_1 \cup C_2)]$ *contains two disjoint cycles* $C_1'$ *and* $C_2'$ *such that* $l_W(C_1') \geq 2, l_W(C_2') \geq 2$ *and* $l(C_1') + l(C_2') < l(C_1) + l(C_2)$.

*Proof* Suppose, for a contradiction, that the lemma fails. Let $s, t, G, W, C_1$ and $C_2$ be chosen with $l(C_1) + l(C_2)$ as small as possible such that the lemma fails for $C_1$ and $C_2$ while the conditions of the lemma are fulfilled. By Lemma 2.4, we see that $V(C_1 \cup C_2) \cap V_1 \nsubseteq W$. Thus $l(C_1) + l(C_2) > 2s + 2t$. First we claim that

$$l(C_1) = 2t. \tag{1}$$

*Proof of (1).* If this is not true, then there exists a vertex $x \in V_1 \cap V(C_1)$ such that $x \notin W$. Clearly, $x^+, x^- \notin W$. Let $G' = G - x - x^+ + x^-x^{++}$, $C_1' = C_1 - x - x^+ + x^-x^{++}$. Obviously, $l_W(C_1') = l_W(C_1)$ and $l(C_1') = l(C_1) - 2$. And we also have $\sum_{x\in V(C_1')\cap W}(d(x, C_2) + d(x^+, C_2)) \geq tl(C_2)/2 + 1$ in $G'$. Thus by the minimality of $l(C_1) + l(C_2)$, the lemma holds for $C_1'$ and $C_2$, that is, $G'[V(C_1' \cup C_2)]$ contains two disjoint cycles $Q'$ and $Q''$ such that $l_W(Q') \geq 2, l_W(Q'') \geq 2$ and $l(Q') + l(Q'') < l(C_1') + l(C_2)$. If $x^-x^{++} \notin E(Q' \cup Q'')$, then $Q'$ and $Q''$ are the two required cycles in $G[V(C_1 \cup C_2)]$. If $x^-x^{++} \in E(Q' \cup Q'')$, then we readily obtain the two required disjoint cycles of $G[V(C_1 \cup C_2)]$ from $Q'$ and $Q''$ by replacing the edge $x^-x^{++}$ with the path $x^-xx^+x^{++}$, a contradiction. Hence $l(C_1) = 2t$. □

Then we claim that the following (2) and (3) hold.

$$\text{For each } v \in V(C_2) \cap V_1 \quad \text{with} \quad |V(C_2) \cap W - v| \geq 2, \quad d(v, C_1) + d(v^+, C_1) > t. \tag{2}$$

$$\text{For each } v \in V(C_1) \cap V_1 \quad \text{with} \quad |V(C_1) \cap W - v| \geq 3, \quad \text{if } t - 1 \geq s, \quad d(v, C_2) + d(v^+, C_2) > l(C_2)/2. \tag{3}$$

*Proofs of (2) and (3).* We only need to show that for each $v \in V(C_i) \cap V_1$ with $|V(C_i) \cap W - v| \geq 4 - i$, we have $d(v, C_{3-i}) + d(v^+, C_{3-i}) > l(C_{3-i})/2$, where $i = 1, 2$. On the contrary, assume that $d(v, C_{3-i}) + d(v^+, C_{3-i}) \leq l(C_{3-i})/2$ for some $v \in V(C_i) \cap V_1$ with $|V(C_i) \cap W - v| \geq 4 - i$. Let $G' = G - v - v^+ + v^-v^{++}$, $C_i' = C_i - v - v^+ + v^-v^{++}$. Obviously, $l_W(C_i') \geq 4 - i$ and $l(C_i') = l(C_i) - 2$. If $i = 2$, then $\sum_{x\in V(C_1)\cap W}(d(x, C_2') + d(x^+, C_2')) \geq tl(C_2)/2 + 1 - t = tl(C_2')/2 + 1$ in $G'$. If $i = 1$, then $\sum_{x\in V(C_1')\cap W}(d(x, C_2) + d(x^+, C_2)) \geq tl(C_2)/2 + 1 - l(C_2)/2 = (t - 1)l(C_2)/2 + 1$ in $G'$. Both of the above cases satisfy the condition of Lemma 2.5. By the minimality of $l(C_1) + l(C_2)$, the lemma holds for $C_i'$ and $C_{3-i}$. By the similar argument of (1), (2) and (3) hold. □

By (1) and $l(C_1) + l(C_2) > 2s + 2t$, we know $l(C_2) > 2s$. Let $C_1 = x_1x_1^+ \cdots x_tx_t^+x_1$ and $C_2 = y_1y_1^+ \cdots y_my_m^+y_1$ with $l(C_2) = 2m$ and $x_1, y_1 \in V_1$. Clearly, $m \geq 3$. We claim that

$$s = 2. \tag{4}$$

*Proof of (4).* On the contrary, suppose $s \geq 3$. Thus for each $y \in V(C_2) \cap V_1$, we have $|V(C_2) \cap W - y| \geq 2$, so we see $d(y, C_1) + d(y^+, C_1) > t$ by (2). Note that $l(C_2) > 2s$, there exists a $y \in V(C_2) \cap V_1$ such that $y \notin W$. Let $G' = G - y - y^+ + y^- y^{++}$, $C_2' = C_2 - y - y^+ + y^- y^{++}$. Obviously, $l_W(C_2') \geq 2$, $l(C_2') = l(C_2) - 2$ and $\sum_{x \in V(C_1)} d(x, C_2') > tl(C_2')/2$ in $G'$. By the minimality of $l(C_1) + l(C_2)$, the lemma holds for $C_1$ and $C_2'$. By the similar argument of (1), the Eq. (4) holds. □

$$t = 3. \tag{5}$$

*Proof of (5).* On the contrary, suppose $t \geq 4$. Thus for each $x \in V(C_1) \cap V_1$, we have $|V(C_1) \cap W - x| \geq 3$ and $t - 1 \geq 3 \geq s$, so we see $d(x, C_2) + d(x^+, C_2) > l(C_2)/2$ by (3). For some vertex $x \in V(C_1) \cap V_1$, let $G' = G - x - x^+ + x^- x^{++}$, $C_1' = C_1 - x - x^+ + x^- x^{++}$. Obviously, $l_W(C_1') = t - 1 \geq 3$ and $l(C_1') = l(C_1) - 2$. And we also have $\sum_{x \in V(C_1') \cap W} (d(x, C_2) + d(x^+, C_2)) > (t-1)l(C_2)/2$ in $G'$. Thus by the minimality of $l(C_1) + l(C_2)$, the lemma holds for $C_1'$ and $C_2$. By the similar argument of (1), the Eq. (5) holds. □

$$\text{For each } y \in V_1 \cap V(C_2), \quad \text{if } y \notin W, \text{ then } d(y^+, C_1) + |N(y, C_1) \cap N(y^{++}, C_1)| \geq 4. \tag{6}$$

*Proof of (6).* On the contrary, suppose that $d(y^+, C_1) + |N(y, C_1) \cap N(y^{++}, C_1)| \leq 3$ for some $y \in V_1 \cap V(C_2)$ and $y \notin W$. We identify $y$, $y^+$ and $y^{++}$ as a new vertex $y_0$, obtaining a new graph $G'$ where the neighborhood of $y_0$ contains all the neighbors of $y$ and $y^{++}$ except $y^+$. Then $C_2$ becomes a new cycle $C_2' = C_2 - y - y^+ - y^{++} + y_0 + y_0 y^- + y_0 y^{+++}$ with $l(C_2') = l(C_2) - 2$ and $l_W(C_2) = l_W(C_2')$ ($y_0 \in W$ if $y^{++} \in W$, otherwise $y_0 \notin W$). Note that $\sum_{x \in V(C_1)} d(x, C_2) \geq tl(C_2)/2 + 1$. By (5) and $d(y^+, C_1) + |N(y, C_1) \cap N(y^{++}, C_1)| \leq 3$, we obtain $\sum_{x \in V(C_1)} d(x, C_2') \geq 3l(C_2)/2 + 1 - 3 = 3l(C_2')/2 + 1$ in $G'$. Thus by the minimality of $l(C_1) + l(C_2)$, the lemma holds for $C_1$ and $C_2'$, that is, $G'[V(C_1 \cup C_2')]$ contains two disjoint cycles $Q'$ and $Q''$ such that $l_W(Q') \geq 2$, $l_W(Q'') \geq 2$ and $l(Q') + l(Q'') < l(C_1) + l(C_2')$. If $y_0 \notin V(Q' \cup Q'')$, then $Q'$ and $Q''$ are the two required cycles. Then we may assume that $y_0 \in V(Q' \cup Q'')$. By symmetry, say $y_0 \in V(Q')$. Let $u y_0 v$ be a path of $Q'$. If $\{u, v\} \subseteq N(y', Q')$ for some $y' \in \{y, y^{++}\}$, then we readily obtain the two required disjoint cycles of $G[V(C_1 \cup C_2)]$ from $Q'$ and $Q''$ by replacing the vertex $y_0$ with $y'$, a contradiction. Otherwise, we readily obtain the two required disjoint cycles of $G[V(C_1 \cup C_2)]$ from $Q'$ and $Q''$ by replacing the vertex $y_0$ with the path $yy^+y^{++}$, a contradiction. □

By the similar argument of (6) (identify $y^-$, $y^+$ and $y$ as a new vertex $y_0$, obtaining a new graph $G'$ where the neighborhood of $y_0$ contains all the neighbors of $y^-$ and $y^+$ except $y$), we have the following statement:

$$\text{for each } y \in V_1 \cap V(C_2), \quad \text{if } y \notin W, \text{ then } d(y, C_1) + |N(y^-, C_1) \cap N(y^+, C_1)| \geq 4. \tag{7}$$

By (5), $C_1 = x_1x_1^+x_2x_2^+x_3x_3^+x_1$. Note that $C_2 = y_1y_1^+ \cdots y_my_m^+y_1$, where $m \geq 3$. By (4), there exists a vertex $y \in V_1 \cap V(C_2)$ such that $y \notin W$. Choose $y$ such that $|\{y, y^{++}\} \cap W|$ is minimum. We may assume $y = y_1$. According to (6) and (7), we find $d(y_1^+, C_1) + |N(y_1, C_1) \cap N(y_2, C_1)| \geq 4$ and $d(y_1, C_1) + |N(y_m^+, C_1) \cap N(y_1^+, C_1)| \geq 4$. Clearly, $d(y_1^+, C_1) \geq 1$.

If $d(y_1^+, C_1) = 3$, then $|N(y_1, C_1) \cap N(y_2, C_1)| \geq 1$. Assume $y_1x_3^+, y_2x_3^+ \in E$. Also, if $d(y_1^+, C_1) = 2$, then $|N(y_1, C_1) \cap N(y_2, C_1)| \geq 2$. Assume $y_1^+x_2, y_1^+x_3 \in E$. Obviously, $|N(y_1, C_1) \cap N(y_2, C_1) - x_2^+| \geq 1$, by symmetry, we may assume $y_1x_3^+, y_2x_3^+ \in E$. Then in both cases $G[V(C_1 \cup C_2)]$ contains two required disjoint cycles $C_1'$ and $C_2'$ with $C_1' = y_1^+x_2x_2^+x_3y_1^+$ and $C_2' = y_1x_3^+C_2[y_2, y_m^+]y_1$, where $l(C_1') = 4$ and $l(C_2') = 2m$, a contradiction. Thus $d(y_1^+, C_1) = 1$ and $d(y_1, C_1) = d(y_2, C_1) = 3$, say $y_1^+x_3 \in E$. Since $d(y_1, C_1) + |N(y_m^+, C_1) \cap N(y_1^+, C_1)| \geq 4$, we have $|N(y_m^+, C_1) \cap N(y_1^+, C_1)| \geq 1$. Then $y_m^+x_3 \in E$.

If $y_2 \notin W$, then $y_1^+, y_2, y_2^+$ satisfy (7), thus we have $|N(y_1^+, C_1) \cap N(y_2^+, C_1)| \geq 1$, hence $y_2^+x_3 \in E$. It follows that $G[V(C_1 \cup C_2)]$ contains two required disjoint cycles $C_1'$ and $C_2'$ with $C_1' = y_1x_2^+x_2x_1^+x_1x_3^+y_1$ and $C_2' = x_3C_2[y_2^+, y_m^+]x_3$, where $l(C_1') = 6$ and $l(C_2') = 2m - 2$, a contradiction. Thus $y_2 \in W$. By the choice of $y$, we find $y_m \in W$.

First suppose $m \geq 4$. Then $y_{m-1} \notin W$, and thus $y_{m-1}, y_{m-1}^+, y_m$ satisfy (6). Clearly, $d(y_{m-1}^+, C_1) \geq 1$. Since $x_1$ and $x_2$ are symmetric, we only need to consider the case $y_{m-1}^+x_2 \in E$ or $y_{m-1}^+x_3 \in E$. Let $C_1' = y_2x_1^+x_1x_3^+y_2$. Then $C_2' = y_{m-1}^+x_2x_2^+x_3y_m^+y_my_{m-1}^+$ or $C_2' = y_{m-1}^+x_3y_m^+y_my_{m-1}^+$. Clearly, $l_W(C_1') \geq 2$, $l_W(C_2') \geq 2$ and $l(C_1') + l(C_2') < l(C_1) + l(C_2)$, a contradiction.

Then suppose $m = 3$. Since $\sum_{x \in V(C_1)} d(x, C_2) \geq tl(C_2)/2 + 1 = 10$, $d(y_1^+, C_1) = 1$, and $d(y_1, C_1) = d(y_2, C_1) = 3$, we have $d(y_3, C_1) + d(y_3^+, C_1) + d(y_2^+, C_1) \geq 3$. If $d(y_k^+, C_1) \geq 2$ for some $k \in \{2, 3\}$, as $x_1 = x_{3+1}$, we say $\{x_i, x_{i+1}\} \subseteq N(y_k^+, C_1)$. This implies that $G[V(C_1) \cup \{y_2, y_k^+\}]$ contains two disjoint cycles $C_1'$ and $C_2'$ such that $C_1' = y_k^+x_ix_i^+x_{i+1}y_k^+$ and $C_2' = y_2x_{i+1}^+x_{i+2}x_{i+2}^+y_2$, where $l_W(C_1') \geq 2$, $l_W(C_2') \geq 2$, a contradiction. Thus $d(y_k^+, C_1) \leq 1$ for all $k \in \{2, 3\}$, so we have $d(y_3, C_1) \geq 1$. Say $y_3x_i^+ \in E$ for some $i \in \{1, 2, 3\}$. If $i \in \{2, 3\}$, then $G[V(C_1 \cup C_2)]$ contains two disjoint cycles $y_3x_i^+x_3y_3^+y_3$ and $y_2x_{i+1}^+x_{i+2}x_{i+2}^+y_2$, again a contradiction. Thus $i = 1$, and $d(y_k^+, C_1) = 1$ for all $k \in \{2, 3\}$. Then $y_2^+x_j \in E$ for some $j \in \{1, 2, 3\}$. It follows that $G[V(C_1 \cup C_2)]$ contains two disjoint cycles $y_3x_1^+x_jy_2^+y_3$ and $y_2x_2^+x_3x_3^+y_2$ if $j \in \{1, 2\}$ and $G[V(C_1 \cup C_2)]$ contains two disjoint cycles $y_3y_3^+x_3y_2^+y_3$ and $y_2x_1^+x_2x_2^+y_2$ if $j = 3$, a contradiction. □

## 3 Proof of Theorem 1.5

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n$, and let $W$ be a subset of $V_1$ with $|W| \geq 2k$, and $d(x) + d(y) \geq n + k$ for all $x \in W$, $y \in V_2$ with $xy \notin E$, where $k$ is a positive integer. Suppose, for a contradiction, that $G$ does not contain $k$ disjoint cycles of $W$-length at least 2. We may assume that $G + xy$ contains $k$ disjoint cycles of $W$-length at least 2 for each pair of nonadjacent vertices $x \in V_1$ and $y \in V_2$ of $G$. Thus $G$ contains $k - 1$ disjoint cycles $C_1, \ldots, C_{k-1}$ of $W$-length at least 2. We choose such a set of cycles $C_1, \ldots, C_{k-1}$ that

$$\sum_{i=1}^{k-1} l(C_i) \text{ is minimum.} \tag{8}$$

Subject to (8), we choose $C_1, \ldots, C_{k-1}$ such that

$$\sum_{i=1}^{k-1} l_W(C_i) \text{ is minimum.} \tag{9}$$

Subject to (8) and (9), we choose $C_1, \ldots, C_{k-1}$ and a path $P$ in $G - V(\bigcup_{i=1}^{k-1} C_i)$ such that

$$|V(P) \cap W| \text{ is maximum.} \tag{10}$$

Subject to (8), (9) and (10), we finally choose $C_1, \ldots, C_{k-1}$ and $P$ such that

$$l(P) \text{ is minimum.} \tag{11}$$

Set $H = \bigcup_{i=1}^{k-1} C_i$, $D = G - V(H)$, $W_0 = W \cap V(D)$ and $|V(D)| = 2d$. Let $D_0 = D - V(P)$ and $P = x_1 x_2 \ldots x_{2p+1}$. By (11), $\{x_1, x_{2p+1}\} \subseteq W_0$.

**Claim 3.1** $l_W(C_i) = 2$ for all $i \in \{1, 2, \ldots k-1\}$.

*Proof* On the contrary, suppose that Claim 3.1 fails. We may assume $l_W(C_1) \geq l_W(C_i)$ for all $i \in \{1, 2, \ldots, k-1\}$. Then $l_W(C_1) \geq 3$. Set $t = l_W(C_1)$. We may assume $V(C_1) \cap W = \{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$, where $i_j < i_{j+1}$ for each $1 \leq j \leq t-1$. Let $L_1 = \{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$ and $L_2 = \{u_{i_1}^+, u_{i_2}^+, \ldots, u_{i_t}^+\}$. First we claim that

$$N(u_{i_j}, D) \cap N(u_{i_k}, D) = \emptyset \quad \text{for each} \quad j \neq k. \tag{12}$$

$$N(u_{i_j}^+, D) \cap N(u_{i_k}^+, D) = \emptyset \quad \text{for each} \quad j \neq k. \tag{13}$$

In fact, if there exists a pair $j, k$ such that $N(u_{i_j}, D) \cap N(u_{i_k}, D) \neq \emptyset$, we may assume $u_{i_j} u, u_{i_k} u \in E$, where $u \in V(D)$, then we replace $C_1$ with $C_1' = uC_1[u_{i_p}, u_{i_q}]u$ if $|C_1[u_{i_p}, u_{i_q}]| \leq |C_1[u_{i_q}, u_{i_p}]|$ with $\{p, q\} = \{j, k\}$, where $l(C_1') < l(C_1)$, this is contrary to (8). And if there exists a pair $j, k$ such that $N(u_{i_j}^+, D) \cap N(u_{i_k}^+, D) \neq \emptyset$, we may assume $u_{i_j}^+ u, u_{i_k}^+ u \in E$. If $u \in W_0$, then we replace $C_1$ with $C_1' = uC_1[u_{i_p}^+, u_{i_q}^+]u$ if $|C_1[u_{i_p}^+, u_{i_q}^+]| \leq |C_1[u_{i_q}^+, u_{i_p}^+]|$ with $\{p, q\} = \{j, k\}$, where $l(C_1') < l(C_1)$, contradicting (8). If $u \notin W_0$, then we replace $C_1$ with $C_1' = uC_1[u_{i_q}^+, u_{i_p}^+]u$ if $l_W(C_1[u_{i_p}^+, u_{i_q}^+]) \leq l_W(C_1[u_{i_q}^+, u_{i_p}^+])$ with $\{p, q\} = \{j, k\}$, where $l(C_1') \leq l(C_1)$ and $2 \leq l_W(C_1') < l_W(C_1)$ as $l_W(C_1) \geq 3$, contradicting (8) or (9).

Thus we have $\sum_{u \in L_1} d(u, D) \leq d$ and $\sum_{u \in L_2} d(u, D) \leq d$ according to (12) and (13). By (8), it is easy to see that $d(u_{i_a}, G[V(C_1)]) = d(u_{i_a}^+, G[V(C_1)]) = 2$ and $u_{i_a} u_{i_{a+1}}^+ \notin E$ for each $1 \leq a \leq t$. So we have $\sum_{u \in L_1 + L_2} d(u, G[V(D \cup C_1)]) \leq$

$2d + 4t$ and $\sum_{u \in L_1+L_2} d(u) \geq t(n+k)$. Then we have $\sum_{u \in L_1+L_2} d(u, H - C_1) \geq t(n+k) - 2d - 4t \geq t\sum_{i=2}^{k-1} l(C_i)/2 + t(k-1) + (t-2)d \geq t\sum_{i=2}^{k-1} l(C_i)/2 + 1$ as $l(C_1) \geq 6$ and $t \geq 3$. This implies that there exists a cycle $C_i \in H - C_1$, say $C_2$, such that $\sum_{u \in L_1+L_2} d(u, C_2) \geq tl(C_2)/2 + 1$. By Lemma 2.5, $G[V(C_1 \cup C_2)]$ contains two disjoint cycles $C_1'$ and $C_2'$ such that $l_W(C_1') \geq 2, l_W(C_2') \geq 2$ and $l(C_1') + l(C_2') < l(C_1) + l(C_2)$, contradicting (8). □

By Claim 3.1, we observe that $|V(P)| \geq 1$. If $|V(P)| = 1$, we say $P = x_{2p+1}$.

**Claim 3.2** $d(x_{2p+1}, P) \leq 1$, and if $x_1$ exists, then $d(x_1, P) \leq 1$.

*Proof* On the contrary, suppose $d(x_{2p+1}, P) \geq 2$, we may assume $\{x_{2i}, x_{2p}\} \subseteq N(x_{2p+1}, P)$. Note that $D$ does not contain a cycle with $W$-length at least two, $l_W(P[x_{2i}, x_{2p}]) = 0$. We obtain a short path by replacing $P$ with $P' = P[x_1, x_{2i}]x_{2p+1}$, this contradicts (11) while (8)–(10) hold. By symmetry, it is easy to see if $x_1$ exists, then $d(x_1, P) \leq 1$. □

**Claim 3.3** We can choose $D_0$ such that $d(x_{2p+1}, D_0) \neq 0$, and if $|D_0| \geq 2$ and $x_1$ exists, then $d(x_1, D_0) \neq 0$.

*Proof* Suppose that $d(x_{2p+1}, D_0) = 0$, then there exists a $y \in V_2 \cap D_0$ such that $x_{2p+1}y \notin E$. By Claim 3.2, $d(x_{2p+1}, P) \leq 1$. Thus $d(x_{2p+1}, D) + d(y, D) \leq 1 + d - 1 = d$. Since $x_{2p+1}y \notin E$ and $x_{2p+1} \in W$, we have $d(x_{2p+1}, H) + d(y, H) \geq n + k - d = \sum_{i=1}^{k-1} l(C_i)/2 + (k-1) + 1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $d(x_{2p+1}, C_1) + d(y, C_1) \geq l(C_1)/2 + 2$. By Lemma 2.3 and (8), we have $l(C_1) = 4$ and $d(x_{2p+1}, C_1) = d(y, C_1) = 2$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. Then we find a $D_0$ such that $d(x_{2p+1}, D_0) \neq 0$ by replacing $C_1$ and $D_0$ with $C_1' = yu_1u_2u_3y$ and $D_0' = D_0 - y + u_4$. We may assume $x_{2p+1}y \in E$. If $|V(D_0)| \geq 2$, then $|V(D_0) - y| \geq 1$. By a similar argument( replacing $x_{2p+1}$ with $x_1$), we can find a $D_0$ such that $d(x_1, D_0 - y) \neq 0$ and $d(x_{2p+1}, D_0) \neq 0$. □

Let $D_0$ be chosen satisfying Claim 3.3, so there exists a vertex in $D_0 \cap V_2$, say $y$, such that

$$x_{2p+1}y \in E. \tag{14}$$

**Claim 3.4** $V(P) \supseteq W_0$.

*Proof* On the contrary, suppose $V(P) \not\supseteq W_0$. Let $x_0 \in W_0 \cap V(D_0)$. According to (10) and (14), $x_0y \notin E$. First we claim that $d(x_0, P) + d(y, P) \leq p + 1$. Otherwise, $d(x_0, P) + d(y, P) \geq p + 2$, i.e., $d(x_0, P - x_{2p+1}) + d(y, P - x_{2p+1}) \geq p + 1$, then $x_{2i-1}y, x_{2i}x_0 \in E$ for some $1 \leq i \leq p$. Let $P' = P[x_1, x_{2i-1}]y\overleftarrow{P}[x_{2i}, x_{2p+1}]x_0$. Obviously, $l_W(P') > l_W(P)$, this is contrary to (10).

We divide the proof of the claim into two cases.

**Case 1.** $d(x_0, D_0 - y) = 0$.

By the claim above, we have $d(x_0, P) + d(y, P) \leq p + 1$. Thus $d(x_0, D) + d(y, D) \leq p + 1 + d - p - 2 = d - 1$ as $x_0y \notin E$. Since $x_0y \notin E$ and $x_0 \in W$, we have $d(x_0, H) + d(y, H) \geq n + k - (d-1) = \sum_{i=1}^{k-1} l(C_i)/2 + k + 1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $d(x_0, C_1) + d(y, C_1) \geq l(C_1)/2 + 2$.

By Lemma 2.3 and (8), we have $l(C_1) = 4$ and $d(x_0, C_1) = d(y, C_1) = 2$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. We replace $C_1$ and $P$ with $C_1' = x_0u_2u_3u_4x_0$ and $P' = Pyu_1$, then $l_W(P') > l_W(P)$, this contradicts (10) while (8)–(9) hold.

**Case 2.** $d(x_0, D_0 - y) \neq 0$, i.e., there exists a vertex $y_0 \in V(D_0 - y) \cap V_2$ such that $x_0y_0 \in E$.

By (10), we see $N(x_{2p+1}, D_0) \cap N(x_0, D_0) = \emptyset$, $N(y, D_0) \cap N(y_0, D_0) = \emptyset$. In particular, we have $x_{2p+1}y_0, x_0y \notin E$. Set $L = \{x_{2p+1}, y, x_0, y_0\}$. Thus we have $\sum_{x\in L} d(x, D_0) \leq d - p + d - p - 1 = 2d - 2p - 1$ and $\sum_{x\in L} d(x) \geq 2(n + k)$. Recall that $d(x_0, P) + d(y, P) \leq p + 1$, and $d(x_{2p+1}, P) \leq 1$ by Claim 3.2, we have $\sum_{x\in L} d(x, D) \leq 2d - 2p - 1 + 1 + p + p + 1 = 2d + 1$. Thus $\sum_{x\in L} d(x, H) \geq 2(n + k) - 2d - 1 = \sum_{i=1}^{k-1} l(C_i) + 2(k - 1) + 1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $\sum_{x\in L} d(x, C_1) \geq l(C_1) + 3$. Then $d(x_{2p+1}, C_1) + d(y_0, C_1) \geq l(C_1)/2 + 2$ or $d(x_0, C_1) + d(y, C_1) \geq l(C_1)/2 + 2$. By Lemma 2.3 and (8), we have $l(C_1) = 4$. Thus $\sum_{x\in L} d(x, C_1) \geq 7$. It follows that $d(x_{2p+1}, C_1) + d(x_0, C_1) \geq 3$ and $d(y', C_1) = 2$ for some $y' \in \{y, y_0\}$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. We may assume $x_{2p+1}u_4, x_0u_4 \in E$ as $d(x_{2p+1}, C_1) + d(x_0, C_1) \geq 3$. Then $G[V(P \cup C_1) \cup \{y, x_0, y_0\}]$ contains a cycle $C_1'$ and a path $P'$ such that $C_1' = y'u_1u_2u_3y'$ and $P' = Pu_4x_0$, where $l_W(P') > l_W(P)$, a contradiction. □

**Claim 3.5** $|W_0| \leq 2$.

*Proof* On the contrary, suppose $|W_0| \geq 3$. Say $\{x_1, x_{2a+1}, x_{2p+1}\} \subseteq W_0$, where $1 < a < p$ and $V(P(x_{2a+1}, x_{2p+1})) \cap W = \emptyset$. By our assumption, we know $D$ does not contain a cycle of $W$-length at least 2, thus we have $d(y, P[x_1, x_{2a+1}]) = 0$ as $x_{2p+1}y \in E$, and $|N(x_{2p+1}, D_0) \cap N(x_{2a+1}, D_0)| = |N(y, D_0) \cap N(x_{2a}, D_0)| = 0$. In particular, $x_{2a+1}y, x_{2p+1}x_{2a} \notin E$. According to (11) and Claim 3.2, we have $d(x_{2a+1}, P(x_{2a+2}, x_{2p}]) = 0$ and $d(x_{2p+1}, P) \leq 1$. Set $L = \{x_{2p+1}, y, x_{2a+1}, x_{2a}\}$. We see $\sum_{x\in L} d(x, D) \leq d - p + d - p - 1 + 1 + p - a + a + 1 + p = 2d + 1$. Hence $\sum_{x\in L} d(x, H) \geq 2(n + k) - 2d - 1 = \sum_{i=1}^{k-1} l(C_i) + 2(k - 1) + 1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $\sum_{x\in L} d(x, C_1) \geq l(C_1) + 3$. On the other hand, by Lemma 2.1 (3) and (8), we obtain $\sum_{x\in L} d(x, C_1) \leq 4 + 4 = 8$. Then $l(C_1) + 3 \leq 8$, i.e., $l(C_1) = 4$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. Thus $\sum_{x\in L} d(x, C_1) \geq 7$. It follows that $d(x_{2p+1}, C_1) + d(x_{2a+1}, C_1) \geq 3$ and $d(y', C_1) = 2$ for some $y' \in \{y, x_{2a}\}$. We may assume $x_{2p+1}u_2, x_{2a+1}u_2 \in E$. It follows that $G[V(P \cup C_1) \cup \{y\}]$ contains two disjoint cycles $C_1'$ and $C'$ such that $C_1' = y'u_1u_4u_3y'$ and $C' = u_2P[x_{2a+1}, x_{2p+1}]u_2$, where $l_W(C') \geq 2$, a contradiction. □

By Claims 3.1, 3.5 and $|W| \geq 2k$, we have $|W| = 2k$ and $|W_0| = 2$. Then $W_0 = \{x_1, x_{2p+1}\}$. By our assumption and (14), $x_1y \notin E$. By Claim 3.3, we know if $|D_0| \geq 2$, then $d(x_1, D_0) \neq 0$. We divide the proof of the theorem into two cases.

**Case 1.** $|D_0| \geq 2$, then there exists a $y_1 \in V(D_0 - y) \cap V_2$, such that $x_1y_1 \in E$.

By our assumption, we know $D$ does not contain a cycle $C$ such that $l_W(C) \geq 2$. Then we have $d(y, D_0) + d(y_1, D_0) \leq d - p - 1$, $d(x_1, D_0) + d(x_{2p+1}, D_0) \leq d - p$ and $x_1y, x_{2p+1}y_1 \notin E$. Thus, we also have $d(y, P) \leq p$, $d(y_1, P) \leq p$. Set $L = \{x_1, y_1, x_{2p+1}, y\}$. Then $\sum_{x\in L} d(x, D_0) \leq 2d - 2p - 1$. By Claim 3.2, we have $d(x_1, P) = d(x_{2p+1}, P) = 1$. Then $\sum_{x\in L} d(x, D) \leq 2d - 2p - 1 + 2p + 2 = 2d + 1$.

Thus $\sum_{x\in L} d(x, H) \geq 2(n+k) - 2d - 1 = \sum_{i=1}^{k-1} l(C_i) + 2(k-1) + 1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $\sum_{x\in L} d(x, C_1) \geq l(C_1)+3$. On the other hand, by Lemma 2.1 (3) and (8), we obtain $\sum_{x\in L} d(x, C_1) \leq 4+4 = 8$. Then $l(C_1)+3 \leq 8$, i.e., $l(C_1) = 4$. Then $\sum_{x\in L} d(x, C_1) \geq 7$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. Clearly, $d(y', C_1) = 2$ for some $y' \in \{y, y_1\}$ and $d(x_{2p+1}, C_1)+d(x_1, C_1) \geq 3$, say $x_{2p+1}u_2, x_1u_2 \in E$. It follows that $G[V(P \cup C_1) \cup \{y, y_1\}]$ contains two disjoint cycles $C_1'$ and $C'$ such that $C_1' = y'u_1u_4u_3y'$ and $C' = u_2P[x_1, x_{2p+1}]u_2$, where $l_W(C') \geq 2$, a contradiction.

**Case 2.** $|D_0| = 1$, thus $|D| = 2p+2$.

According to $x_1y \notin E$ and Claim 3.2, we have $d(x_1, D) = 1$ and $d(y, D) \leq p$. Thus $d(x_1, H)+d(y, H) \geq n+k-p-1 = \sum_{i=1}^{k-1} l(C_i)/2+(k-1)+1$. This implies that there exists a cycle $C_i \in H$, say $C_1$, such that $d(x_1, C_1) + d(y, C_1) \geq l(C_1)/2 + 2$. By Lemma 2.3 and (8), we have $l(C_1) = 4$ and $d(x_1, C_1) = d(y, C_1) = 2$. Let $C_1 = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. We replace $C_1$ and $P$ with $C_1' = x_1u_2u_3u_4x_1$ and $P' = x_{2p+1}yu_1$, then by (11) we have $|P| = 3$. So we have that $D = x_1x_2x_3y$ be a 4-path.

By our assumption, $G[V(D \cup C_1)]$ does not contain two disjoint cycles $C'$ and $C''$ such that $l_W(C') \geq 2, l_W(C'') \geq 2$. Thus we see $d(x_2, C_1) = d(x_3, C_1) = 0$. So we have $x_3u_2, x_2u_1 \notin E$. Set $L = \{x_1, x_2, x_3, y, u_1, u_2\}$. It is easy to see that $\sum_{x\in L} d(x, D + C_1) = 3+2+2+3+3+3 = 16$. Recall that $x_1y, x_3u_2, x_2u_1 \notin E$, we have $\sum_{x\in L} d(x, H - C_1) \geq 3(n+k) - 16 = 3\sum_{i=2}^{k-1} l(C_i)/2+3(k-2)+2$. This implies that there exists a cycle $C_i \in H - C_1$, say $C_2$, such that $\sum_{x\in L} d(x, C_2) \geq 3l(C_2)/2 + 4$. On the other hand, we see $G[V(D \cup C_1) - \{p, q\}]$ contains a cycle of $W$-length at least two with $\{p, q\} \in \{\{x_1, u_2\}, \{x_2, x_3\}, \{u_1, y\}\}$, then by Lemma 2.1 (3), (8) and $x_1u_2, x_2x_3, u_1y \in E$, we obtain $\sum_{x\in L} d(x, C_2) \leq 4+4+4 = 12$. Then $3l(C_2)/2 + 4 \leq 12$, i.e., $l(C_2) = 4$. Thus $\sum_{x\in L} d(x, C_2) \geq 10$. Let $C_2 = v_1v_2v_3v_4v_1$ with $v_1 \in V_1$. By our assumption and by Lemma 2.2, $\sum_{x\in D} d(x, C_2) \leq 6$ and if $\sum_{x\in D} d(x, C_2) = 6$, then $d(x_1, C_2) = 0$ or $d(y, C_2) = 0$. Recall that $\sum_{x\in L} d(x, C_2) \geq 10$, we see $d(u_1, C_2) = d(u_2, C_2) = 2$ and $d(x_1, C_2) = 0$ or $d(y, C_2) = 0$. It follows that $G[V(D \cup C_1 \cup C_2)]$ contains three disjoint cycles $x_1u_2u_3u_4x_1$, $v_1x_2x_3yv_1$ and $u_1v_2v_3v_4u_1$ or $x_1u_2u_3u_4x_1$, $x_2v_1v_2v_3x_2$ and $v_4x_3yu_1v_4$, the last contradiction completes the proof of the main theorem.

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