Graphs and Combinatorics (2017) 33:1065–1079
DOI 10.1007/s00373-017-1794-3

ORIGINAL PAPER

Bounds for Bipartite Rainbow Ramsey Numbers

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Received: 7 May 2016 / Revised: 12 May 2017 / Published online: 23 May 2017

Abstract Given bipartite graphs G and H, the bipartite rainbow Ramsey number $BRR(G; H)$ is the minimum integer N such that any edge-coloring of $K_{N,N}$ with any number of colors contains either a monochromatic copy of G or a rainbow copy of H. It is known that $BRR(G; H)$ exists if and only if G is a star or H is a forest consisting of stars. For fixed $t \geq 3$, $s \geq (t-1)! + 1$ and large n, we shall show that $BRR(K_{t,s}; K_{1,n}) = \Theta(n^t)$ and $BRR(K_{1,n}; K_{t,t}) = \Theta(n)$. We also improve the known bounds for $BRR(C_{2m}; K_{1,n})$, $BRR(K_{1,n}; C_{2m})$, $BRR(B_{s,t}; K_{1,n})$ and $BRR(K_{1,n}; B_{s,t})$, where $B_{s,t}$ is a broom consisting of $s+t$ edges obtained by identifying the center of star $K_{1,s}$ with an end-vertex of a path P_{1+t}. Particularly, we have $BRR(C_{2m}; K_{1,n}) \geq (1-o(1))n^{m/(m-1)}$ for $m = 2, 3, 5$ and large n.

Keywords Bipartite rainbow Ramsey number · Edge-coloring · Even cycle · Broom

1 Introduction

Let G be a graph. A monochromatic coloring of G is an edge-coloring of G by a single color, and a rainbow coloring of G is an edge-coloring of G whose edges have

Supported by NSFC 11201342.

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pairwise distinct colors. The Ramsey number $R_k(G)$ is the smallest integer N such that in any k-coloring of the edges of K_N, there is a monochromatic copy of G.

For graphs G and H, the rainbow Ramsey number $RR(G; H)$ is defined to be the minimum integer N such that any edge-coloring of K_N using any number of colors contains either a monochromatic copy of G or a rainbow copy of H, see Eroh [5]. Jamison et al. [9] proved that $RR(G; H)$ exists if and only if G is a star or H is a forest consisting of stars. Results for bounding $RR(G; H)$ with various types of parameters can be found in literature, see [2,8,10,13].

Given two bipartite graphs G and H, the bipartite rainbow Ramsey number $BRR(G; H)$ is the minimum integer N such that any edge-coloring of $K_{N,N}$ with any number of colors contains either a monochromatic copy of G or a rainbow copy of H. For an extended survey regarding bounds for rainbow Ramsey numbers and bipartite rainbow Ramsey numbers, see [7].

The following two bounds were obtained by Eroh and Oellermann [6].

Lemma 1 [6] *Let G and H be connected bipartite graphs. Then $BRR(G; H)$ exists if and only if G or H is a star.*

Lemma 2 [6] *Let G_n and B_m be bipartite graphs such that G_n is connected and has n vertices in the larger part, and B_m has m edges. If $BRR(G_n; B_m)$ exists, then $BRR(G_n; B_m) \geq (n-1)(m-1)+1$.*

Moreover, they proved that $3n-2 \leq BRR(K_{1,n}; C_4) \leq 6n-8$, where $K_{1,n}$ is a star with n edges and C_4 is a 4-cycle. Later, Balister et al. [3] restated the bipartite rainbow Ramsey number in terms of matrices. By a construction, they found $BRR(K_{1,n}; C_4) = 3n-2$, verifying that the lower bound is the exact value. We shall consider $BRR(K_{t,s}; K_{1,n})$ and $BRR(K_{1,n}; K_{t,t})$.

We need another definition in the proofs. Given graphs G and H, Erdős et al. [4] defined the anti-Ramsey number $AR(G; H)$ to be the maximum number k of colors such that there exists an edge-coloring of G with exactly k colors in which every copy of H in G is not rainbow colored. Let P_{1+t} be a path with t edges, and $B_{s,t}$ a broom consisting of $s+t$ edges obtained by identifying the center of a star $K_{1,s}$ with an end-vertex of P_{1+t}. Jiang and West [11] derived bounds for $AR(K_n; B_{s,t})$.

In Sect. 2, we show $BRR(K_{t,s}; K_{1,n}) = \Theta(n^t)$ for fixed $t \geq 3$, $s \geq (t-1)!+1$ and large n. And in Sect. 3, we give $t^2(n-1)+1 \leq BRR(K_{1,n}; K_{t,t}) \leq t^3(n-1)+t-1$ for $n > t \geq 3$. In last two sections, we consider $BRR(C_{2m}; K_{1,n})$, $BRR(K_{1,n}; C_{2m})$, $BRR(B_{s,t}; K_{1,n})$ and $BRR(K_{1,n}; B_{s,t})$. Particularly, we have $BRR(C_{2m}; K_{1,n}) \geq (1-o(1))n^{m/(m-1)}$ for $m = 2, 3, 5$ and large n.

2 Bounding $BRR(K_{t,s}; K_{1,n})$

To prove the existence of $BRR(K_{t,t}; K_{1,n})$, Eroh and Oellermann [6] showed that for any positive integers t and n,

$$(t-1)(n-1)+1 \leq BRR(K_{t,t}; K_{1,n}) \leq (t-1)(n-1)^{(t-1)(n-1)+1}+1. \quad (1)$$

We shall improve Eq. (1) as follows.

Theorem 1 *For fixed integers t and s with $t \geq 3$, $s \geq (t-1)! + 1$,*

$$BRR(K_{t,s}; K_{1,n}) = \Theta(n^t).$$

We need a relationship between Ramsey numbers and bipartite rainbow Ramsey numbers.

Lemma 3 *Let G be a complete bipartite graph with order $|G| \geq 3$. Then for any integer $n \geq 4$,*

$$R_{n-2}(G) \leq BRR(G; K_{1,n}).$$

Proof Let $N = R_{n-2}(G) - 1$ and K_N be a complete graph with vertex set $\{a_1, \ldots, a_N\}$. Then there is an edge-coloring of K_N with $n-2$ colors containing no monochromatic copy of G. Consider $K_{N,N}$ on bipartition $U = \{u_1, \ldots, u_N\}$ and $V = \{v_1, \ldots, v_N\}$. For $i \neq j$, color the edge $u_i v_j$ in $K_{N,N}$ by the color of $a_i a_j$ in K_N. Color the edges $\{u_i v_i \mid 1 \leq i \leq N\}$ by a new color, which form a monochromatic matching of N edges. Since the total number of colors is $n-1$, there is no rainbow copy of $K_{1,n}$ in $K_{N,N}$.

Suppose that $G = K_{t,s}$ and there is a monochromatic G in $K_{N,N}$. Let $\{u_{p_1}, \ldots, u_{p_t}, v_{q_1}, \ldots, v_{q_s}\}$ be the vertex set of G in $K_{N,N}$. Since $|G| \geq 3$, then $G \neq K_{1,1}$. And G is a monochromatic copy of $K_{t,s}$, we see that $p_i \neq q_j$ for any $1 \leq i \leq t$ and $1 \leq j \leq s$. Then the edge set $\{a_{p_i} a_{q_j} \mid 1 \leq i \leq t, 1 \leq j \leq s\}$ forms a monochromatic copy of $K_{t,s}$ in K_N, yielding a contradiction. □

The following was obtained by Alon et al. [1].

Lemma 4 *Let $t \geq 2$ and $s \geq (t-1)! + 1$ be fixed integers. Then*

$$R_n(K_{t,s}) = \Theta(n^t).$$

Given positive integers t, s, n and b, define $a_{t,s}(n; b)$ to be the smallest integer a such that in any $b \times a$ matrix A either there is a $t \times s$ sub-matrix B whose elements are all the same or there are at least n distinct elements in some row or column. Observe that for $b \leq (n-1)(t-1)$, $a_{t,s}(n; b)$ is undefined: consider any number of columns, each filled with at most $n-1$ symbols repeated at most $t-1$ times(using the same $n-1$ symbols in distinct columns).

For positive integers b, t and n with $b > (n-1)(t-1)$, given a b-tuple $z = (z_1, \ldots, z_b)$, let $q(z, t)$ be the number of subsets $T \subseteq \{1, \ldots, b\}$ with $|T| = t$ such that all the elements z_i, $i \in T$, are the same. Set $q(n, b, t)$ to be the minimum value of $q(z, t)$ over all b-tuples z for which the distinct elements of z are less than n in z. If $b = p(n-1) + r$, $0 \leq r < n-1$, then an optimal b-tuple z contains $n-r-1$ entries repeated p times and r entries repeated $p+1$ times. So we get

$$q(n, b, t) = (n-r-1)\binom{p}{t} + r\binom{p+1}{t}.$$

Lemma 5 *For positive integers t, s, n and b with $b > (n-1)(t-1)$,*

$$a_{t,s}(n;b) \leq 1 + \binom{b}{t}(n-1)(s-1)\frac{1}{q(n,b,t)}.$$

Proof Assume that A is a $b \times a$ extremal matrix with $a = a_{t,s}(n;b) - 1$ such that A has no $t \times s$ sub-matrix whose elements are all the same and the number of the distinct elements in each row or column are less than n.

Every column of A has at least $q(n,b,t)$ t-tuples of the same elements, so A has at least $q(n,b,t)a$ t-tuples of the same elements in its columns. Therefore at least $q(n,b,t)a/\binom{b}{t}$ of these t-tuples are placed along the same set of t rows. Since A has no $t \times s$ submatrix whose elements are all the same and the distinct elements in each row are less than n, we obtain

$$q(n,b,t)a/\binom{b}{t} \leq (n-1)(s-1),$$

implying the required inequality. □

Now we consider the bounds for $BRR(K_{t,s}; K_{1,n})$.

Proof of Theorem 1 The lower bound follows from Lemmas 3 and 4.

For the upper bound, assume that A is a $b \times a$ matrix with $a = a_{t,s}(n;b)$. Set $b = (s-1)(n-1)^t$. Then

$$q(n,b,t) = (n-1)\binom{(s-1)(n-1)^{t-1}}{t}.$$

By Lemma 5, we obtain that for large n, $a_{t,s}(n;b)$ is at most

$$1 + \binom{b}{t}(n-1)(s-1)\frac{1}{q(n,b,t)} \leq 1 + \frac{(n-1)(s-1)\binom{(s-1)(n-1)^t}{t}}{(n-1)\binom{(s-1)(n-1)^{t-1}}{t}}$$
$$\leq 1 + (s-1)\frac{\Big((s-1)(n-1)^t \times \big((s-1)(n-1)^t - 1\big) \times \ldots \times \big((s-1)(n-1)^t - t + 1\big)\Big)/t!}{\Big((s-1)(n-1)^{t-1} \times \big((s-1)(n-1)^{t-1} - 1\big) \times \cdots \times \big((s-1)(n-1)^{t-1} - t + 1\big)\Big)/t!}$$
$$\leq 1 + (s-1)\left(\frac{(s-1)(n-1)^t}{(s-1)(n-1)^{t-1} - t + 1}\right)^t \leq 1 + (s-1)\left(\frac{(n-1)^t}{(n-1)^{t-1} - \frac{t-1}{s-1}}\right)^t.$$

Let $\delta = \frac{t-1}{s-1}$ and $\epsilon = \frac{\delta(n-1)}{(n-1)^{t-1}-\delta}$. Then we have $\delta = \frac{\epsilon(n-1)^{t-1}}{n-1+\epsilon}$ and we obtain that

$$a_{t,s}(n;b) \leq 1 + (s-1)\left(\frac{(n-1)^t}{(n-1)^{t-1} - \delta}\right)^t$$
$$= 1 + (s-1)\left(\frac{(n-1)^t}{(n-1)^{t-1} - \frac{\epsilon(n-1)^{t-1}}{n-1+\epsilon}}\right)^t$$

$$= 1 + (s-1)\left(\frac{n-1}{1-\frac{\epsilon}{n-1+\epsilon}}\right)^t = 1 + (s-1)(n-1+\epsilon)^t,$$

which implies that $a_{t,s}(n;b) < (s-1)n^t$ for large n.

Then we obtain that for large n, in any edge-coloring of $K_{(s-1)(n-1)^t,(s-1)n^t}$ with any number of colors, either there is a monochromatic copy of $K_{t,s}$, or there is a rainbow copy of $K_{1,n}$. Since $K_{(s-1)(n-1)^t,(s-1)n^t}$ is a subgraph of $K_{(s-1)n^t,(s-1)n^t}$, we have in any edge-coloring of $K_{(s-1)n^t,(s-1)n^t}$ with any number of colors, either there is a monochromatic copy of $K_{t,s}$, or there is a rainbow copy of $K_{1,n}$, which implies $BRR(K_{t,s}; K_{1,n}) \leq (s-1)n^t$. □

3 Bounding $BRR(K_{1,n}; K_{t,t})$

To prove the existence of $BRR(K_{1,n}; K_{t,t})$, Eroh and Oellermann [6] showed that for integers $n \geq 2$ and $t \geq 1$,

$$(n-1)(t^2-1)+1 \leq BRR(K_{1,n}; K_{t,t}) \leq \lceil \frac{1}{2}t^2(t-1)(tn+n-t-3)+2 \rceil. \quad (2)$$

For $t = 2$, Balister et al. [3] proved the lower bound is the exact value. We shall improve the upper bound in Eq. (2) as follows with similar proof from Balister et al. [3].

Theorem 2 *For any integer $n \geq 4$,*

$$BRR(K_{1,n}; K_{3,3}) \leq 17n - 15. \quad (3)$$

And for integers n and t with $n > t \geq 3$,

$$BRR(K_{1,n}; K_{t,t}) \leq t^3(n-1) + t - 1. \quad (4)$$

For the proofs, we need some definitions. Given positive integers n, t and b, define $a_n(t,t;b)$ be the smallest integer a such that in any $b \times a$ matrix either some entry is repeated at least n times in some row or column or there is a $t \times t$ sub-matrix with distinct elements. Observe that for $b \leq (n-1)(t-1)$, $a_n(t,t;b)$ is undefined: consider any number of columns, each filled with $t-1$ symbols repeated $n-1$ times(using distinct symbols in distinct columns). For positive integers b, n and t with $b > (n-1)(t-1)$, given $z = (z_1, \ldots, z_b)$, let $p(z,t)$ be the number of t-tuple subsets $T \subseteq \{1, \cdots, b\}$ such that the t elements z_i, $i \in T$, are all distinct. Let $p(n,b,t)$ be the minimum value of $p(z,t)$ over all b-tuples z for which every element of z is repeated less than n times in z. It is well known that if $b = q(n-1)+r$, $0 \leq r < n-1$, then z contains q entries repeated $n-1$ times and one entry repeated r times. Hence

$$p(n,b,t) = \binom{q}{t}(n-1)^t + \binom{q}{t-1}(n-1)^{t-1}r.$$

The following was obtained by Balister et al. [3].

Lemma 6 *For positive integers n, t and b with $b > (n-1)(t-1)$,*

$$a_n(t,t;b) \le 1 + \binom{b}{t}(t^2-t+1)(t-1)(n-1)\frac{1}{p(n,b,t)}.$$

Proof of the upper bound (3) Assume that A is a $b \times a$ matrix with $a = a_n(3,3;b)$. Then A has either some entry repeated at least n times in some row or column, or a 3×3 sub-matrix with distinct elements. Set $b = 17(n-1)+2$. Then

$$p(n,b,3) = \binom{17}{3}(n-1)^3 + 2\binom{17}{2}(n-1)^2.$$

By Lemma 6, we obtain

$$\begin{aligned} a_n(3,3;b) &\le 1 + 14\binom{17(n-1)+2}{3}(n-1)\frac{1}{p(n,b,3)} \\ &\le 1 + \frac{119(n-1)+14}{120(n-1)+48}(17(n-1)+1) \le 17n-15. \end{aligned}$$

Then we have $a_n(3,3;17n-15) \le 17n-15$. Hence $BRR(K_{1,n};K_{3,3}) \le 17n-15$.

$\square$

Proof of the upper bound (4) Assume that A is a $b \times a$ matrix with $a = a_n(t,t;b)$. Set $b = t^3(n-1)+t-1$. Since $n > t$, then we have

$$p(n,b,t) = \binom{t^3}{t}(n-1)^t + (t-1)\binom{t^3}{t-1}(n-1)^{t-1}.$$

By Lemma 6, we obtain that

$$\begin{aligned} a_n(t,t;b) &\le 1 + \binom{t^3(n-1)+t-1}{t}(t^2-t+1)(t-1)(n-1)\frac{1}{p(n,b,t)} \\ &\le 1 + \frac{\left(t^3(n-1)+t-1\right)^{t-2}(t^2-t+1)(t-1)}{(n-1)^{t-3}(t^3-t+2)^{t-2}((t^3-t+1)(n-1)+t(t-1))}\left(t^3(n-1)+t-2\right) \\ &\le 1 + \left(\frac{t^3+t-1}{t^3-t+2}\right)^{t-2}\frac{(t^2-t+1)(t-1)(n-1)}{(t^3-t+1)(n-1)+t(t-1)}\left(t^3(n-1)+t-2\right). \end{aligned}$$

Set functions $g(t) = \frac{t^3-t+1}{(t-1)(t^2-t+1)}$, $h(t) = \left(\frac{t^3+t-1}{t^3-t+2}\right)^{t-2}$, and we have $a_n(t,t;b) \le 1 + \frac{h(t)}{g(t)}\left(t^3(n-1)+t-2\right)$.

For $t \ge 3$,

$$h(t) = \left(1 + \frac{2t-3}{t^3-t+2}\right)^{t-2} \le e^{\frac{(t-2)(2t-3)}{t^3-t+2}}$$

$$\leq 1 + \frac{(t-2)(2t-3)}{t^3-t+2} + \frac{e}{2}\frac{(t-2)^2(2t-3)^2}{(t^3-t+2)^2}$$
$$\leq 1 + \frac{(t-2)(2t-3)}{t^3-t+2} + 2\frac{(t-2)^2(2t-3)^2}{(t^3-t+2)^2}.$$

Then we have

$$\begin{aligned} g(t) - h(t) &\geq \frac{t^3-t+1}{(t-1)(t^2-t+1)} - 1 - \frac{(t-2)(2t-3)}{t^3-t+2} - 2\frac{(t-2)^2(2t-3)^2}{(t^3-t+2)^2} \\ &\geq \frac{8t^4-24t^3+35t^2-27t+10}{(t^3-t+2)(t^3-2t^2+2t-1)} - 2\frac{(t-2)^2(2t-3)^2}{(t^3-t+2)^2} \\ &\geq \frac{8t^4-24t^3+35t^2-27t+10}{(t^3-t+2)^2} - 2\frac{(t-2)^2(2t-3)^2}{(t^3-t+2)^2} \\ &\geq \frac{32t^3-111t^2+141t-62}{(t^3-t+2)^2}. \end{aligned}$$

So $g(t) \geq h(t)$ for $t \geq 3$. And hence $a_n(t, t; b) \leq t^3(n-1) + t - 1$. □

4 $BRR(C_{2m}; K_{1,n})$ and $BRR(K_{1,n}; C_{2m})$

Here we shall show the lower bound for $BRR(C_{2m}; K_{1,n})$ as follows.

Theorem 3 *For $m = 2, 3, 5$, if $n \to \infty$, then*

$$BRR(C_{2m}; K_{1,n}) \geq (1 - o(1))n^{m/(m-1)}.$$

Let $m \geq 2$ be an integer and $q \geq m$ be a prime power. Let $F(q)$ be the Galois field of q elements, and both X and Y be copies of the Cartesian product $F^m(q)$. Denote by N the number $q^m = |X| = |Y|$. We shall use vectors in $F^{m-1}(q)$ as colors to color the complete bipartite graph $K_{N,N}$ on partite sets X and Y such that there is no monochromatic copy of C_{2m} for $m = 2, 3, 5$. For vertices $A \in X$ and $B \in Y$ with

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

color the edge AB with color $S \in F^{m-1}(q)$ when

$$S = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_{m-1} + b_{m-1} \end{pmatrix} + b_m \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_m \end{pmatrix}.$$

Let us denote by $H_S(m, q)$ the subgraph induced by all edges in the color S.
The following was obtained by Li and Lih [12].

Lemma 7 *Let $S \in F^{m-1}(q)$ and $q \geq m \geq 2$. Then $H_S(m, q)$ contains no monochromatic C_{2m} for $m = 2, 3, 5$.*

Proof of Theorem 3 Let p_1 and p_2 be consecutive primes such that $p_1^{m-1} \leq n - 1 < p_2^{m-1}$. From the Prime Number Theorem, we know $p_1 \sim p_2$ and hence $p_1^{m-1} \sim n$ as $n \to \infty$. By the definition of $H_S(m, p_1)$, we use $p_1^{m-1} \leq n - 1$ colors to color $K_{N,N}$, $N = p_1^m$, such that it contains neither a rainbow copy of $K_{1,n}$ nor a monochromatic copy of C_{2m} by Lemma 7. Thus, $BRR(C_{2m}; K_{1,n})$ is at least $N = p_1^m \geq (1 - o(1))n^{m/(m-1)}$. □

For fixed m, $BRR(C_{2m}; K_{1,n})$ is nonlinear on n. However, $BRR(K_{1,n}; C_{2m})$ is linear on n. Especially, for $m = 2$, $BRR(K_{1,n}; C_4) = BRR(K_{1,n}; K_{2,2}) = 3n - 2$ which is determined completely, see [3]. By borrowing the method of Eroh and Oellermann [6], we obtain the bounds for $BRR(K_{1,n}; C_{2m})$.

Theorem 4 *For any integers $n, m \geq 2$,*

$$(2m - 1)(n - 1) + 1 \leq BRR(K_{1,n}; C_{2m}) \leq 4m(n - 2) + m(m - 1)(n - 1) + 2.$$

Furthermore, for m odd, the lower bound can be improved to $2m(n - 1) + 1$.

Proof For the lower bound, by Lemma 2, $BRR(K_{1,n}; C_{2m}) \geq (2m - 1)(n - 1) + 1$.

For m odd, let $M = 2m(n - 1)$. Consider a $K_{2m,2m}$ on bipartition $U = \{u_0, u_1, \ldots, u_{2m-1}\}$ and $V = \{v_0, v_1, \ldots, v_{2m-1}\}$. We define the color $C(e)$ of each edge e in $K_{2m,2m}$ as follows. For any $i, j \in \{0, 1, \ldots, 2m - 1\}$, let $C(u_i v_j) \equiv i + j \pmod{2m}$. We claim that any pair of adjacent edges are in different colors. If not, suppose $C(u_i v_{j_1}) = C(u_i v_{j_2})$ with $j_1 \neq j_2$, then $j_1 \equiv j_2 \pmod{2m}$. Since $j_1, j_2 \in \{0, 1, \ldots, 2m - 1\}$, we have $j_1 = j_2$, for a contradiction. Now replace each u_i and each v_j with $n - 1$ new vertices to produce a copy of $K_{M,M}$. Thus, there is no monochromatic copy of $K_{1,n}$.

Suppose that there is a rainbow copy of C_{2m} with the edge set $\{u_{i_1} v_{j_1}, v_{j_1} u_{i_2}, u_{i_2} v_{j_2}, \ldots, u_{i_m} v_{j_m}, v_{j_m} u_{i_1}\}$. Divide the set into $E = \{u_{i_1} v_{j_1}, u_{i_2} v_{j_2}, \ldots, u_{i_m} v_{j_m}\}$ and $E' = \{v_{j_1} u_{i_2}, v_{j_2} u_{i_3}, \ldots, v_{j_{m-1}} u_{i_m}, v_{j_m} u_{i_1}\}$.

Then we have

$$\sum_{e \in E} C(e) \equiv \sum_{e' \in E'} C(e') \pmod{2m}. \quad (5)$$

Since the coloring of $K_{M,M}$ uses $2m$ colors and C_{2m} is rainbow, the edges of C_{2m} exactly use all the colors of $\{0, 1, \ldots, 2m - 1\}$.

Then

$$\sum_{e \in E} C(e) + \sum_{e' \in E'} C(e') = \sum_{i=0}^{2m-1} i.$$

However, $\sum_{i=0}^{2m-1} i = m(2m-1)$. For m odd, $\sum_{e\in E} C(e) + \sum_{e'\in E'} C(e')$ is odd, which contradicts Eq. (5). Hence, this coloring of $K_{N,N}$ contains no rainbow copy of C_{2m}.

For the upper bound, let $N = 4m(n-2) + m(m-1)(n-1) + 2$. Consider any edge-coloring of $K_{N,N}$ that contains no monochromatic copy of $K_{1,n}$. Then each color appears at most $n-1$ times at each vertex. Denote by $N(C_{2m})$ the number of C_{2m} in $K_{N,N}$ and $N'(C_{2m})$ the number of C_{2m} that are not rainbow colored in $K_{N,N}$.

Then we have

$$N(C_{2m}) = \binom{N}{m}\binom{N}{m}\frac{(m!)^2}{4m}. \tag{6}$$

We now estimate the value of $N'(C_{2m})$. If C_{2m} is not rainbow colored, there are at least two edges in the same color. Let $N_1'(C_{2m})$ be the number of C_{2m} containing two adjacent edges in the same color and $N_2'(C_{2m})$ be the number of C_{2m} containing two nonadjacent edges in the same color.

We have

$$N'(C_{2m}) \leq N_1'(C_{2m}) + N_2'(C_{2m}).$$

Suppose the two edges uv and uw are adjacent with the same color. There are $2N$ choices for u and then N choices for v, in the other partite set. Since at most $n-1$ edges are incident with u in the same color, there are at most $n-2$ choices for w. Since the edge uw might have been chosen first, we have counted each pair of adjacent edges in the same color twice. This makes a total of at most $N^2(n-2)$ choices for $\{u, v, w\}$. There are $\binom{N-1}{m-1}\binom{N-2}{m-2}$ ways to choose the remaining vertices from $K_{N,N}$. Along with uv and uw, the chosen $2m-3$ vertices can construct at most $(m-1)!(m-2)!$ even cycles C_{2m} in $K_{N,N}$.

Then

$$N_1'(C_{2m}) \leq N^2(n-2)\binom{N-1}{m-1}\binom{N-2}{m-2}(m-1)!(m-2)!. \tag{7}$$

Suppose the two edges uv and xy are nonadjacent with the same color. We may assume that u, x are in the same partite set and v, y are in the other partite set. There are N choices for u, N choices for v, and then $N-1$ choices for x. Since at most $n-1$ edges are incident with x in the same color as edge uv, there are at most $n-1$ choices for y. Since the edge xy might have been chosen first, we have counted each pair of nonadjacent edges in the same color twice. This makes a total of at most $\frac{1}{2}N^2(N-1)(n-1)$ choices for $\{u, v, x, y\}$. There are $\binom{N-2}{m-2}\binom{N-2}{m-2}$ ways to choose the remaining vertices from $K_{N,N}$. Along with uv and xy, the chosen $2m-4$ vertices can construct at most $\frac{(m-2)!m!}{2m}$ even cycles C_{2m} in $K_{N,N}$.

Then

$$N_2'(C_{2m}) \leq \frac{1}{4}N^2(N-1)(n-1)\binom{N-2}{m-2}\binom{N-2}{m-2}(m-2)!(m-1)!. \tag{8}$$

By Eqs. (6), (7) and (8), we obtain that

$$\begin{aligned}
N_1'(C_{2m}) + N_2'(C_{2m}) &\le N^2(n-2)\binom{N-1}{m-1}\binom{N-2}{m-2}(m-1)!(m-2)! \\
&\quad + \frac{1}{4}N^2(N-1)(n-1)\binom{N-2}{m-2}\binom{N-2}{m-2}(m-2)!(m-1)! \\
&= \binom{N}{m}\binom{N}{m}\frac{m^2(m-1)}{N-1}(n-2)(m-1)!(m-2)! \\
&\quad + \frac{1}{4}\binom{N}{m}\binom{N}{m}\frac{m^2(m-1)^2}{N-1}(n-1)(m-1)!(m-2)! \\
&= \binom{N}{m}\binom{N}{m}(m!)^2\frac{n-2+\frac{1}{4}(m-1)(n-1)}{N-1} \\
&= \binom{N}{m}\binom{N}{m}(m!)^2\frac{n-2+\frac{1}{4}(m-1)(n-1)}{4m(n-2)+m(m-1)(n-1)+1} \\
&< \binom{N}{m}\binom{N}{m}\frac{(m!)^2}{4m} \; = \; N(C_{2m}).
\end{aligned}$$

So $N'(C_{2m}) < N(C_{2m})$ and thus there is a rainbow copy of C_{2m} in $K_{N,N}$. □

5 $BRR(B_{s,t}; K_{1,n})$ and $BRR(K_{1,n}; B_{s,t})$

In the section, we consider the bounds for bipartite rainbow Ramsey numbers of two graphs where one is a broom and the other is a star. We shall bound the bipartite rainbow Ramsey number $BRR(B_{s,t}; K_{1,n})$ and $BRR(K_{1,n}; B_{s,t})$.

Theorem 5 *For any integers $n, s, t \ge 2$,*

$$max\left\{(n-1)\left(s+\left\lceil\frac{t}{2}\right\rceil-1\right), 2(n-2)\left(\left\lceil\frac{t}{2}\right\rceil-1\right)\right\} \\ +1 \le BRR(B_{s,t}; K_{1,n}) \le (2s+t-3)(n-1).$$

Proof of the lower bound in Theorem 5 By Lemma 2 we know that $BRR(B_{s,t}; K_{1,n}) \ge (n-1)\left(s+\left\lceil\frac{t}{2}\right\rceil-1\right)+1$ since $B_{s,t}$ is a bipartite graph for the largest partite set has $s+\left\lceil\frac{t}{2}\right\rceil$ vertices. So it suffices to show that $BRR(B_{s,t}; K_{1,n}) \ge 2(n-2)\left(\left\lceil\frac{t}{2}\right\rceil-1\right)+1$.

Let $N = 2(n-2)\left(\left\lceil\frac{t}{2}\right\rceil-1\right)$. We give a coloring of $K_{N,N}$ that contains neither a monochromatic copy of $B_{s,t}$ nor a rainbow copy of $K_{1,n}$ as follows. Let V and V' be the two partite sets of $K_{2(n-2),2(n-2)}$. Then we divide the set V into two sets A and B with $A = \{a_1, a_2, \ldots, a_{n-2}\}$ and $B = \{b_1, b_2, \ldots, b_{n-2}\}$, and divide the set V' into two sets C and D with $C = \{c_1, c_2, \ldots, c_{n-2}\}$ and $D = \{d_1, d_2, \ldots, d_{n-2}\}$. For i, $1 \le i \le n-2$, color all edges that join $a_i \in A$ and a vertex in C with color i; color all edges that join $b_i \in B$ and a vertex in D with color $i+n-2$; color all edges that join $c_i \in C$ and a vertex in B with color $i+2(n-2)$; color all edges that join $d_i \in D$ and a vertex in A with color $i+3(n-2)$. The bipartite graph $K_{2(n-2),2(n-2)}$ is colored with

$4(n-2)$ colors, and each vertex is incident with exactly $n-1$ different colors. Now replace each vertex in $K_{2(n-2),2(n-2)}$ with $\lceil \frac{t}{2} \rceil - 1$ new vertices to obtain a coloring of $K_{N,N}$. Since there are also exactly $n-1$ colors incident with each vertex in $K_{N,N}$, there is no rainbow copy of $K_{1,n}$. Each color induces a copy of $K_{\lceil \frac{t}{2} \rceil -1, (\lceil \frac{t}{2} \rceil -1)(n-2)}$. However, $B_{s,t}$ is a bipartite graph with one partite set containing $\lfloor \frac{t}{2} \rfloor + 1$ vertices and the other partite set containing $s + \lceil \frac{t}{2} \rceil$ vertices. Hence, this coloring of $K_{N,N}$ contains no monochromatic copy of $B_{s,t}$. □

For the proof of the upper bound for $BRR(B_{s,t}; K_{1,n})$, we establish the following lemma by borrowing the method of Eroh and Oellermann [6].

Lemma 8 *For any integers $s, t \geq 2$, if a bipartite graph G has average degree at least $2s + t - 3$, then G has $B_{s,t}$ as a subgraph.*

Proof Suppose $t = 2$, and let G be a bipartite graph with average degree at least $2s-1$. Denote V the vertex set of degree at least $2s-1$. If there is no $B_{s,2}$ as a subgraph in G, then for any $v \in V$, the neighbors of vertex v do not have any neighbors other than v in G. Thus, G consists of a star forest in which each star center vertex has at least $2s-1$ neighbors and a subgraph with maximum degree at most $2s-2$. Then the average degree of G is at most $2s-2$, which produces a contradiction. Hence there is a copy of $B_{s,2}$ in G.

Now we proceed by induction on t. Suppose $t \geq 3$, and let G be a bipartite graph with average degree at least $2s + t - 3$. Let H be a minimal subgraph with average degree at least $2s + t - 3$ in G in the sense that any proper subgraph in H has average degree less than $2s + t - 3$. By the inductive hypothesis, we may assume that H has a subgraph $B_{s,t-1}$ with the vertex set $\{u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t\}$ where $\{u_1, u_2, \ldots, u_s, v_1\}$ is the vertex set of the star $K_{1,s}$ with the center vertex v_1 and $\{v_1, v_2, \ldots, v_t\}$ is the vertex set of the path P_t with endpoints v_1 and v_t. If the vertex v_t is adjacent to any vertex not in the $B_{s,t-1}$, then it contains $B_{s,t}$. We may assume that v_t is not adjacent to any vertex except the vertices of the broom, so the degree of v_t is at most $t/2$ for t even and at most $s + (t-1)/2$ for t odd.

Let A be the vertex set $\{v_2, v_3, \ldots, v_t\}$ on the path of the broom $B_{s,t-1}$, B be the vertex set $\{v_1, u_1, u_2, \ldots, u_s\}$ which is the star of the broom $B_{s,t-1}$ and C be the set of remaining vertices in H. Thus, the vertex set $V(H) = A \cup B \cup C$. For two vertex sets U and V, we denote $|E(U, V)|$ the number of edges between U and V. Then we prove the following two assertions.

Claim 1 *For t odd, $|E(A, C)| \geq \frac{t-1}{2}\left(s + \frac{t-1}{2} - 3\right) + 1$.*

Suppose that $|E(A, C)| \leq \frac{t-1}{2}\left(s + \frac{t-1}{2} - 3\right)$. Based on the parity of t, $|E(A)|$ is at most $\frac{(t-1)^2}{4}$ for t odd and $\frac{t(t-2)}{4}$ for t even. $|E(A, B)|$ is at most $\frac{t-1}{2}(s+1)$ for t odd and $\frac{t}{2}(s+1) - s$ for t even.

Since the average degree of H

$$d(H) = \frac{2(|E(B \cup C)| + |E(A)| + |E(A, C)| + |E(A, B)|)}{|V(H)|},$$

we have

$$d(H) \leq \frac{d(B \cup C)|V(B \cup C)| + \frac{(t-1)^2}{2} + (t-1)(s + \frac{t-1}{2} - 3) + (t-1)(s+1)}{|V(H)|}.$$

Since $d(H) \geq (2s + t - 3)$, we have

$$2s + t - 3 \leq d(B \cup C)\frac{|V(B \cup C)|}{|V(H)|} + (2s + t - 3)\frac{|V(A)|}{|V(H)|}.$$

Since $|V(H)| = |V(A)| + |V(B \cup C)|$, we have $d(B \cup C) \geq 2s + t - 3$. Thus we can obtain a proper subgraph of H with average degree at least $2s + t - 3$, which contradicts our choice of H, completing the proof of Claim 1.

Claim 2 *If $d_H(v_t) = s + \frac{t-1}{2}$ for t odd, then we have a copy of $B_{s,t}$ as a subgraph in H.*

Since $d_H(v_t) = s + \frac{t-1}{2}$, then v_t is adjacent to $\{u_1, u_2, \ldots, u_s, v_2, v_4, \ldots, v_{t-1}\}$ for t odd. Let $A_1 = \{v_2, v_4, \ldots, v_{t-1}\}$ and $A_2 = \{v_3, v_5, \ldots, v_{t-2}\}$. For any $v_i \in A_1$, if $d_C(v_i) \geq s$, v_i and its neighbors can induce a copy of $K_{1,s}$ in H. Along with the path $P_{t+1} = v_{i+1}v_{i+2} \ldots v_t u_1 v_1 v_2 \ldots v_{i-1}$, we can have a copy of $B_{s,t}$. Thus, we may assume that $d_C(v_i) \leq s - 1$ for $v_i \in A_1$, then $|E(A_1, C)| \leq (s-1)\frac{t-1}{2}$. Since $d_C(v_t) = 0$, by Claim 1, we can find

$$|E(A_2, C)| = |E(A, C)| - |E(A_1, C)| \geq \frac{(t-3)^2}{4}.$$

Case 1 $t \geq 5$. Then there is at least one edge between A_2 and C. Assume that this edge joins $v_{i_0} \in A_2$ to the vertex $w \in C$. Since i_0 is odd, the vertex v_t must be adjacent to v_{i_0-1}. Hence $v_1 v_2 \ldots v_{i_0-1} v_t v_{t-1} v_{t-2} \ldots v_{i_0+1} v_{i_0} w$ forms a copy of P_{t+1}. Along with the copy of $K_{1,n}$ in B, we again have a copy of $B_{s,t}$.

Case 2 $t = 3$. From Claim 1 and the previous case, we know that $|E(A, C)| = d_C(v_2) \geq s - 1$ for $t = 3$. However, $d_C(v_2) \leq s - 1$, otherwise we have a copy of $B_{s,3}$. Thus, we may assume that $d_C(v_2) = s - 1$. Let the neighbors of vertex v_2 in C be $w_1, w_2, \ldots, w_{s-1}$. Since v_2 is also adjacent to v_1 and v_3, we have $d_H(v_2) = s + 1$.

If $d_C(v_1) \geq 1$, let w be the vertex in C which is adjacent to v_1. Then the vertex set $\{v_1, u_2, u_3, \ldots, u_s, w\}$ can induce a copy of $K_{1,s}$. Along with the path $P_4 = v_1 u_1 v_3 v_2$, we have a copy of $B_{s,3}$. So we may assume that $d_C(v_1) = 0$.

If u_1 is adjacent to some vertex w' in $C \backslash \{w_1, w_2, \ldots, w_{s-1}\}$, the vertex set $\{v_2, v_3, w_1, w_2, \ldots, w_{s-1}\}$ induces a copy of $K_{1,s}$. Along with the path $P_4 = v_2 v_1 u_1 w'$, we again have a copy of $B_{s,3}$. Hence, we assume that $d_C(u_1) \leq s - 1$.

Let A' denote the vertex set $\{u_1, v_1, v_2, v_3\}$, B' denote the vertex set $\{u_2, u_3, \ldots, u_s\}$. Then we have

$$d(H) \leq \frac{d(B' \cup C)|V(B' \cup C)| + 2|E(A')| + 2|E(A', C)| + 2|E(A', B')|}{|V(H)|}.$$

Since $d(H) \geq 2s$, $|E(A')| = 4$, $|E(A', C)| \leq 2s - 2$ and $|E(A', B')| = 2s - 2$, we have

$$2s \leq d(B' \cup C)\frac{|V(B' \cup C)|}{|V(H)|} + 2s\frac{|V(A')|}{|V(H)|}.$$

Since $|V(H)| = |V(A')| + |V(B' \cup C)|$, we have $d(B' \cup C) \geq 2s$. Then we can obtain a proper subgraph of H with average degree at least $2s$, which again contradicts our choice of H, completing the proof of Claim 2.

We continue the proof of Lemma 8. From Claim 2, we may assume that $d_H(v_t) \leq s + \frac{t-1}{2} - 1$, including the case that t is even. We have

$$\begin{aligned} d(H\backslash\{v_t\}) &\geq \frac{2(|E(H)| - (s + \frac{t-1}{2} - 1))}{|V(H)| - 1} = \frac{2|E(H)| - (2s + t - 3)}{|V(H)| - 1} \\ &\geq \frac{(2s + t - 3)|V(H)| - (2s + t - 3)}{|V(H)| - 1} = 2s + t - 3. \end{aligned}$$

Thus, we have a proper subgraph of H with average degree at least $2s + t - 3$, which again contradicts our choice of H. There must be some subgraph $B_{s,t}$ in H, so in G. □

Proof of the upper bound in Theorem 5 Let $N = (2s + t - 3)(n - 1)$. Consider any edge-coloring of $G = K_{N,N}$. Suppose this edge-coloring of $K_{N,N}$ contains no rainbow copy of $K_{1,n}$. Let G_c be the subgraph induced by all edges in color c, V_c the set of vertices incident with edges of color c, and C_v the set of colors incident with vertex v. We denote $d_c(v)$ the degree of vertex v in G_c. Then for any v, we have $|C_v| \leq n - 1$, and

$$d(G_c) = \frac{\sum_{v \in V_c} d_c(v)}{|V_c|}.$$

So

$$\sum_c \frac{\sum_{v \in V_c} d_c(v)}{|V_c|} \geq \frac{\displaystyle\sum_c \sum_{v \in V_c} d_c(v)}{\displaystyle\sum_c |V_c|} = \frac{\displaystyle\sum_{v \in V(G)} d(v)}{\displaystyle\sum_{v \in V(G)} |C_v|} \geq \frac{2N^2}{2N(n-1)} = 2s + t - 3.$$

Thus, there must be some color c such that $d(G_c) \geq 2s + t - 3$. By Lemma 8, we can obtain a copy of $B_{s,t}$ in G_c, and, hence we have a monochromatic copy of $B_{s,t}$ in G. □

Now we determine the bounds for $BRR(K_{1,n}; B_{s,t})$. For $t = 1$, $B_{s,t} = K_{1,s+1}$ and we know that $BRR(K_{1,n}; B_{s,1}) = (n - 1)s + 1$, see [6]. Now we show the value of $BRR(K_{1,n}; B_{s,2})$.

Lemma 9 *For any positive integers n and s,*

$$BRR(K_{1,n}; B_{s,2}) = (n-1)(s+1) + 1.$$

Proof By Lemma 2, we know that $BRR(K_{1,n}; B_{s,2}) \geq (n-1)(s+1)+1$. Let $N = (n-1)(s+1)+1$. Consider an edge-coloring of $K_{N,N}$ with any number of colors. If there is no monochromatic copy of $K_{1,n}$, then at least $s+2$ colors are present at each vertex. We can take a rainbow copy of $K_{1,s+1}$ from the coloring $K_{N,N}$. Let u and v denote the center vertex and any other vertex of this rainbow $K_{1,s+1}$, respectively. Then there are at least $s+2$ colors incident with v.

If at least $s+2$ colors are incident with v in $K_{N,N}\backslash\{u\}$, there is at least one edge incident with v in some color that does not yet appear in the rainbow $K_{1,s+1}$. If $s+1$ colors are incident with v in $K_{N,N}\backslash\{u\}$, then the color of edge uv does not appear in these $s+1$ colors, so we can obtain at least one edge incident with v in some color that does not yet appear in the rainbow $K_{1,s+1}$. Along with the $K_{1,s+1}$, we have a rainbow $B_{s,2}$. □

The next theorem provides bounds for $BRR(K_{1,n}; B_{s,t})$.

Theorem 6 *For any positive integers n, s and t,*

$$(n-1)(s+t-1)+1 \leq BRR(K_{1,n}; B_{s,t}) \leq (n-1)(s+t-1)+s+\frac{t+1}{2}.$$

Proof The assertion is obvious for $n = 1$, so we assume $n \geq 2$. Since $BRR(K_{1,n}; B_{s,1}) = (n-1)s+1$, the assertion is also trivial for $t = 1$. Then we suppose $t \geq 2$.

The lower bound follows from Lemma 2. For the upper bound, let $N = (n-1)(s+t-1)+s+(t+1)/2$. For $t = 2$, from Lemma 9 we know that $BRR(K_{1,n}; B_{s,2}) \leq (n-1)(s+1)+s+1$. We proceed by induction on t. Consider an edge-coloring of $K_{N,N}$ that does not contain a monochromatic copy of $K_{1,n}$. We may assume that there is a rainbow copy of $B_{s,t-1}$. Let F be the rainbow copy of $B_{s,t-1}$ in $K_{N,N}$ and $V(F)$ be the vertex set of F. Denote u and v the center of $K_{1,s}$ and the another endpoint of path P_t in F. To prove there is a rainbow copy of $B_{s,t}$, we consider the parity of t.

Case 1 If t is even, u and v are in the different partite sets of $K_{N,N}$. Then v has $N - t/2$ neighbors in $K_{N,N}\backslash V(F)$. Since there is no monochromatic copy of $K_{1,n}$, there are at least

$$\left\lceil \frac{(n-1)(s+t-1)+s+(t+1)/2-t/2}{n-1} \right\rceil \geq s+t$$

colors incident with v. There are $s+t-1$ colors in F. Thus, at least one edge incident with v in some color does not yet appear in the rainbow F. We obtain a rainbow copy of $B_{s,t}$ in $K_{N,N}$ by combining this edge and the broom F.

Case 2 If t is odd, u and v are in the same partite set of $K_{N,N}$. Then v has $N - s - (t-1)/2$ neighbors in $K_{N,N}\backslash V(F)$. Since there is no monochromatic $K_{1,n}$, there are at least

$$\left\lceil \frac{(n-1)(s+t-1)+s+(t+1)/2-s-(t-1)/2}{n-1} \right\rceil \geq s+t$$

colors incident with v, which gives a rainbow copy of $B_{s,t}$ similarly. □

Acknowledgements The authors are grateful to the referees for their invaluable comments, such as notations, language and proofs, which have greatly improved the presentation of the paper.

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