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The Complexity of the Vertex 3-Colorability Problem for Some Hereditary Classes Defined By 5-Vertex Forbidden Induced Subgraphs

D. S. Malyshev[1]

Abstract We completely determine the complexity status of the vertex 3-colorability problem for the problem restricted to all hereditary classes defined by at most 3 forbidden induced subgraphs each on at most 5 vertices. We also present a complexity dichotomy for the problem and the family of all hereditary classes defined by forbidding an induced *bull* and any set of induced subgraphs each on at most 5 vertices.

Keywords Vertex 3-colorability problem · Computational complexity · Polynomial-time algorithm

1 Introduction

A *coloring* is an arbitrary mapping of colors to vertices (or edges) of some graph such that any adjacent vertices (or edges) receive distinct colors. In other words, a *coloring* of a graph G is an arbitrary mapping $c\colon V(G) \longrightarrow \mathbb{N}$ ($c\colon E(G) \longrightarrow \mathbb{N}$) such that $c(v_1) \neq c(v_2)$ for any adjacent vertices v_1 and v_2 ($c(e_1) \neq c(e_2)$ for any adjacent edges e_1 and e_2). A coloring is a *k-coloring* if the set $\mathbb{N}$ is replaced by $\overline{1, k}$. We will use this term only for the vertex case.

The *edge k-colorability problem* (abbreviated as the EDGE k-COL problem) is to verify whether edges of a given graph can be colored in k colors. The *vertex k-colorability problem* (abbreviated as the k-COL problem) is to verify whether vertices of a given graph can be colored in k colors. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors in vertex colorings of G. The *coloring problem*

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(abbreviated as the COL problem), for a given graph and a number k, is to determine whether its chromatic number is at most k or not. For any $k \geq 3$, the EDGE k-COL and k-COL problems are NP-complete. The COL problem is also NP-complete.

A graph H is a *subgraph* of a graph G if H can be obtained from G by deletion of vertices and edges. A graph H is an *induced subgraph* of a graph G if H is obtained from G by deletion of vertices. A *class* is a set of simple graphs closed under isomorphism. A class of graphs is *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and the graphs in $\mathcal{X}$ are said to be $\mathcal{Y}$-free. If $\mathcal{Y} = \{H\}$, then we will write "H-free" instead of "$\{H\}$-free". We say that a graph H is H_s-*free* if it does not contain H as a subgraph.

The COL problem for H-free graphs can be solved in polynomial time if H is an induced subgraph of a P_4 or a $P_3 + K_1$, and it is NP-complete in all other cases [11]. A study of forbidden induced pairs was also initiated in [11]. However, when we forbid two induced subgraphs, the situation becomes more difficult. For example, now the computational complexity of the COL problem is not known even for some hereditary classes defined by two forbidden induced subgraphs each on at most 4 vertices [13]. Some recent results about the complexity of the COL problem restricted to several families of hereditary classes defined by small forbidden induced structures are presented in the papers [5,7,9,15,17].

The situation for the k-COL problem is not clear even when only one induced subgraph is forbidden. The complexity of the 3- COL problem is known for all classes of the form $Free(\{H\})$, where $|V(H)| \leq 6$ [2]. A similar result for H-free graphs with $|V(H)| \leq 5$ was recently obtained for the 4- COL problem [6]. For each fixed k, the k-COL problem can be solved in polynomial time for P_5-free graphs [8]. The 3- COL problem can be solved in polynomial time for P_7-free graphs [1]. For every $k \geq 5$, the k-COL problem is NP-complete in the class of P_6-free graphs [10]. Additionally, the 4- COL problem is NP-complete for P_7-free graphs [10]. On the other hand, at the present time the complexity status of the k-COL problem is open for P_8-free graphs and $k = 3$, for P_6-free graphs and $k = 4$.

So, there exist many gaps in understanding the complexity of the k-COL and the COL problems for hereditary classes. There are two ways to increase knowledge in the field. The first of them is to limit the number of forbidden induced subgraphs, the second one is to limit the size of forbidden structures. Bounding the number of forbidden induced structures or their size produces a family of hereditary classes. Possible progress in each of the research directions is to obtain a (partial) complete complexity dichotomy for larger values of the bound.

In this paper, we consider the 3- COL problem and hereditary classes defined by forbidden induced subgraphs each on at most 5 vertices. There are complexity dichotomies for the problem and the families $\{Free(\mathcal{S})|$ each graph in $\mathcal{S}$ has at most 4 vertices$\}$, $\{Free(\{H_1, H_2\})|$ $\max(|V(H_1)|, |V(H_2)|) \leq 5\}$ [16]. Prior to our study, there was no a complexity dichotomy for the problem and the family $\{Free(\mathcal{S})|$ $\mathcal{S}$ has at most 3 graphs each on at most 5 vertices$\}$. In this paper, we present a complexity dichotomy for the 3- COL problem in the families:

$\{Free(\{H_1, H_2, H_3\})| \max_{i \in \overline{1,3}} |V(H_i)| \leq 5\}$ and $\{Free(\mathcal{S})|$ each graph in $\mathcal{S}$ has at most 5 vertices and $bull \in \mathcal{S}\}$.

2 Notation and Formulation of the Main Result

For a vertex x of a graph, $deg(x)$ means its degree, $N(x)$ is its neighborhood, $N[x]$ denotes its closed neighborhood. For a graph G, $\Delta(G)$ is maximum degree of its vertices. The *sum* $G_1 + G_2$ is the disjoint union of graphs G_1 and G_2 with non-intersected sets of vertices.

As usual, P_n, C_n, K_n, O_n, and $K_{p,q}$ stand, respectively, for a simple path with n vertices, a chordless cycle with n vertices, a complete graph with n vertices, an empty graph with n vertices, and a complete bipartite graph with p vertices in the first part and q vertices in the second. A *k-fan* F_k is a graph obtained by connecting a vertex x to all vertices of a simple path $(x_1, \ldots, x_k)$. A 3-*fan* is also called a *diamond*. A *k-wheel* W_k is a graph obtained by connecting a vertex x to all vertices of a simple cycle $(x_1, \ldots, x_k)$.

The graphs *bull*, *cricket*, *butterfly*, *crown* have a vertex set $\{x_1, x_2, x_3, x_4, x_5\}$. The edge set for a *b*ull is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_5\}$, for a *c*ricket is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_1x_5\}$, for a *butterfly* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_1x_5, x_4x_5\}$, for a *crown* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5\}$. A *fish* is the graph obtained by identifying a vertex of a K_3 with a degree 2 vertex of a *diamond*. The graphs *diamond*, *bull*, *cricket*, *butterfly*, *crown*, *fish* are depicted in Figure 1.

A *spindle* is the graph having vertices $x_1, x_2, x_3, x_4, x_5, y, z$ and the edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, yx_1, yx_2, yx_3, zx_4, zx_5, zx_1$. A *kite* is the graph obtained by adding a new vertex to a *diamond* and an edge connecting the new vertex with a degree 2 vertex of the *diamond*. A *dart* is the graph obtained by adding a new vertex to a *diamond* and an edge connecting the new vertex with a degree 3 vertex of the *diamond*. A *banner* is the graph obtained by adding a new vertex to a C_4 and an edge connecting the new vertex with a vertex of the cycle. A *house* is the graph obtained by adding a new vertex to a C_4 and two edges connecting the new vertex with adjacent vertices of the cycle. The graphs *spindle*, *kite*, *dart*, *banner*, *house* are depicted in Fig. 2.

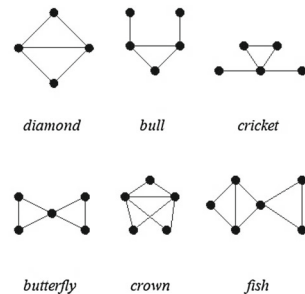**Fig. 1** The graphs *diamond*, *bull*, *cricket*, *butterfly*, *crown*, *fish*

diamond   bull   cricket

butterfly   crown   fish

Fig. 2 The graphs *spindle*, *kite*, *dart*, *banner*, *house*

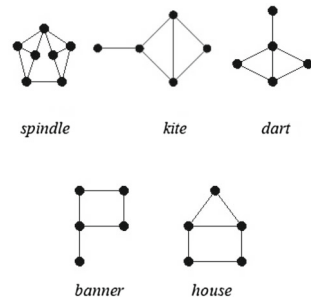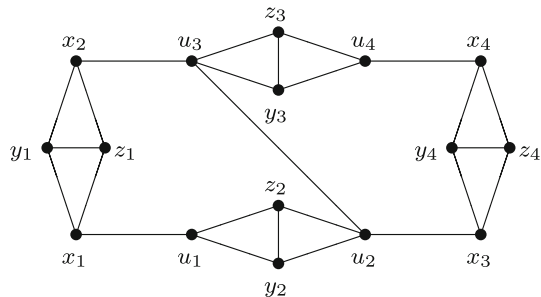spindle   kite   dart

banner   house

Fig. 3 The graph *badge*

Let us define the following 6 classes of graphs:

- $\mathcal{X}_1^*$ is the set of all forests
- $\mathcal{X}_2^*$ is the set of line graphs of all forests of maximum degree at most 3
- $\mathcal{X}_3^*$ is the set of all graphs in which any 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup$ {*cricket, kite, diamond* $+ K_1$}.
- $\mathcal{X}_4^*$ is the set of all graphs in which any 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup$ {*kite, diamond* $+ K_1$, *butterfly, crown*}.
- $\mathcal{X}_5^*$ is the set of all graphs in which any 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup$ {*kite, diamond* $+ K_1$, *house*, $C_4 + K_1$, F_4, W_4, *bull, dart, crown*}.
- $\mathcal{X}_6^*$ is the set of all graphs in which any 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup$ {*cricket, bull, house, banner*, $C_4 + K_1$, C_5}.

All these classes are hereditary. The main result of this paper can be formulated as follows. Let $\mathcal{S}$ be a set of graphs each on at most 5 vertices such that either $\mathcal{S}$ has at most 3 graphs or *bull* $\in \mathcal{S}$. Then the 3-COL problem is polynomial for $Free(\mathcal{S})$ whenever it does not include each of the mentioned 6 classes; otherwise, it is NP-complete.

3 Some New Results on NP-Completeness of the 3- COL Problem

A *badge* is the graph drawn in Fig. 3.

Lemma 1 *The graph badge is 3-colorable. Moreover, in any 3-coloring of a badge, the vertices* x_1, x_2, x_3, x_4 *receive the same color.*
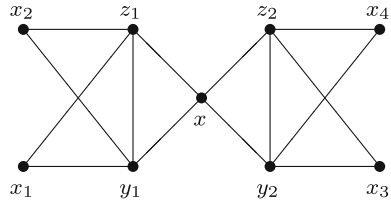
Fig. 4 The graph *spider*
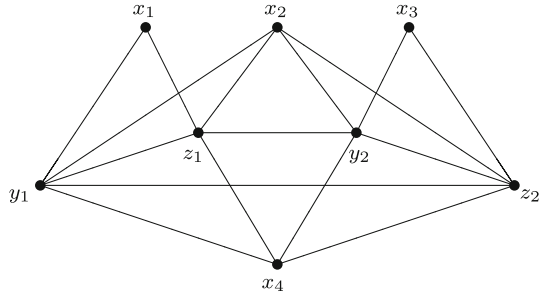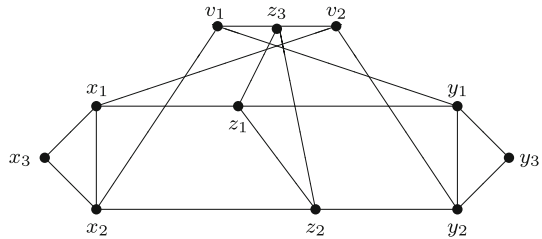
Fig. 5 The graph *chevron*

Fig. 6 The graph *castle*

Proof Firstly, we show that a *badge* is 3-colorable. Assign 1 as the color of $x_1, x_2, x_3, x_4, y_2, y_3$, 2 as the color of y_1, y_4, u_1, u_2, z_3, 3 as the color of z_1, z_2, z_4, u_3, u_4. The resultant mapping is a 3-coloring. Let c be an arbitrary 3-coloring of a *badge*. Then, $c(x_1) = c(x_2), c(x_3) = c(x_4), c(u_1) = c(u_2), c(u_3) = c(u_4)$. Clearly, $c(x_1) \neq c(u_1), c(u_2) \neq c(u_3), c(x_2) \neq c(u_3)$. Hence, the vertices x_1, u_2, u_3 have pairwise distinct colors. Therefore, $c(x_1) = c(x_2) = c(x_3) = c(x_4)$.   □

A *spider* is the graph drawn in Fig. 4.
A *chevron* is the graph drawn in Fig. 5.
The following lemma is easy to prove.

Lemma 2 *The graph spider is 3-colorable. Additionally, x, x_1, x_2, x_3, x_4 have the same color in any 3-coloring of a spider. A chevron has the unique 3-coloring, where $\{x_1, x_2, x_3, x_4\}, \{y_1, y_2\}$, and $\{z_1, z_2\}$ are the color classes.*

The graph *castle* is drawn in Fig. 6.

Lemma 3 *The graph castle is 3-colorable. Moreover, in any 3-coloring of a castle, the vertices x_3 and y_3 receive the same color.*

Proof Firstly, we show that a *castle* is 3-colorable. Assign 1 as the color of x_1, z_3, y_1, 2 as the color of x_2, z_1, y_2, 3 as the color of x_3, z_2, y_3, v_1, v_2. The resultant mapping is a 3-coloring.

Assume that there is a 3-coloring c of a *castle* such that $c(x_3) \neq c(y_3)$. We also assume that $c(x_1) = 1$ and $c(x_2) = 2$. There are the only four possible cases for the value of $(c(y_1), c(y_2))$, they are $(1, 3), (3, 2), (2, 3), (3, 1)$. The first two cases are equivalent, the second two are also equivalent. In the first two cases, there are two vertices in $\{x_1, x_2, y_1, y_2\}$ with the same indices having the same color. In the second two cases, vertices in the set with the same color have different indices. Suppose that $c(y_1) = 1$ and $c(y_2) = 3$. Hence, $c(z_2) = 1, c(v_1) = 3, c(v_2) = 2$. Therefore, $c(z_3) = 1$. We have a contradiction, as z_2 and z_3 have the same color. Now, suppose that $c(y_1) = 2$ and $c(y_2) = 3$. Hence, $c(z_2) = 1, c(z_1) = 3$, and $c(z_3) = 2$. The vertex v_2 must have the color 2, as $c(x_1) = 1$ and $c(y_2) = 3$. We have a contradiction, as v_2 and z_3 receive the same color. So, the initial assumption was false. □

Let G be a graph, x be a vertex of G, whose neighborhood consists of 4 vertices v_1, v_2, v_3, v_4. Let H be an arbitrary graph in $\{badge, spider, chevron\}$. An H-*implantation* to x is to delete x from G, add H and the edges $x_1 v_1, x_2 v_2, x_3 v_3, x_4 v_4$. A *castle-implantation* to x is to delete x from G, add a *castle* and the edges $v_1 x_3, v_2 x_3, v_3 y_3, v_4 y_3$. By Lemmas 1–3, all the resultant graphs are 3-colorable whenever G is 3-colorable.

Lemma 4 *The 3-*COL *problem is NP-complete for each of the classes* $\mathcal{X}_3^*, \mathcal{X}_4^*, \mathcal{X}_5^*, \mathcal{X}_6^*$.

Proof The 3- COL problem is NP-complete for the class $\mathcal{Y}$ of all connected graphs having degrees of all vertices equal to 4 [4]. Let us consider a graph $G \in \mathcal{Y}$ and an arbitrary graph $H \in \{badge, spider, chevron\}$. We simultaneously apply an H-implantation to all vertices of G. The resultant graph G' belongs to $\mathcal{X}_3^*$ or $\mathcal{X}_4^*$ or $\mathcal{X}_5^*$ if $H = badge$ or $H = spider$ or $H = chevron$, respectively. To justify this fact, let us consider an induced subgraph G^* of G' on at most 5 vertices. If G^* is not connected, then $G^* \in \mathcal{X}_1^* \cup \mathcal{X}_2^*$, except the cases, when $G^* = diamond + K_1$ or $G^* = C_4 + K_1, H = chevron$. If G^* is connected, then it contains at most one edge in $E(G) \cap E(G')$ or it is isomorphic to a P_5. The graph P_5 belongs to $\mathcal{X}_1^*$. Clearly, G^* belongs to one of the classes $\mathcal{X}_3^*, \mathcal{X}_4^*, \mathcal{X}_5^*$ (depending on the choice of H) whenever $|E(G^*) \cap E(G)| \leq 1$. The graph G' is 3-colorable if and only if G is 3-colorable, by Lemmas 1 and 2. Hence, the 3- COL problem for $\mathcal{Y}$ can be polynomially reduced to the same problem for each of the classes $\mathcal{X}_3^*, \mathcal{X}_4^*, \mathcal{X}_5^*$. Therefore, it is NP-complete for each of them.

The 3- COL problem is NP-complete for the class $\mathcal{Z}$ of all connected triangle-free graphs having degree of every vertex at most 4 [14]. Let $G \in \mathcal{Z}$. We will simultaneously apply a *castle*-implantation to all degree 4 vertices of G. Similar to the reasonings from the previous paragraph one can prove that the resultant graph G'' belongs to $\mathcal{X}_6^*$. By Lemma 3, G'' is 3-colorable if and only if G is 3-colorable. Thus, the 3- COL problem for $\mathcal{Z}$ can be polynomially reduced to the same problem for $\mathcal{X}_6^*$. Therefore, it is NP-complete for $\mathcal{X}_6^*$. □

4 Irreducible Graphs and their Properties

An *odd wheel* is an arbitrary graph in $\{W_3, W_5, W_7, \ldots\}$. A graph is said to be *odd wheel-free* if it does not contain an induced *odd wheel*. This property can be checked in polynomial-time. Since any *odd wheel* is not 3-colorable, a necessary condition for a graph to be 3-colorable is to be *odd wheel*-free. Note that a graph is *odd wheel*-free if and only if it is *odd wheel$_s$-free*, i.e. it does not contain an *odd wheel* as a subgraph. This property can be checked in polynomial time because it is equivalent to the property that every neighborhood induces a bipartite subgraph.

If a graph contains a *spindle* as a subgraph (not necessarily induced), then the graph is not 3-colorable. Hence, a necessary condition for a graph to be 3-colorable is to be *spindle$_s$-free*. This property can also be tested in polynomial time.

Two non-adjacent vertices of a graph are called *twins* if they have equal neighborhoods. Two vertices are said to be *quasi-twins* if the neighborhood of one of them is included in the neighborhood of the second one. If G is a graph, $x, y \in V(G)$, and $N(x) \subseteq N(y)$, then $\chi(G) = \chi(GP\backslash\{x\})$. Indeed, one can arrange a color of y from a 3-coloring of $G\backslash\{x\}$ to the vertex x to produce a 3-coloring of G.

A *cut-vertex* of a connected graph G is a vertex x, whose removal disconnects the graph. An *x-block* of G is any subgraph of G induced by all vertices of a connected component of $G\backslash\{x\}$ and x. Verifying that a given vertex is a cut-vertex of G and computing all the corresponding blocks can be done in linear time by the depth first search algorithm. If $G_1^x, \ldots, G_s^x$ are all x-blocks of G, then $\chi(G) = \max_{i \in \overline{1,s}}(\chi(G_i^x))$.

Clearly, if a vertex x of a graph G has degree at most 2, then G is 3-colorable if and only if $G\backslash\{x\}$ is 3-colorable. Moreover, by Brooks' Theorem [3], a graph G is 3-colorable whenever $\Delta(G) \leq 3$ and G is not isomorphic to a K_4.

If $G_1, \ldots, G_s$ are all connected components of a graph G, then $\chi(G) = \max_{i \in \overline{1,s}}(\chi(G_i))$.

The properties, decompositions, and compressions above lead to consider *irreducible graphs*, i.e. connected *odd wheel*- and *spindle$_s$-free* graphs without pairs of quasi-twins, cut-vertices, vertices of degree at most 2 having maximum degree at least 4. The following lemma is clear.

Lemma 5 *For any hereditary class $\mathcal{X}$, the* 3- COL *problem for $\mathcal{X}$ can be polynomially reduced to the same problem for the set of all irreducible graphs in $\mathcal{X}$.*

By $R(p, q)$ we denote the corresponding *Ramsey number*, i.e. minimum number ν such that every graph with ν vertices contains an O_p or a K_q as an induced subgraph. As any irreducible graph must be K_4-free, the following lemma is true.

Lemma 6 *For any irreducible $K_{1,p}$-free graph G, the inequality $\Delta(G) \leq R(p, 3) - 1$ holds.*

As $R(4, 3) = 9$, for any irreducible $K_{1,4}$-free graph G, the inequality $4 \leq \Delta(G) \leq 8$ is true. Any vertex of maximum degree of any irreducible $K_{1,4}$-free graph must belong to some triangle of the graph. Indeed, the contrary would imply that in some irreducible $K_{1,4}$-free graph the neighbourhood of some vertex of maximum degree

induce an empty graph and this empty graph has at most three vertices. Hence, the graph cannot be irreducible.

Lemma 7 *There are no irreducible* $\{K_{1,4}, diamond, bull\}$*-free graphs and* $\{K_{1,4}, diamond, cricket, butterfly\}$*-free graphs.*

Proof Let G be an irreducible $\{K_{1,4}, diamond\}$-free graph, x be a vertex of G of maximum degree. Then, there is a triangle (x, y, z), vertices x' and x'' each adjacent to x and simultaneously non-adjacent to y and z. If $x'x'' \in E(G)$, then G is not *butterfly*-free, otherwise G is not *cricket*-free. Hence, there are no irreducible graphs in $Free(\{K_{1,4}, diamond, cricket, butterfly\})$. As G is irreducible, each of the degrees of y and z must be at least 3, simultaneously. Hence, there are vertices y' and z' such that $y' \in N(y)\backslash(N(x) \cup N(z))$ and $z' \in N(z)\backslash(N(x) \cup N(y))$. If G is *bull*-free, then $y'z', y'x', y'x'', z'x', z'x''$ are edges of G. Hence, the vertices x', x'', y', z' induce a *diamond* or a K_4 in G. We have a contradiction. Thus, there are no irreducible $\{K_{1,4}, diamond, bull\}$-free graphs. $\square$

Lemma 8 *Every irreducible* $\{kite, cricket, butterfly\}$*-free graph is diamond-free. Every irreducible* $\{K_{1,4}, kite, bull\}$*-free graph on at least 8 vertices is diamond-free.*

Proof Assume that there is an irreducible $\{kite, butterfly, cricket\}$-free graph G containing a *diamond* induced by vertices x', x'', y, z, where $x'x'' \notin E(G)$. As x' and x'' are not quasi-twins, there are vertices $z' \in N(x')\backslash N(x'')$ and $y' \in N(x'')\backslash N(x')$. Each of the vertices y' and z' belongs to $(N(y)\backslash N(z)) \cup (N(z)\backslash N(y))$, as G is irreducible and *kite*-free. Without loss of generality, $z'y \in E(G)$. As G is $\{kite, butterfly\}$-free, $y'z \in E(G)$. Otherwise, $y' \in N(y)\backslash N(z)$ and $y'z' \notin E(G)$, as G is *odd wheel*-free and y, x', z', y', x'' induce a *butterfly*. As z' and z are not quasi-twins, there is a vertex $y'' \in N(z')\backslash N(z)$. Clearly, y'' and x' are adjacent, otherwise $y''y \in E(G)$ and $y''x'' \in E(G)$, as G is $\{kite, butterfly\}$-free and x', z', y'', x'', z, y induce a W_5. Similarly, there is a vertex $z'' \in (N(x'') \cap N(y'))\backslash(N(y) \cup N(z))$. To avoid an induced *kite*, $x''y'', y'z', x'z''$ are edges of G. To avoid a *cricket* induced by x'', y, z, y'', z'', the vertices y'' and z'' must be adjacent. Then, x', y, z, y'', z'' induce a *butterfly*. We have a contradiction with the assumption.

Let G be an irreducible $\{K_{1,4}, kite, bull\}$-free graph having at least 8 vertices. We show that G cannot contain an induced cycle $(x_1, x_2, x_3, x_4, x_5, x_6)$ and a vertex x adjacent to all vertices of the cycle. Assume that G contains such a cycle and such a vertex x. As G is connected and $|V(G)| \geq 8$, there is a vertex outside V' having a neighbor in V', where $V' \triangleq \{x, x_1, x_2, x_3, x_4, x_5, x_6\}$. As G is $K_{1,4}$-free, for every vertex outside V' having a neighbor in V', it has a neighbor in $V'\backslash\{x\}$. Let y be such a vertex. Let us show that y is adjacent to each of the vertices $x_1, x_2, x_3, x_4, x_5, x_6$. Suppose that x_1 and y are adjacent, but y and x_2 are not adjacent. If $yx \notin E(G)$, then yx_4 and yx_5 are edges of G, to avoid an induced *bull*. Hence, x, y, x_5, x_4, x_2 induce a *kite*. If x and y are adjacent, then $yx_4 \notin E(G)$ and $yx_6 \notin E(G)$, as G is *odd wheel*-free. Hence, x, y, x_2, x_4, x_6 induce a $K_{1,4}$. We have a contradiction in both cases. Clearly, $N(x) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, as a possible neighbor of x distinct from $x_1, x_2, x_3, x_4, x_5, x_6$ must be adjacent to each of the 6 vertices and to x, by the previous reasonings from this paragraph and the fact that G is $K_{1,4}$-free.

Hence, G would not be *odd wheel*-free. Therefore, $N(x) \subseteq N(y)$, i.e. x and y are quasi-twins. We have a contradiction with the assumption about the existence of a 6-cycle and a vertex.

Assume that there is an irreducible $\{K_{1,4}, kite, bull\}$-free graph G on at least 8 vertices containing an induced *diamond*. Let x', x'', y, z, y', z' have the same meaning as in the first paragraph up to and including the hypothesis that $z'y \in E(G)$. We show in this paragraph that $yy' \notin E(G)$, i.e. $y'z \in E(G)$. Suppose the opposite. Clearly, $z'y' \notin E(G)$, otherwise z', x', z, x'', y', and y induce a W_5. As z and z' are not quasi-twins, there is a vertex $z_1 \in N(z')\backslash N(z)$. Clearly, $z_1 \neq y'$. The vertex z_1 cannot be adjacent to x', otherwise x', y, z, z_1, y' or x', z', z_1, z, y' induce a *bull*. Hence, $z_1y \in E(G)$. As z and y' are not quasi-twins, there is a vertex $z_2 \in N(y')\backslash N(z)$. Clearly, $z_2 \neq z', x''z_2 \notin E(G)$, and $z_2y \in E(G)$. Clearly, $z_1 \neq z_2$, otherwise $(z, x', z', z_1, y', x'')$ is an induced 6-cycle and y is adjacent to all its vertices. Since G is *odd wheel*-free, $z_1z_2 \notin E(G)$. Then, y, z_1, z_2, x', x'' induce a $K_{1,4}$.

So, y' is adjacent to z and x''. As z and z' are not quasi-twins, there is a vertex $y'' \in N(z')\backslash N(z)$. We may assume that y'' is adjacent to x' and z', by a similar argument as above. As G is *spindle$_s$*-free, $y'y'' \notin E(G)$. Then, x', y, z, y', y'' induce a *bull*. We have a contradiction.   □

Lemma 9 *Every irreducible $\{K_{1,4}, bull, cricket\}$-free graph containing either a F_4 or a W_4 as an induced subgraph has at most 457 vertices.*

Proof Let G be an irreducible $\{K_{1,4}, bull, cricket\}$-free graph. Assume that G contains an induced F_4. We use the same notation for vertices of the subgraph as in the definition. We will show that every vertex of G lies from x at distance at most 2. Assume that there is a vertex z of G lying from x at distance 3. Let (x, x', y, z) be a length 3 induced path and $V' \triangleq \{x_1, x_2, x_3, x_4\}$. Clearly, $y \notin N[x]$ and $N(z) \cap N[x] = \emptyset$. One may assume that $x' \in V'$. Suppose the opposite. Then, $x' \notin V'$ and $N(y) \cap (\{x\} \cup V') = \emptyset$. Clearly, $x'x_1 \notin E(G)$ and $x'x_4 \notin E(G)$, otherwise both $x'x_1$ and $x'x_4$ are edges of G, to avoid an induced *bull*. It is impossible, as G is *odd wheel*-free. To avoid an induced *cricket*, $x'x_2$ and $x'x_3$ are edges of G. Hence, G is not *odd wheel*-free. We have a contradiction. So, $x' \in V'$. To avoid a *bull* induced by a subset of $V' \cup \{x, y, z\}$, either $N(y) \cap V' = V'$ or $N(y) \cap V' = \{x_1, x_4\}$ is true. The first case is impossible, as G is *cricket*-free. As G is irreducible, y is not a cut-vertex of G. Hence, there is an induced path connecting x and z of length at least 3. Its third vertex must be distinct from y and simultaneously adjacent to x_1 and x_4. It is impossible, as G is *cricket*-free and *spindle$_s$*-free. We have a contradiction with the assumption. As every vertex of G lies from x at distance at most 2 and $\Delta(G) \leq 8$, $|V(G)| \leq 1 + 8 + 7 \cdot 8 = 65$.

Assume that G has an induced W_4 and G is F_4-free. We also use the same notation for vertices of the subgraph W_4 as in the definition. As x_1 and x_3 are not quasi-twins, there is a vertex $y_1 \in N(x_1)\backslash N(x_3)$. As G is $\{K_4, F_4\}$-free, $y_1 \in N(x_1)\backslash(N(x_2) \cup N(x_3) \cup N(x_4) \cup N(x))$. Similarly, for any $i = \overline{2, 4}$, there is a vertex $y_i \in N(x_i)\backslash(\bigcup_{j=1, j \neq i}^{4} N(x_j) \cup N(x))$. As G is *bull*-free, $y_1y_2, y_2y_3, y_3y_4, y_4y_1$ are edges of G. Similarly, $y_1y_3 \notin E(G)$ and $y_2y_4 \notin E(G)$. We will show that every

vertex of G lies from x at distance at most 3. Assume that there is a vertex y lying at distance 4 from x. Consider an induced path (x, a_1, a_2, a_3, y). We may consider that $a_1 \in V'$. Otherwise, as G is {$bull$, K_4, F_4}-free, there are the only following cases: a) $N(a_1) \cap V' = \emptyset$, b) $N(a_1) \cap V' = \{x_1, x_3\}$, c) $N(a_1) \cap V' = \{x_2, x_4\}$. Moreover, $N(a_2) \cap V' = \emptyset$ and $N(a_3) \cap V'' = \emptyset$, where $V'' \triangleq \{y_1, y_2, y_3, y_4\}$. In case a, $a_1 y_1, a_1 y_2, a_1 y_3, a_1 y_4$ are edges of G, as G is $bull$-free. Since $G \in Free(\{bull, K_4, F_4\})$, $a_2 y_1 \notin E(G), a_2 y_2 \notin E(G), a_2 y_3 \notin E(G), a_2 y_4 \notin E(G)$. Hence, x, a_1, a_2, y_1, y_2 induce a $cricket$. Let us consider case b, case c is similar. As G is {F_4, $bull$}-free, $a_1 y_1 \notin E(G), a_1 y_3 \notin E(G), a_2 y_1 \in E(G), a_2 y_3 \in E(G)$. Hence, y_1, y_3, a_1, a_2, a_3 induce a $K_{1,4}$.

So, we assume that $a_1 \in V'$. Suppose that $a_1 = x_1$. We may also consider that $a_2 = y_1$. Otherwise, $N(a_3) \cap V'' = \emptyset, x, x_1, x_2, y_1, a_2$ induce a graph in {F_4, $cricket$} or the graph $bull$ is induced by one of the sets $\{x_1, x_2, y_1, a_2, a_3\}$ and $\{y_1, x_1, a_2, x, a_3\}$. As G is {$K_{1,4}$, $bull$, $cricket$}-free, $a_3 y_1, a_3 y_2, a_3 y_3, a_3 y_4$ are edges of G. Hence, y_1, y_2, a_1, a_3, a_4 induce a $bull$. We have a contradiction with the assumption. So, distance between x and any vertex of G is at most 3. Hence, $|V(G)| \le 1 + 8 + 7 \cdot 8 + 7 \cdot 7 \cdot 8 = 457$.   □

Lemma 10 *The* 3-COL *problem can be solved in polynomial time for each of the classes* $Free(\{K_{1,4}, bull, cricket, crown\})$ *and* $Free(\{K_{1,4}, bull, cricket, butterfly\})$.

Proof Let G be an irreducible {$K_{1,4}$, $bull$, $cricket$}-free graph. By Lemma 9, we may assume that it is {F_4, W_4}-free. Let us show that $\Delta(G) = 4$. Let v be a vertex of G having neighbors v_1, v_2, v_3, v_4, v_5. As G is {$K_{1,4}$, K_4, $cricket$}-free, there are three of them inducing a P_3 in G. Suppose that (v_1, v_2, v_3) is the induced 3-path. As G is {K_4, F_4, W_4}-free, neither v_4 nor v_5 belongs to $N(v_1) \cup N(v_3)$. Hence, v, v_1, v_3, v_4, v_5 induce a $cricket$ or a $K_{1,4}$. We have a contradiction.

Suppose that G contains an induced $fish$. Let y_1 and y_2 be the degree 3 vertices of the fish, x_2 be its degree 4 vertex, (x_1, y_1, y_2) and (x_2, z_1, z_2) are its triangles. We will show that $N(y_1) \backslash \{y_2\} = N(y_2) \backslash \{y_1\}$. Assume the opposite. Without loss of generality, one may assume that there is a vertex $y \notin \{x_1, x_2, y_2\}$ adjacent to y_1 and non-adjacent to y_2. It cannot be adjacent to each of the vertices x_1 and x_2, as G is {F_4, W_4}-free. As G is $bull$-free, $y z_1$ and $y z_2$ are edges of G. As G is irreducible and {F_4, W_4}-free, there is a vertex $x \in N(x_1) \backslash (N(x_2) \cup N(y_1) \cup N(y_2))$. To avoid an induced $bull$, $yx \in E(G)$. To avoid an induced $cricket$, x must be adjacent to at least one of the vertices z_1 and z_2. Hence, x, y, x_2, z_1, z_2 induce a F_4 or a W_4. Therefore, $N(y_1) \backslash \{y_2\} = N(y_2) \backslash \{y_1\}$ is true. Similarly, if there is a common neighbor of z_1 and z_2 distinct from x_2, then $N(z_1) \backslash \{z_2\} = N(z_2) \backslash \{z_1\}$. Such a neighbor must exist, otherwise, there are vertices $z' \in N(z_1) \backslash N(z_2)$ and $z'' \in N(z_2) \backslash N(z_1)$, as G is irreducible. Recall that the degree of x_2 is 4. Hence, as G is $bull$-free, $z' z'', z' y_1, z' y_2, z'' y_1, z'' y_2$ are edges of G. We have a contradiction, as G is K_4-free. If $deg(y_1) = deg(y_2) = 3$, then the graph H^* formed by deleting x_2, y_1, y_2 and adding the edges $x_1 z_1$ and $x_1 z_2$ is 3-colorable if and only if it is so for G. It is not hard to see that H^* is also {$K_{1,4}$, $bull$, $cricket$}-free. So, for any induced $fish$, we may assume that a $diamond$ included in it must be included in an induced $crown$.

Let a_1, a_2, b_1, b_2 be vertices of G inducing a *diamond* such that $a_1 a_2 \notin E(G)$. Assume that this *diamond* is not included in an induced *crown*. Let $N_1 \triangleq (N(a_1) \cup N(a_2)) \backslash \{a_1, a_2, b_1, b_2\}$ and $N_2 \triangleq (N(b_1) \cup N(b_2)) \backslash \{a_1, a_2, b_1, b_2\}$. As G is $\{K_4, F_4, W_4\}$-free, none of the elements of N_1 is adjacent to an element of $\{b_1, b_2\}$, none of the elements of N_2 is adjacent to an element of $\{a_1, a_2\}$. As G is irreducible, then $\max(deg(a_1), deg(a_2)) \geq 3$. As G is *cricket*-free, if $deg(a_1) = 4$, then its two neighbors each distinct from b_1 and b_2 must be adjacent. Moreover, one of them is adjacent to a_2, otherwise there is a *diamond* included in an induced *fish* that is not included in an induced *crown*. As G is irreducible, $deg(a_2) = 4$. Therefore, if $\max(deg(a_1), deg(a_2)) = 4$, then there are vertices $u_1 \in N(a_1) \backslash N(a_2), u_2 \in N(a_2) \backslash N(a_1), u \in N(a_1) \cap N(a_2)$. As G is *cricket*-free, uu_2 is an edge of G. As G is *bull*-free, u_1 and u_2 are adjacent. Hence, G is not *spindle$_s$*-free. So, $deg(a_1) = deg(a_2) = 3$.

Suppose that N_2 has two elements or one element simultaneously non-adjacent to b_1 and b_2 or $N_2 = \emptyset$. Our aim is to show that G is 3-colorable if and only if $H^{**} \triangleq G \backslash \{a_1, a_2, b_1, b_2\}$ is 3-colorable. Since G is *bull*-free, if N_2 has two elements, then they must be adjacent. Assume that H^{**} has a 3-coloring. There is a color (say, 1) distinct from the colors of the vertices in N_1. Assign 1 as the color of a_1 and a_2. If $|N_2| = 2$, then its elements have the colors 1 and 2 or 1 and 3 or 2 and 3. In each of the three cases b_1 and b_2 can be colored in 2 and 3 such that the resultant coloring of G is a 3-coloring. Similarly, G has a 3-coloring whenever $|N_2| \leq 1$.

By the previous reasonings and Lemmas 5 and 9, the 3-COL problem for $Free(\{K_{1,4}, bull, cricket, crown\})$ can be polynomially reduced to the 3-COL problem for $Free(\{K_{1,4}, bull, diamond, cricket, crown\})$. By Lemma 7, the problem is polynomial for $Free(\{K_{1,4}, bull, cricket, crown\})$.

Assume that G is an irreducible $\{K_{1,4}, bull, cricket, butterfly, F_4, W_4\}$-free graph. Let a_1, a_2, b_1, b_2, b_3 be vertices of G inducing a *crown* such that (a_1, a_2, b_1), $(a_1, a_2, b_2), (a_1, a_2, b_3)$ are the triangles of the subgraph. Let $N \triangleq \bigcup_{i=1}^{3} N(b_i) \backslash \{a_1, a_2\}$. We call the set N the *neighborhood* of the corresponding induced *crown*. It is easy to see that $deg(b_1) = deg(b_2) = deg(b_3) = 3$, i.e. $|N| \leq 3$. As G is *bull*-free, N does not induce a K_3. It is easy to verify that G is 3-colorable whenever N has at most 2 elements and $H' \triangleq G \backslash \{a_1, a_2, b_1, b_2, b_3\}$ has a 3-coloring. If N has 3 elements, then G is 3-colorable if and only if there is a 3-coloring of H' in which the elements of N receive at most two distinct colors. The graph G' is obtained from H' by adding a new vertex adjacent to all elements of N. In other words, G' can be obtained from G by contracting $\{a_1, a_2, b_1, b_2, b_3\}$ into a single vertex. Notice that the three edges between $\{b_1, b_2, b_3\}$ and N in G also exists in G'. Clearly, G' is 3-colorable if and only if there is a 3-coloring of H' in which the elements of N receive at most two distinct colors. In other words, G is 3-colorable if and only if G' is 3-colorable.

By the reasonings above, we may assume that every induced *diamond* of G is contained in an induced subgraph *crown* of G, every induced subgraph *crown* of G has neighborhood with 3 elements. Any two induced copies of a *crown* have no common vertices. Let G^* be the graph obtained by contracting vertices into a single vertex in every induced copy of a *crown* in the graph G. As G is 3-colorable if and only

if G' is 3-colorable, it is so for G and G^*. As G is $\{K_{1,4}, cricket, butterfly, F_4, W_4\}$-free, every degree 4 vertex v of G is a degree 3 vertex of an induced $diamond$. In other words, v is a degree 4 vertex of an induced $crown$. Hence, $\Delta(G^*) \leq 3$, as $\Delta(G) = 4$. By Brooks' Theorem, G^* is 3-colorable. Hence, G is 3-colorable. Therefore, the 3- COL problem for $Free(\{K_{1,4}, bull, cricket, butterfly\})$ can be polynomially reduced to the 3- COL problem for $Free(\{K_{1,4}, bull, diamond, cricket, butterfly\})$. Hence, by Lemma 7, the *lemma* holds.   □

5 Main Result

Recall that $\mathcal{X}_1^*$ is the set of all forests, $\mathcal{X}_2^*$ is the set of all line graphs of forests of maximum degree at most 3. For any $i \in \overline{3, 6}$, the class $\mathcal{X}_i^*$ is the set of all graphs in which any 5 vertices induce a subgraph in:

- $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, kite, diamond + K_1\}$ for $i = 3$
- $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, diamond + K_1, butterfly, crown\}$ for $i = 4$
- $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{kite, diamond + K_1, house, C_4 + K_1, F_4, W_4, bull, dart, crown\}$ for $i = 5$
- $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{cricket, bull, house, banner, C_4 + K_1, C_5\}$ for $i = 6$

Theorem 1 *Let $\mathcal{S}$ be a set of graphs each on at most 5 vertices such that either $\mathcal{S}$ has at most 3 graphs or $bull \in \mathcal{S}$. Then, the 3- COL problem is NP-complete for $\mathcal{X} = Free(\mathcal{S})$ if $\mathcal{X}$ includes at least one of the classes $\mathcal{X}_1^*$–$\mathcal{X}_6^*$. It is polynomial-time solvable for each of the remaining cases.*

Proof It is known that the 3-COL and the EDGE 3- COL problems are NP-complete for $Free(\{C_3, C_4, \ldots, C_k\})$ for every $k \geq 3$ [12]. Hence, for every $k \geq 3$, the 3- COL problem is NP-complete for the class of line graphs of all $\{C_3, C_4, \ldots, C_k\}$-free graphs of maximum degree at most 3. Therefore, the 3- COL problem is NP-complete for $\mathcal{X}$ if $\mathcal{X}_1^* \cap \mathcal{S} = \emptyset$ or $\mathcal{X}_2^* \cap \mathcal{S} = \emptyset$. In other words, when $\mathcal{X} \supseteq \mathcal{X}_1^*$ or $\mathcal{X} \supseteq \mathcal{X}_2^*$. The 3- COL problem is NP-complete for each of the classes $\mathcal{X}_3^*, \mathcal{X}_4^*, \mathcal{X}_5^*, \mathcal{X}_6^*$, by Lemma 4. Hence, the problem is NP-complete for $\mathcal{X}$ if it includes at least one of the classes $\mathcal{X}_1^*$–$\mathcal{X}_6^*$.

Assume that $\mathcal{X}_1^* \not\subseteq \mathcal{X}, \mathcal{X}_2^* \not\subseteq \mathcal{X}, \mathcal{X}_3^* \not\subseteq \mathcal{X}, \mathcal{X}_4^* \not\subseteq \mathcal{X}, \mathcal{X}_5^* \not\subseteq \mathcal{X}, \mathcal{X}_6^* \not\subseteq \mathcal{X}$. If $G_1 \in \mathcal{X}_1^*$ and $G_2 \in \mathcal{X}_2^*$ are arbitrary graphs each having at most 5 vertices, $\{G_1, G_2\} \neq \{K_{1,4}, bull\}$ and $\{G_1, G_2\} \neq \{K_{1,4}, butterfly\}$, then the 3- COL problem is polynomial-time solvable for $Free(\{G_1, G_2\})$ [16]. Hence, we may also assume that a $K_{1,4}$ is the unique forest in $\mathcal{S}$, a $bull$ or a $butterfly$ is an element of $\mathcal{S}$, $(\mathcal{X}_2^* \backslash \{bull, butterfly\}) \cap \mathcal{S} = \emptyset$. Suppose that $bull \in \mathcal{S}$. As $\mathcal{X}_3^* \not\subseteq \mathcal{X}$, $\mathcal{S}$ contains an element G', which is an induced subgraph of a graph in $\{cricket, kite, diamond + K_1\}$. Hence, $G' \in \{cricket, kite, diamond + K_1, diamond\}$. Clearly, any $H + K_1$-free graph G is H-free or it contains at most $(\Delta(G) + 1)|V(H)|$ vertices. By this fact, Lemmas 7 and 8, the problem is polynomial for $\mathcal{X}$ if $G' \in \{kite, diamond + K_1, diamond\}$. If $cricket \in \mathcal{S}$, then $\{butterfly, crown\} \cap \mathcal{S} \neq \emptyset$, as $\mathcal{X}_4^* \not\subseteq \mathcal{X}$. By Lemma 10, the problem is polynomial for $\mathcal{X}$ in both possible cases. Suppose that $butterfly \in \mathcal{S}$ and $bull \notin \mathcal{S}$. As $\mathcal{X}_3^* \not\subseteq \mathcal{X}$, $\mathcal{S}$ contains a graph $G'' \in \{cricket, kite, diamond + K_1, diamond\}$.

In other words, $\mathcal{S} = \{K_{1,4}, butterfly, G''\}$. If $G'' = cricket$, then $\mathcal{X}$ includes $\mathcal{X}_5^*$. If $G'' \in \{kite, diamond + K_1, diamond\}$, then $\mathcal{X}$ includes $\mathcal{X}_6^*$. The last cases are impossible. □

6 Concluding Remarks and Problems for Future Work

In this paper, we have presented a complexity dichotomy for the 3- COL problem in the families: $\{Free(\{H_1, H_2, H_3\})| \max_{i \in \overline{1,3}} |V(H_i)| \leq 5\}$ and $\{Free(\mathcal{S})|$ each graph in $\mathcal{S}$ has at most 5 vertices and $bull \in \mathcal{S}\}$. More precisely, if $\mathcal{X}$ is a class in the families, then the 3- COL problem is NP-complete for $\mathcal{X}$ if $\mathcal{X} \supseteq \mathcal{X}_i^*$ for some $i \in \overline{1, 6}$, otherwise the problem can be solved in polynomial time for graphs $\mathcal{X}$. This result has a natural consequence for the COL problem, as this problem becomes NP-complete for any class in the families that contains one of the 6 classes of graphs. However, we cannot claim polynomial-time solvability of the COL problem for the remaining classes in the families, as some places in the proofs from Section 5 heavily use the assumption that the vertices may only be colored with at most 3 different colors. Clarification of the complexity of the COL problem for at least some of these classes is an interesting research problem for future work. Concerning the 3- COL problem, the next natural step pushing the research forward is to obtain a complexity dichotomy for all hereditary classes defined by four forbidden induced structures each on at most 5 vertices. Perhaps, such a dichotomy will already give a complete classification for all hereditary classes defined by forbidding induced subgraphs each on at most 5 vertices.

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