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ORIGINAL PAPER

Spanning Tree Decompositions of Complete Graphs Orthogonal to Rotational 1-Factorizations

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Abstract In Krussel et al. (ARS Comb 57:77–82, 2000), Krussel, Marshall, and Verall proved that whenever $2n - 1$ is a prime of the form $8m + 7$, there exists a spanning tree decomposition of K_{2n} orthogonal to the 1-factorization GK_{2n}. In this paper, we develop a technique for constructing spanning tree decompositions that are orthogonal to rotational 1-factorizations of K_{2n}. We apply our results to show that, for *every* $n > 2$, there exists a spanning tree decomposition orthogonal to GK_{2n}. We include similar applications to other rotational families of 1-factorizations, and provide directions for further research.

Keywords Spanning tree · 1-factorization · Complete graph · Matching

Mathematics Subject Classification 05C05 · 05C15 · 05C70

1 Introduction

A **graph** $G = (V, E)$ consists of a finite set V of **vertices** and a set E of 2-element subsets of V called **edges**. A **1-factor** in G is a set of edges that partition V, and a **1-factorization** is a partition of E into 1-factors. The 1-factorizations of the complete graph K_{2n} are well-studied and we refer to [5,6] for an introduction to the topic and a survey of many known results.

The number of non-isomorphic 1-factorizations of K_{2n} grows very rapidly with n (see [2]), but one well-known family is denoted GK_{2n}, and can be easily described as

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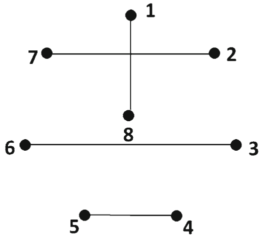
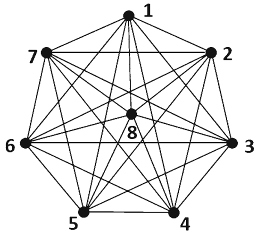
follows (see Example 1.1 below). Place vertices $1, 2, \ldots, 2n-1$ in a circle around a central vertex labeled $2n$. Add the edge $\{1, 2n\}$ and the edges $\{i+1, 2n-i\}$ for $0 < i < n$ to form a 1-factor. Rotating this 1-factor around vertex $2n$ gives a partition of the edges of K_{2n} into exactly $2n-1$ different 1-factors. This partition is the 1-factorization known as GK_{2n}.

Example 1 A 1-factor (left) of the complete graph K_8 (right) can be rotated to partition the edge set, giving the 1-factorization GK_8.



starting 1-factor for GK_8



complete graph K_8

A 1-factorization of K_{2n} is said to be **rotational** (also called *1-rotational*) if it is stabilized by a permutation of the vertices that fixes one vertex and cyclically permutes the rest. The 1-factorization GK_8 illustrated above is an example, since the permutation $\rho = (1, 2, 3, \ldots, 7)$ fixes vertex 8 and cyclically permutes the other vertices. In doing so, ρ also cyclically permutes the seven 1-factors of this 1-factorization, thereby stabilizing the set of them. So GK_8 is rotational, and analogously, we see that GK_{2n} is rotational for any n. For rotational 1-factorizations (and the more general notions of *pyramidal* and *k-pyramidal* 1-factorizations), we refer the reader to [4,5].

Given any 1-factorization $\mathcal{F}$ of K_{2n}, we say that a subgraph G is **orthogonal** to $\mathcal{F}$ if each 1-factor of $\mathcal{F}$ shares at most one edge with G. For example, the star graph, in which vertex 8 is adjacent to vertices 1 through 7, is a spanning tree for K_8 that is orthogonal to the 1-factorization GK_8 described above.

To avoid trivialities, let us assume that $n > 2$. In [1], Brualdi and Hollingsworth conjectured that for any 1-factorization $\mathcal{F}$ of K_{2n}, there exists a partition of K_{2n} into n disjoint spanning trees that are orthogonal to $\mathcal{F}$. In that same paper, they were able to prove that for any 1-factorization $\mathcal{F}$ of K_{2n}, there exist at least two disjoint spanning trees that are orthogonal to $\mathcal{F}$. In [3], Krussel et al. proved that there were at least three such trees. Regarding the 1-factorization GK_{2n}, however, they proved a much stronger result, stated as follows.

Theorem 1 [3, Thm. 3] *If $2n-1$ is a prime number of the form $8m+7$ for some integer m, then there exists a full set of n disjoint spanning trees for K_{2n} that are orthogonal to GK_{2n}.*

In this paper, we develop a technique for finding spanning tree decompositions that are orthogonal to rotational 1-factorizations of K_{2n}. Applying our technique, we prove the following.

Theorem 2 *For any integer $n > 2$, there exists a full set of n disjoint spanning trees for K_{2n} that are orthogonal to GK_{2n}.*

Because our methods exploit the rotational nature of GK_{2n}, we believe that there are similar applications to other rotational families of 1-factorizations. In Sect. 7, we prove Theorem 2 and offer directions for further application of the results.

2 Terminology for Edges in Rotational Subgraphs of K_{2n}

We will be considering many different subgraphs of the complete graph K_{2n}. We draw such graphs in **rotational form**, meaning vertices 1 through $2n-1$ are spaced evenly around a circle in clockwise manner, and vertex $2n$ is placed at the center. An example of a subgraph drawn in rotational form appears in Example 1 above.

Definition 1 Suppose K_{2n} is drawn in rotational form. For any edge $\{a, b\}$, if $2n \notin \{a, b\}$, we define its **length** by

$$\text{length}\{a, b\} = \min\{|a-b|, 2n-1-|a-b|\},$$

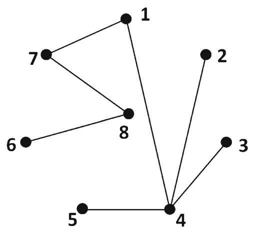
and if $2n \in \{a, b\}$, we say the edge has length 0.

Definition 2 Suppose K_{2n} is drawn in rotational form. For any edge $\{a, b\}$ of nonzero length, we define its **center** (denoted center$\{a, b\}$) to be the unique vertex $x \neq 2n$ such that

$$\text{length}\{a, x\} = \text{length}\{x, b\}.$$

For any edge $\{a, 2n\}$ of length 0, we define the center to be a.

Example 2 A subgraph G of K_8 is drawn in rotational form. The length and center are given for each edge.



subgraph G

edge	1, 4	1, 7	2, 4	3, 4	4, 5	6, 8	7, 8
length	3	1	2	1	1	0	0
center	6	4	3	7	1	6	7

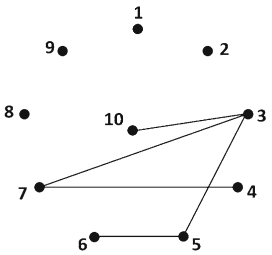
Note that, in rotational form, any edge of K_{2n} is uniquely determined by its center and length.

3 Starter Graphs and Rotational Families

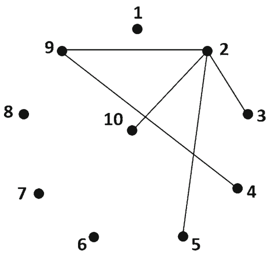
To obtain a rotational decomposition of K_{2n}, we begin with starter graphs, which are graphs drawn in rotational form that contain one edge of each length.

Definition 3 Fix any integer $n > 0$ and any n-tuple of integers $(c_0, \ldots, c_{n-1})$ satisfying $0 < c_l < 2n$ for each l $(0 \leq l < n)$. We define the **starter graph**, denoted $SG(c_0, \ldots, c_{n-1})$, to be the subgraph of K_{2n} with a single edge of length l and center c_l, for each l $(0 \leq l < n)$.

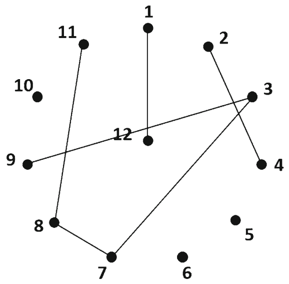
Example 3 The following are examples of starter graphs.



$SG(3, 1, 4, 1, 5)$



$SG(2, 7, 1, 8, 2)$



$SG(1, 2, 3, 4, 5, 6)$

Definition 4 Suppose G is a subgraph of K_{2n} with edge set E. Let ρ denote the permutation of the vertex set that cyclically permutes $(1, 2, 3, \ldots, 2n-1)$ and fixes $2n$. We define $\rho(G)$ to be the subgraph of K_{2n} with edge set

$$\rho(E) = \{\{\rho(a), \rho(b)\} \mid \{a, b\} \in E\}.$$

Equivalently, vertices a, b are adjacent in $\rho(G)$ if and only if the vertices $\rho^{-1}(a)$, $\rho^{-1}(b)$ are adjacent in G. Observe that, for any edge $\{a, b\}$ and integer i, the edge $\{\rho^i(a), \rho^i(b)\}$ has the same length as $\{a, b\}$ and has center $c + i$, where c is the center of $\{a, b\}$.

Using the above, we can now introduce the notion of a rotational family of subgraphs.

Definition 5 Fix any integers $n, d > 0$ and let G be any subgraph of K_{2n}. We define the **rotational family** $\mathcal{F}_G^d$ **generated by** G to be the set

$$\mathcal{F}_G^d := \{G, \rho(G), \ldots, \rho^{d-1}(G)\}$$

where ρ is the permutation given in Definition 4.

Proposition 1 *Fix any integer $n > 0$ and any n-tuple of integers $(c_0, \ldots, c_{n-1})$ satisfying $0 < c_l < 2n$ for each l $(0 \leq l < n)$. Let $G = SG(c_0, \ldots, c_{n-1})$. Then the following hold.*

(i) *The rotational family $\mathcal{F}_G^{2n-1}$ forms a decomposition of K_{2n}.*
(ii) *If G is a 1-factor, then $\rho^i(G)$ is a 1-factor for every integer i, and the rotational family $\mathcal{F}_G^{2n-1}$ forms a rotational 1-factorization of K_{2n}.*

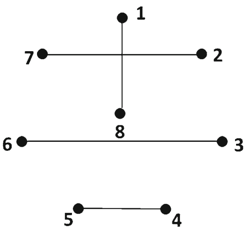
Proof (i). By way of contradiction, fix any l $(0 \leq l < n)$ and suppose that for some i, j $(0 \leq i < j < 2n-1)$, the graphs $\rho^i(G)$ and $\rho^j(G)$ in $\mathcal{F}_G^{2n-1}$ share an edge of length l. Then $c_l + i \equiv c_l + j \pmod{2n-1}$, contradicting our choice of i, j. So the graphs in $\mathcal{F}_G^{2n-1}$ are pairwise edge-disjoint and partition the edges of K_{2n}.
(ii). By Definition 4, the map ρ is an isomorphism between G and $\rho(G)$. So every graph in $\mathcal{F}_G^{2n-1}$ is isomorphic to G, and the result follows. □

4 Three Families of Rotational 1-Factorizations

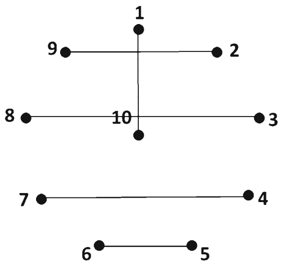
We will illustrate our results using the following three families of rotational 1-factorizations, the first two of which are fairly common in the literature (see [5,8]).

Definition 6 Fix any integer $n > 0$. The 1-factorization GK_{2n} of K_{2n} is the rotational family $\mathcal{F}_G^{2n-1}$ generated by $G = SG(c_0, \ldots, c_{n-1})$ where

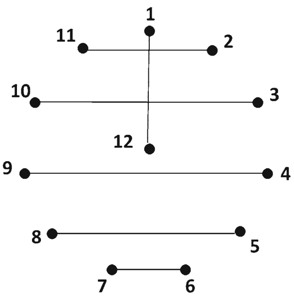
$$c_l = 1 \text{ for all } l \ (0 \le l < n).$$



GK_8 starter
$SG(1, 1, 1, 1)$



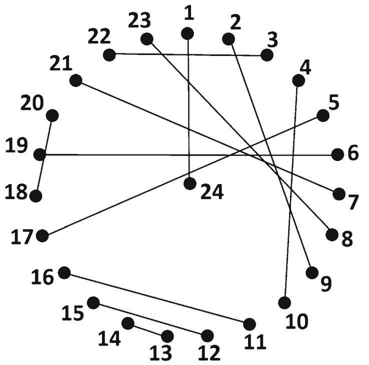
GK_{10} starter
$SG(1, 1, 1, 1, 1)$



GK_{12} starter
$SG(1, 1, 1, 1, 1, 1)$

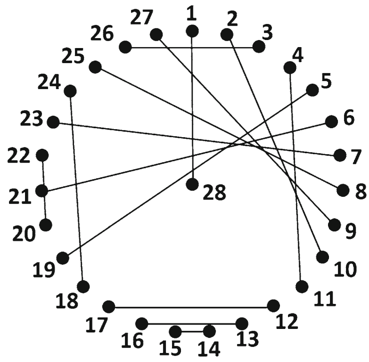
Definition 7 (Adapted from [8]) Fix any integer $n > 0$. The 1-factorization WK_{2n} of K_{2n} is the rotational family $\mathcal{F}_G^{2n-1}$ generated by $G = SG(c_0, \ldots, c_{n-1})$ where c_l is given by the following table, depending on the form of n and l. (When appropriate, entries are to be read modulo $2n - 1$.)

	$l < \lceil n/2 \rceil$			$l \ge \lceil n/2 \rceil$	
	$l \equiv_4 0$	$l \equiv_4 2$	$l \equiv_4 1, 3$	$l = \lceil n/2 \rceil + 2i$	$l = \lceil n/2 \rceil + 2i + 1$
$n = 8k + 0$	1	$12k + 1$	1	$4k - 3i$	$12k - 2 - 3i$
$n = 8k + 1$	1	$12k + 1$	1	$12k + 1 - 3i$	$4k - 1 - 3i$
$n = 8k + 2$	1	$12k + 3$	2	$12k + 4 - 3i$	$4k + 1 - 3i$
$n = 8k + 3$	1	$12k + 5$	1	$4k + 1 - 3i$	$12k + 2 - 3i$
$n = 8k + 4$	1	$12k + 7$	2	$4k + 3 - 3i$	$12k + 5 - 3i$
$n = 8k + 5$	1	$12k + 7$	$16k + 9$	$12k + 6 - 3i$	$4k - 3i$
$n = 8k + 6$	1	$12k + 9$	1	$12k + 9 - 3i$	$4k + 2 - 3i$
$n = 8k + 7$	1	$12k + 11$	$16k + 13$	$4k + 2 - 3i$	$12k + 7 - 3i$



WK_{24} starter

$SG(1, 2, 19, 2, 1, 2, 7, 17, 4, 14, 1, 11)$

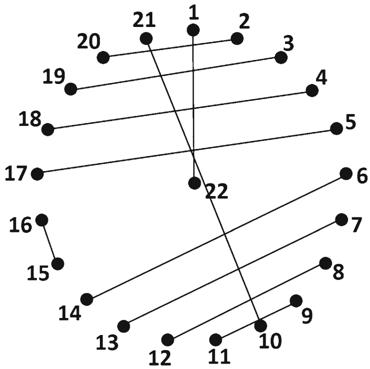


WK_{28} starter

$SG(1, 1, 21, 1, 1, 1, 21, 21, 6, 18, 3, 15, 27, 12)$

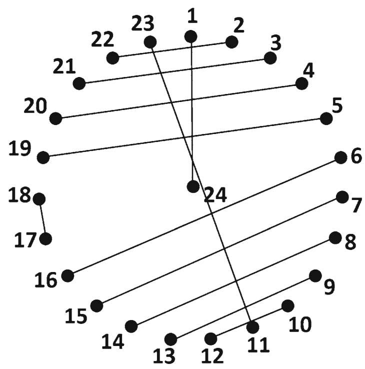
Definition 8 Fix any integer $n > 0$. The 1-factorization HK_{2n} of K_{2n} is the rotational family $\mathcal{F}_G^{2n-1}$ generated by the graph $G = SG(c_0, \ldots, c_{n-1})$ where c_l is given by the following table, depending on the form of n and l:

	$l < 2$		$2 \le l \le n-2$		$l > n-2$
	$l = 0$	$l = 1$	$l \equiv_2 0$	$l \equiv_2 1$	$l = n-1$
$n = 2k + 0$	1	k	$2k - 1$	$2k$	$3k - 1$
$n = 2k + 1$	1	k	$2k$	$2k + 1$	k



HK_{22} starter

$SG(1, 5, 10, 11, 10, 11, 10, 11, 10, 11, 5)$



HK_{24} starter

$SG(1, 6, 11, 12, 11, 12, 11, 12, 11, 12, 11, 17)$

5 Opposing Pair Graphs and Their Rotations

To construct spanning trees that are orthogonal to a given rotational 1-factorization, we use rotational families of graphs that are built using opposing pairs of edges. We begin this section by defining these terms.

Definition 9 Suppose K_{2n} is drawn in rotational form and let e_1, e_2 be any pair of edges with centers c_1, c_2. We define the **distance** between them to be

$$\text{dist}(e_1, e_2) = \text{length}\{c_1, c_2\},$$

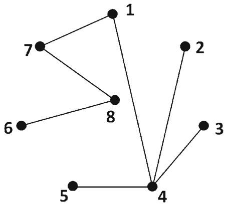
and edges at distance $n - 1$ are said to be **opposing**. Observe that, for any edge pair $\{e_1, e_2\}$ and integer i, we have $\text{dist}(\rho^i(e_1), \rho^i(e_2)) = \text{dist}(e_1, e_2)$.

Definition 10 Suppose K_{2n} is drawn in rotational form and let e_1, e_2 be any pair of edges with centers c_1, c_2. If $c_1 \neq c_2$, we define the **direction** of the pair e_1, e_2 to be the vertex

$$\text{dir}(e_1, e_2) = \text{center}\{c_1, c_2\}.$$

For the case $c_1 = c_2$, we set $\text{dir}(e_1, e_2) = c_1$. Observe that, for any edge pair $\{e_1, e_2\}$ and integer i, we have $\text{dir}(\rho^i(e_1), \rho^i(e_2)) = \text{dir}(e_1, e_2) + i$.

Example 4 A subgraph G of K_8 is drawn in rotational form below:



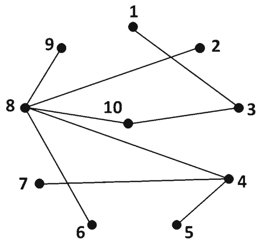
Graph G

edge pair	$\{1,7\},\{3,4\}$	$\{1,7\},\{4,5\}$	$\{3,4\},\{4,5\}$	$\{6,8\},\{7,8\}$
distance	3	3	1	1
direction	2	6	4	3

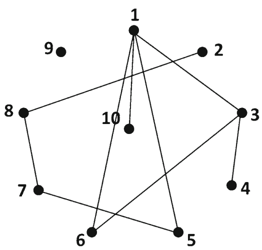
With this terminology in place, we can now define the graphs of interest.

Definition 11 Fix any integers $n > t \geq 0$ and any n-tuple of integers $(d_0, \ldots, d_{n-1})$ satisfying $0 < d_i < 2n$ for each i $(0 \leq i < n)$. We define the **opposing pair graph**, denoted $OPG_t(d_0, \ldots, d_{n-1})$, to be the subgraph of K_{2n} with a single edge of length t and center d_t, and, for each $i \neq t$, an opposing pair of edges of length i and direction d_i. We refer to t as the **exceptional length** of the OPG.

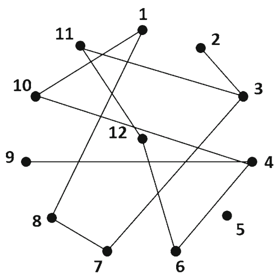
Example 5 The following are examples of opposing pair graphs:



$OPG_4(1, 2, 9, 3, 6)$



$OPG_0(1, 1, 4, 7, 1)$



$OPG_3(3, 5, 8, 7, 2, 4)$

Let us next observe that any opposing pair graph can be expressed as a union of a pair of starter graphs.

Proposition 2 *Fix any integers $n > t \geq 0$ and any n-tuple of integers $(d_0, \ldots, d_{n-1})$ satisfying $0 < d_i < 2n$ for each i $(0 \leq i < n)$. Let $G = OPG_t(d_0, \ldots, d_{n-1})$. Then*

$$G = SG(a_0, \ldots, a_{n-1}) \cup SG(b_0, \ldots, b_{n-1}),$$

where $a_i, b_i \equiv d_i \pm \lfloor n/2 \rfloor \pmod{2n-1}$, respectively, for each $i \neq t$, and where $a_t = b_t = d_t$.

Proof Let X^-, X^+ be the two starter graphs in the union above. For each $i \neq t$, the graph X^- has a single edge of length i and center a_i, and X^+ has a single edge of length i and center b_i. These two edges form an opposing pair of length i and direction d_i, as desired. When $i = t$, both graphs X^- and X^+ share a single edge of length t and center d_t. So $X^- \cup X^+$ gives the desired opposing pair graph G. □

Notice that any opposing pair graph in K_{2n} has exactly $2n - 1$ edges, which is the same as the number of edges required for a spanning tree of K_{2n}. Indeed, the graph $OPG_4(1, 2, 9, 3, 6)$, which is the left-most graph depicted in Example 5, is a spanning tree for K_{10}. In general, however, an opposing pair graph need not be acyclic. By a standard result about spanning trees [7, p. 68], if S is *any* set of $2n - 1$ edges in K_{2n}, then S will be a spanning tree for K_{2n} iff S is acyclic and iff S forms a connected graph on the vertex set.

If we find an opposing pair graph that forms a spanning tree for K_{2n}, then it is useful to note that each of its rotations will be spanning trees as well. But unlike the starter graphs in Proposition 1, the rotations of an opposing pair graph do not form a decomposition of K_{2n}. If we stop rotating in time, however, we can come fairly close, as the following result indicates.

Proposition 3 *Fix any integers $n > t \geq 0$ and any n-tuple of integers $(d_0, \ldots, d_{n-1})$ satisfying $0 < d_i < 2n$ for each i $(0 \leq i < n)$. Let $G = OPG_t(d_0, \ldots, d_{n-1})$. Then the following hold.*

(i) *The graphs in the rotational family $\mathcal{F}_G^{n-1}$ are pairwise edge-disjoint.*
(ii) *The set of edges in K_{2n} that are not contained in any graph of $\mathcal{F}_G^{n-1}$ form a subgraph with exactly $2n - 1$ edges.*
(iii) *If G is a spanning tree for K_{2n}, then every rotation of G is a spanning tree for K_{2n}. In particular, every graph in the rotational family $\mathcal{F}_G^{n-1}$ is a spanning tree for K_{2n}.*

Proof (i). By way of contradiction, suppose that for some $0 \leq i < j < n - 1$, the graphs $\rho^i(G)$ and $\rho^j(G)$ share an edge of length $x \neq t$. Then by Proposition 2,

$$d_x \pm \lfloor n/2 \rfloor + i \equiv d_x \pm \lfloor n/2 \rfloor + j \pmod{2n-1}.$$

But then $j - i \equiv 0, \pm(n-1)$, which is impossible for these values of i, j. Similarly, if $\rho^i(G)$ and $\rho^j(G)$ share an edge of length t, then $d_t + i \equiv d_t + j$ contradicting our choice of i, j.

(ii). Each of the $n - 1$ graphs in $\mathcal{F}_G^{n-1}$ has $2n - 1$ edges, so by (i), their union has $(n-1)(2n-1)$ edges. Since K_{2n} has $n(2n-1)$ edges, the result follows.

(iii). By Definition 4, the map ρ is an isomorphism between G and $\rho(G)$. So every graph in $\mathcal{F}_G^{n-1}$ is isomorphic to G, and the result follows. □

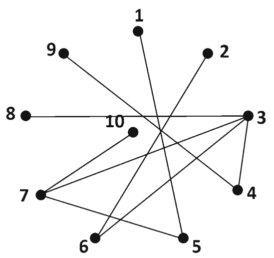
The graph with $2n - 1$ edges, mentioned in part (ii) of the above proposition, deserves special attention. We make the following definition.

Definition 12 Fix any integers $n > t \geq 0$ and any n-tuple of integers $(d_0, \ldots, d_{n-1})$ satisfying $0 < d_i < 2n$ for each i $(0 \leq i < n)$. Let $G = OPG_t(d_0, \ldots, d_{n-1})$. We define the graph $\widetilde{G}$ by

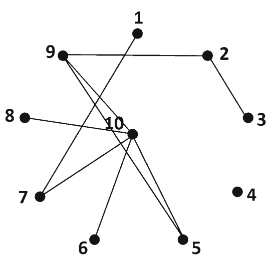
$$\widetilde{G} = K_{2n} \backslash \left(\bigcup_{H \in \mathcal{F}_G^{n-1}} H \right).$$

In other words, $\widetilde{G}$ is the complementary graph in K_{2n} of the union of the rotational family $\mathcal{F}_G^{n-1}$.

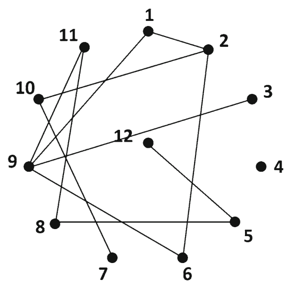
Example 6 The complementary graphs $\widetilde{G}$ for the graphs of Example 5:



The graph $\widetilde{G}$ for $OPG_4(1, 2, 9, 3, 6)$



The graph $\widetilde{G}$ for $OPG_0(1, 1, 4, 7, 1)$



The graph $\widetilde{G}$ for $OPG_3(3, 5, 8, 7, 2, 4)$

Note that when $G = OPG_t(d_0, \ldots, d_{n-1})$, the graph $\widetilde{G}$ has exactly n edges of length t and a single edge of each length $i \neq t$ (see Example 6). Occasionally it happens that both G and $\widetilde{G}$ are spanning trees for K_{2n}. When this occurs, G is of great use in constructing an orthogonal spanning tree decomposition.

6 Orthogonality and Opposing Pair Graphs

Theorem 3 *Fix any integers $n > t \geq 0$ and fix any two n-tuples of integers $(c_0, \ldots, c_{n-1})$ and $(d_0, \ldots, d_{n-1})$ satisfying $0 < c_i, d_i < 2n$ for each i $(0 \leq i < n)$. Let $S = SG(c_0, \ldots, c_{n-1})$ and $G = OPG_t(d_0, \ldots, d_{n-1})$. Suppose S is a 1-factor. Then the following hold.*

(i) *If G is orthogonal to the 1-factorization $\mathcal{F}_S^{2n-1}$, then every rotation of G is orthogonal to $\mathcal{F}_S^{2n-1}$. In particular, every graph in the rotational family $\mathcal{F}_G^{n-1}$ is orthogonal to $\mathcal{F}_S^{2n-1}$.*

(ii) *If G is orthogonal to the 1-factorization $\mathcal{F}_S^{2n-1}$, then the graph $\widetilde{G}$ is also orthogonal to $\mathcal{F}_S^{2n-1}$.*

Proof (i) If $\rho(G)$ is not orthogonal to $\mathcal{F}_S^{2n-1}$, there exists an integer j such that $\rho^j(S) \in \mathcal{F}_S^{2n-1}$ and where $\rho^j(S)$ shares more than one edge with $\rho(G)$. But then by Definition 4, G shares more than one edge with $\rho^{j-1}(S) \in \mathcal{F}_S^{2n-1}$, so that G is not orthogonal to $\mathcal{F}_S^{2n-1}$. By contrapositive, $\rho(G)$ is orthogonal whenever G is, so by repeated application of ρ, the result follows.

(ii) By (i), each of the $n-1$ rotations of G in $\mathcal{F}_G^{n-1}$ shares one edge with each of the $2n-1$ rotations of S in the 1-factorization $\mathcal{F}_S^{2n-1}$. By Proposition 3(i), these rotations of G are edge-disjoint. This leaves exactly one edge from each 1-factor in $\mathcal{F}_S^{2n-1}$ as the set of edges of $\widetilde{G}$, as desired. □

Our next goal is to characterize which opposing pair graphs $OPG_t(d_0, \ldots, d_{n-1})$ are orthogonal to the 1-factorization generated by a given starter $SG(c_0, \ldots, c_{n-1})$. To obtain the characterization, we begin with the following lemma, which concerns a partition of a set of integers.

Lemma 1 *Fix any integer* $n \geq 2$ *and subsets* $A, B \subseteq \{1, 2, \ldots, 2n-1\}$ *such that* $|A| = |B| = n-1$ *and where*

$$B \equiv \{a + n \mid a \in A\} \pmod{2n-1}.$$

Then the following are equivalent.

(i) $A \cup B = \{1, 2, \ldots, 2n-2\}$.

(ii) $A = \{n, n+1, \ldots, 2n-2\}$ *and* $B = \{1, 2, \ldots, n-1\}$.

Proof Since (ii) clearly implies (i), it remains to consider the converse. Suppose (i) holds. Then $0 \notin A$ and it suffices to show that, for any i $(0 \leq i \leq n-2)$, if $i \notin A$, then $i+1 \notin A$. To this end, suppose $i \notin A$. Then $i+n \notin B$, so $i+n \in A$, forcing $i+2n \in B$. But $i+2n \equiv i+1$, so $i+1 \notin A$. □

With the above lemma in place, we now characterize the opposing pair graphs that are orthogonal to a given rotational 1-factorization. Without loss of generality, we may restrict our attention to opposing pair graphs that satisfy $d_t = c_t$, in view of Theorem 3(i) above.

Theorem 4 *Fix any integers* $n > t \geq 0$ *and fix any two n-tuples of integers* $(c_0, \ldots, c_{n-1})$ *and* $(d_0, \ldots, d_{n-1})$ *satisfying* $0 < c_i, d_i < 2n$ *for each* i $(0 \leq i < n)$. *Let* $S = SG(c_0, \ldots, c_{n-1})$ *and* $G = OPG_t(d_0, \ldots, d_{n-1})$. *Suppose S is a 1-factor and suppose* $d_t = c_t$. *Then the graph G is orthogonal to the 1-factorization* $\mathcal{F}_S^{2n-1}$ *if and only if*

$$\{d_i - c_i + (-1)^n \lfloor n/2 \rfloor\}_{i \neq t} \equiv \{1, 2, \ldots, n-1\} \pmod{2n-1}.$$

Proof First note that an edge of length x and center y is in $\rho^i(S)$ if and only if

$$y - c_x \equiv i \pmod{2n-1}. \tag{1}$$

The graph G has a single edge of length t with center $d_t = c_t$, and so it shares this edge with the starting 1-factor $S = \rho^0(S)$. By Proposition 2, for each length $l \neq t$, G has a pair of edges of length l and centers $d_l \pm \lfloor n/2 \rfloor$. Now G is orthogonal to $\mathcal{F}_S^{2n-1}$ if and only if it shares exactly 1 edge with each of the other rotations $\rho^i(S)$ $(1 \leq i \leq 2n-2)$ in $\mathcal{F}_S^{2n-1}$. By (1), this occurs if and only if

$$A \cup B = \{1, 2, \dots, 2n-2\}$$

where A, B are subsets of $\{1, 2, \dots, 2n-1\}$ satisfying $A \equiv \{d_i - c_i - \lfloor n/2 \rfloor\}_{i \neq t}$ and $B \equiv \{d_i - c_i + \lfloor n/2 \rfloor\}_{i \neq t} \pmod{2n-1}$.

When n is even, $B \equiv \{a + n \mid a \in A\} \pmod{2n-1}$. So by Lemma 1, G is orthogonal to $\mathcal{F}_S^{2n-1}$ if and only if $B = \{1, 2, \dots, n-1\}$ as desired. When n is odd, $A \equiv \{b + n \mid b \in B\} \pmod{2n-1}$. So by swapping the roles of A and B in Lemma 1, G is orthogonal to $\mathcal{F}_S^{2n-1}$ if and only if $A = \{1, 2, \dots, n-1\}$ as desired. □

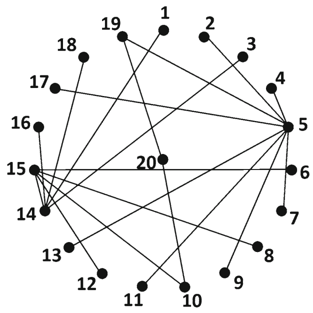
Since there are exactly $(n-1)!$ ways to pair up the elements of the sets appearing on the two sides of the congruence in Theorem 4, we have the following corollary.

Corollary 1 *Fix any integers $n > t \geq 0$ and fix any two n-tuples of integers $(c_0, \dots, c_{n-1})$ and $(d_0, \dots, d_{n-1})$ satisfying $0 < c_i, d_i < 2n$ for each i $(0 \leq i < n)$. Let $S = SG(c_0, \dots, c_{n-1})$ and $G = OPG_t(d_0, \dots, d_{n-1})$. Suppose S is a 1-factor. Then there are exactly $(n-1)!$ different opposing pair graphs orthogonal to the 1-factorization $\mathcal{F}_S^{2n-1}$ that have a single edge of length t and center $d_t = c_t$.*

7 Application to Rotational 1-Factorizations

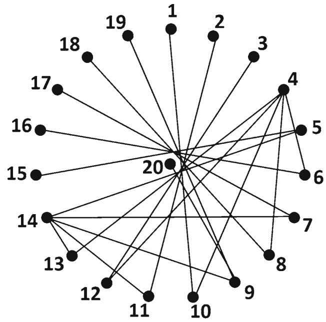
Definition 13 Fix any integer $n > 2$. We define DGK_{2n} to be the spanning tree decomposition of K_{2n} given by $\mathcal{F}_G^{n-1} \cup \{\widetilde{G}\}$, where $\mathcal{F}_G^{n-1}$ is the rotational family generated by the graph $G = OPG_{n-1}(d_0, \dots, d_{n-1})$, and where d_l is given by the following table, depending on the form of n and l. (When appropriate, entries are to be read modulo $2n-1$.)

	$l = 0$	$1 \leq l \leq n-2$	$l = n-1$
$n = 2k + 0$	k	$n - (-1)^l (n - \lceil l/2 \rceil)$	1
$n = 2k + 1$	n	$n - (-1)^l \lceil l/2 \rceil$	1

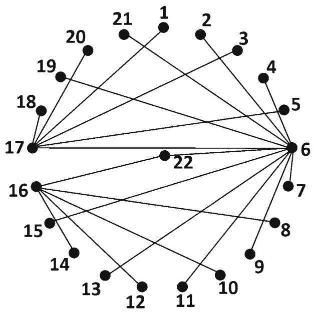


DGK_{20} starter

$OPG_9(5, 19, 1, 18, 2, 17, 3, 16, 4, 1)$

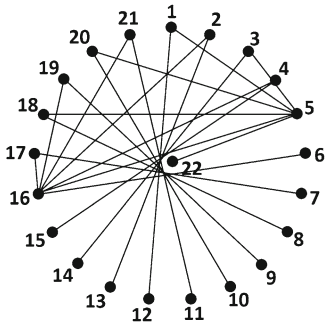


complementary graph $\widetilde{G}$



DGK_{22} starter

$OPG_{10}(11, 12, 10, 13, 9, 14, 8, 15, 7, 16, 1)$



complementary graph $\widetilde{G}$

Proof of Theorem 2 Using the criteria given in Theorems 4 and 3, it is easily verified that the spanning tree decomposition DGK_{2n} of Definition 13 is orthogonal to the 1-factorization GK_{2n} given in Definition 6. We also observe that each graph in DGK_{2n} has exactly $2n - 1$ edges. So to prove that these graphs are trees, it suffices to check that they are connected, which is straightforward using Definition 13. □

We suspect that similar families of spanning tree decompositions can be found that are orthogonal to other families of 1-factorizations, as the following result suggests.

Proposition 4 *For every integer n $(2 < n < 11)$, there exist orthogonal spanning tree decompositions of K_{2n} for the 1-factorizations WK_{2n} and HK_{2n}.*

Proof For each value of n, the table below gives an opposing pair graph G that is a spanning tree K_{2n} and that is orthogonal to WK_{2n} (respectively HK_{2n}). For each of these, the complementary graph $\widetilde{G}$ is also a tree. Accordingly, there is a spanning tree decomposition of K_{2n} given by $\mathcal{F}_G^{n-1} \cup \{\widetilde{G}\}$, where $\mathcal{F}_G^{n-1}$ is the rotational family generated by the graph G, and where each tree in the decomposition is orthogonal to WK_{2n} (respectively HK_{2n}), as desired. □

n	Opposing pair graph for WK_{2n}	Opposing pair graph for HK_{2n}
3	$OPG_2(3, 4, 1)$	$OPG_2(3, 4, 1)$
4	$OPG_3(4, 6, 5, 1)$	$OPG_3(4, 6, 5, 1)$
5	$OPG_4(5, 7, 3, 3, 1)$	$OPG_4(5, 4, 7, 1, 1)$
6	$OPG_5(8, 5, 6, 4, 7, 1)$	$OPG_5(3, 9, 10, 9, 9, 1)$
7	$OPG_6(7, 9, 9, 10, 9, 2, 1)$	$OPG_6(3, 10, 12, 10, 10, 12, 1)$
8	$OPG_7(10, 11, 6, 12, 10, 7, 8, 1)$	$OPG_7(3, 12, 14, 11, 13, 13, 11, 1)$
9	$OPG_8(7, 8, 5, 12, 11, 9, 11, 5, 1)$	$OPG_8(3, 13, 16, 16, 11, 13, 13, 15, 1)$
10	$OPG_9(11, 13, 8, 15, 10, 3, 19, 2, 9, 1)$	$OPG_9(3, 15, 18, 18, 13, 16, 16, 15, 12, 1)$

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