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ORIGINAL PAPER

Dominator Colorings of Certain Cartesian Products of Paths and Cycles

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Abstract A dominator coloring of a graph G is a proper coloring of G with the additional property that every vertex dominates an entire color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of colors among all dominator colorings of G. In this paper, we determine the dominator chromatic numbers of Cartesian product graphs $P_2 \square P_n$ and $P_2 \square C_n$.

Keywords Dominator coloring · Dominator chromatic number · Cartesian product

1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. Any vertex of G is said to *dominate* itself and all its neighbors. A subset D of V is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A *proper coloring* of G is a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ such that adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors needed for a proper coloring of G. A *color class* is the set consisting of all those vertices assigned the same color. If there is only one vertex in some color class, we call it a *singleton color class*.

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A *dominator coloring* of G is a proper coloring such that every vertex of G dominates all vertices of at least one color class (possibly its own class). The *dominator chromatic number*, denoted by $\chi_d(G)$, of G is the minimum number of colors needed for a dominator coloring of G. Obviously, $\chi(G) \le \chi_d(G)$. For a nonegative integer k, a k-dominator coloring is a proper dominator coloring using at most k colors. The definition of dominator coloring was first introduced by Gera et al. [3]. She also proved that computing the dominator chromatic number of a graph is NP-hard for general graphs. In [4], Gera showed that every graph satisfies:

$$\max\{\gamma(G), \chi(G)\} \le \chi_d(G) \le \gamma(G) + \chi(G). \tag{1}$$

For more results on dominator coloring, we refer the reader to [1,2,5,7].

The Cartesian product graph $P_2 \Box P_n$ has vertex set $V(P_2 \Box P_n) = \{v_1, v_2, \ldots, v_n; u_1, u_2, \ldots, u_n\}$ and edge set $E(P_2 \Box P_n) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}$. If we add two edges $v_1 v_n$ and $u_1 u_n$ to $P_2 \Box P_n$, the resulting graph is denoted by $P_2 \Box C_n$. The purpose of this paper is to study the dominator colorings for $P_2 \Box P_n$ and $P_2 \Box C_n$. More precisely, We prove that

Theorem 1 *For* $n \ge 2$, $\chi_d(P_2 \Box P_n) = \begin{cases} 2 & \text{if } n = 2; \\ 4 & \text{if } n = 4; \\ \lfloor n/2 \rfloor + 3 & \text{if } n = 3 \text{ and } n \ge 5. \end{cases}$

Theorem 2 *For* $n \ge 3$, $\chi_d(P_2 \Box C_n) = \begin{cases} 3 & \text{if } n = 3; \\ 2\lfloor n/4 \rfloor + 2 & \text{if } n \equiv 0 \pmod 4; \\ 2\lfloor n/4 \rfloor + 3 & \text{if } n \equiv 1 \pmod 4; \\ 2\lfloor n/4 \rfloor + 4 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$

2 Preliminaries

A *clique* in a graph G is a complete subgraph of G. Let f be a proper coloring of G and $S \subset V$. If color a appears on no other vertices but S, then we say that S *consumes* color a.

Gera [4] pointed out that given a graph G and a subgraph H, the dominator coloring numbers $\chi_d(H)$ can be smaller or larger than $\chi_d(G)$. Next, we present a sufficient condition to guarantee that $\chi_d(H) \le \chi_d(G)$. For any subset $S \subset V$, $G \backslash S$ is the graph obtained from G by removing S and all edges incident with vertices in S. Let $G[S]$ denote the subgraph induced by S in G.

Lemma 1 *Let* $G = (V, E)$ *be a graph and* $S \subset V$. *Write* $V' = \{v_i | v_i \in V \backslash S \text{ and } N(v_i) \cap S \ne \emptyset\}$. *If* $G[V']$ *is a clique, then* $\chi_d(G \backslash S) \le \chi_d(G)$.

Proof Let f be a χ_d-dominator coloring of G. Suppose that the restriction of f to $G \backslash S$ is not a dominator coloring of $G \backslash S$, then it must be the case that the dominated color class of some vertex $v \in V'$ is totally contained in S. Now recolor v by this color. Then $\{v\}$ is a singleton color class. If there exist at least two vertices in V' that dominate the same color class in S, then arbitrarily choose one vertex among them and recolor it by that color. Obviously, the resulting coloring is a χ_d-dominator coloring of $G\backslash S$. □

If we take $S = \{u_{m+1}, \ldots, u_n; v_{m+1}, \ldots, v_n\}$ in $P_2\Box P_n$, then the following is an easy consequence of Lemma 1.

Corollary 1 *If $n \geq m$, then $\chi_d(P_2\Box P_n) \geq \chi_d(P_2\Box P_m)$.*

Lemma 2 *Let f be a χ_d-dominator coloring of $P_2\Box P_n$. If there is no singleton color class under f, then the color classes of $P_2\Box P_n$ must be in the form of $\{v_i, u_{i+1}\}$ or $\{u_i, v_{i+1}\}$, where $i = 1, 2, \ldots, n-1$.*

Proof Without loss of generality, we assume that $f(v_1) = 1$. Since there is no singleton color class, then v_1 dominates color class $\{u_1, v_2\}$. So $f(u_1) = f(v_2)$. Assume that $f(u_1) = f(v_2) = 2$. By exchanging the roles of u_1 and v_1, we have that $f(v_1) = f(u_2) = 1$ and u_1 dominates color class $\{v_1, u_2\}$. Now, colors 1 and 2 cannot be used on other vertex any more. Using a similar argument, we can get that all the color classes are in the form of $\{v_i, u_{i+1}\}$ and $\{u_i, v_{i+1}\}$. □

Corollary 2 *Let f be a χ_d-dominator coloring of $P_2\Box P_n$. If there is no singleton color class under f, then n is even.*

We close this section with some known results.

Lemma 3 *([4]) For the cycle C_n, we have* $\chi_d(C_n) = \begin{cases} \lceil n/3 \rceil, & \text{if } n = 4; \\ \lceil n/3 \rceil + 1, & \text{if } n = 5; \\ \lceil n/3 \rceil + 2, & \text{otherwise}. \end{cases}$

Lemma 4 *([6]) For $n \geq 1$, $\gamma(P_2\Box P_n) = \lfloor (n+2)/2 \rfloor$.*

Lemma 5 *([8]) For $n \geq 3$,* $\gamma(P_2\Box C_n) = \begin{cases} \lceil (n+1)/2 \rceil, & \text{if } n \text{ is not a multiple of 4}; \\ n/2, & \text{if } n \text{ is a multiple of 4}. \end{cases}$

3 Dominator Colorings for $P_2\Box P_n$

In this section, we determine $\chi_d(P_2\Box P_n)$ for all $n \geq 2$. We first consider the cases for $n \leq 4$.

Lemma 6 *For $n \leq 4$, we have* $\chi_d(P_2\Box P_n) = \begin{cases} 2, & n = 2; \\ 4, & n = 3,4. \end{cases}$

Proof Since $P_2\Box P_2$ is isomorphic to C_4, the result follows from Lemma 3. If $n = 3$, then by Lemma 4 and the right side of inequality (1), we have $\chi_d(P_2\Box P_3) \leq 2+2 = 4$. Next, we show that $\chi_d(P_2\Box P_3) \geq 4$.

Assume to the contrary that f is a dominator coloring of $P_2\Box P_3$ using at most three colors. It follows from Corollary 2 that there is at least one singleton color class under f. By symmetry, we distinguish the following two cases.

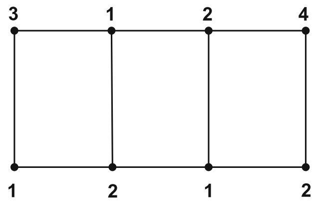

Fig. 1 A 4-dominator coloring of $P_2 \Box P_4$

Case 1. $\{v_1\}$ is a singleton color class.

Without loss of generality, we assume that $f(v_1) = 1$. If $f(u_1) \neq f(v_2)$, then u_2 cannot be properly colored. So assume $f(u_1) = f(v_2) = 2$. Then we have $f(u_2) = 3$, $f(u_3) = 2$ and $f(v_3) = 3$. However, in this case v_3 does not dominate any color class, a contradiction.

Case 2. $\{v_2\}$ is a singleton color class.

Assume that $f(v_2) = 1$. If $f(v_1) \neq f(u_2)$, then u_1 cannot be properly colored. So assume that $f(v_1) = f(u_2) = 2$. It follows that $f(u_1) = f(u_3) = 3$ and $f(v_3) = 2$. However, in this case u_3 does not dominate any color class, a contradiction.

By Corollary 1, we have $\chi_d(P_2 \Box P_4) \geq \chi_d(P_2 \Box P_3) = 4$. On the other hand, a 4-dominator coloring is shown in Fig. 1. Hence $\chi_d(P_2 \Box P_4) = 4$. This completes the proof. □

The rest of this section is devoted to proving Theorem 1 for $n \geq 5$. By Lemma 4 and the right side of inequality (1), we have that $\chi_d(P_2 \Box P_n) \leq \lfloor (n+2)/2 \rfloor + 2 = \lfloor n/2 \rfloor + 3$. Hence, it suffices to show $\chi_d(P_2 \Box P_n) \geq \lfloor n/2 \rfloor + 3$ in the sequel. In our proof, we will use the following claim. First, we give one more notation: for $k \leq n$, define G_k to be the subgraph of $P_2 \Box P_n$ induced by $\{u_1, \ldots, u_k; v_1, \ldots, v_k\}$.

Claim *Let $G = P_2 \Box P_n$ and let f be a dominator coloring of G. If $\{u_k\}$ or $\{v_k\}$ is a singleton color class for some $k \leq n$, then the restriction of f to G_k is a dominator coloring of G_k.*

Proof By symmetry, we assume that $\{u_k\}$ is a singleton color class. Then both u_k and v_k dominate color class $\{u_k\}$ in G_k, and for vertex $v \in V(G_k) \backslash \{u_k, v_k\}$, v dominates the same color class as that in G. Thus the claim follows. □

Lemma 7 *For the path P_5, we have $\chi_d(P_2 \Box P_5) \geq 5$.*

Proof Suppose to the contrary that f is a dominator coloring of $P_2 \Box P_5$ using at most four colors. By Corollary 2, there is at least one singleton color class under f. Let k be the smallest integer such that $\{u_k\}$ or $\{v_k\}$ is a singleton color class of $P_2 \Box P_5$. Then by our claim, the restriction of f on G_k is a dominator coloring of G_k. By symmetry, we distinguish among three cases.

Case 1. $k = 3$.

By Lemma 6, at least four colors appear on G_3. So $f(v_5)$, $f(u_5)$ and $f(u_4)$ are reused colors in G_3, which implies that u_5 does not dominate any color class, a contradiction.

Case 2. $k = 4$.

Again by Lemma 6, four colors all appear on G_4. Assume without loss of generality that $\{u_4\}$ is a singleton color class. Since $f(u_5)$ and $f(v_5)$ are reused colors in $P_2\Box P_4$, then either v_5 dominates color class $\{u_5, v_4\}$ or $\{v_4\}$ is a singleton color class. In both cases, we only have two remaining colors to color G_4. It is easy to check that in this case u_1 does not dominate any color class.

Case 3. $k = 1$.

By symmetry we may assume without loss of generality that $\{u_1\}$ is a singleton color class. Then neither u_5 nor v_5 is a singleton color class, otherwise u_3 can not dominate any color class. Combining the above two cases, we obtain that there is no singleton color class in the form of $\{u_k\}$ or $\{v_k\}$ for $k = 2, 3, 4, 5$. By the same argument as in the proof of Lemma 2 , we have that $\{u_5, v_4\}$, $\{v_5, u_4\}$, $\{u_3, v_2\}$ and $\{v_3, u_2\}$ are four color classes. Then there is no color that can be assigned to vertices v_1 and u_1, a contradiction. □

Lemma 8 *For the path P_6, we have $\chi_d(P_2\Box P_6) \geq 6$.*

Proof Suppose to the contrary that f is a dominator coloring of $P_2\Box P_6$ using at most five colors. If there is no singleton color class under f, then by Lemma 2, six colors are needed for f, a contradiction. So there is at least one singleton color class. Let k be the smallest integer such that $\{u_k\}$ or $\{v_k\}$ is a singleton color class of $P_2\Box P_6$. Then by our claim, the restriction of f on G_k is a proper dominator coloring of G_k. By symmetry, we distinguish among three cases.

Case 1. $k = 5$.

Without loss of generality, we assume that $\{v_5\}$ is a singleton color class, say $f(v_5) = 5$. Then by our claim the restriction of f on G_5 is a dominator coloring of G_5. According to Lemma 7, five colors all appear on G_5. Then $f(u_6)$ and $f(v_6)$ both are reused colors on G_5. Therefore, either $\{u_5\}$ is a singleton color class, or $\{u_5, v_6\}$ consists a color class dominated by u_6.

Subcase 1.1. $\{u_5\}$ is a singleton color class, say $f(u_5) = 4$. Since $N[u_2]$ consumes at least one color, we have at most two colors to color the vertices in set $\{v_3, v_4, u_4, v_6, u_6\}$. Without loss of generality, let $f(v_3) = f(u_4) = 1$ and $f(v_4) = 2$. As $\{f(v_6), f(u_6)\} = \{1, 2\}$, v_3 does not dominate color class $\{f^{-1}(1)\}$ or $\{f^{-1}(2)\}$. So $\{v_2, u_3\}$ consumes color 3. This implies that $\{f(u_1), f(u_2), f(v_1)\} = \{1, 2\}$. However, in this case u_1 does not dominate any color class, a contradiction.

Subcase 1.2. $\{u_5, v_6\}$ consists a color class, say $f(u_5) = f(v_6) = 4$. Since $N[u_4]$ consumes at least one color, we only have two remaining colors to color the vertices in set $\{u_1, u_2, v_1, v_2, v_3\}$. Then we can see that u_1 does not dominate any color class, a contradiction.

Case 2. $k = 3$.

Now the restriction of f on G_3 is a proper dominator coloring of G_3. By Lemma 6, there are at least four colors on G_3. If $\{u_6\}$ is a singleton color, then $f(v_4)$, $f(v_5)$, $f(v_6)$ and $f(u_5)$ are reused colors on G_3. This implies that v_5 does not dominate any color class, a contradiction. So $f(u_6)$ is a reused color in $P_2\Box P_6$. Using a similarly argument, we can prove that $f(u_5)$ is also a reused color in $P_2\Box P_6$. By symmetry, we have that

$f(v_5)$ and $f(v_6)$ both are reused colors in $P_2\Box P_6$. Then $\{v_5, u_6\}$ and $\{v_6, u_5\}$ are two new color classes, a contradiction.

Combining Cases 1 and 2, we know that there is no singleton color class in the form of $\{u_k\}$ or $\{v_k\}$ for $k = 2, 3, 4, 5$.

Case 3. $k = 1$.

Suppose $\{u_1\}$ is a singleton color class. If neither $\{u_6\}$ nor $\{v_6\}$ is a singleton color class, we know that $\{u_5, v_6\}$, $\{v_5, u_6\}$, $\{u_3, v_4\}$ and $\{v_3, u_4\}$ are four color classes. Then there is no color that can be assigned to vertices v_1 and v_2, a contradiction. So at least one of $\{u_6\}$ and $\{v_6\}$ is singleton. If $\{u_6\}$ is singleton, then since $N[v_5]$ consume at least one color class, we only have two colors to color the vertices in set $\{v_1, v_2, v_3, u_2, u_3, u_4\}$. It is easy to check that in this case u_3 does not dominate any color class, a contradiction. The same result holds when $\{v_6\}$ is singleton. □

Lemma 9 *For $n \geq 7$, we have $\chi_d(P_2\Box P_n) \geq \lfloor n/2\rfloor + 3$.*

Proof Our proof proceeds by induction on n. For $n = 7$, by Corollary 1 and Lemma 8, we have $\chi_d(P_2\Box P_7) \geq 6$.

So the lemma holds for $n = 7$. Assume that $\chi_d(P_2\Box P_n) \geq \lfloor n/2\rfloor + 3$ holds for $n < k$.

When $n = k$, if $n = 2t + 1$ is odd, then $\chi_d(P_2\Box P_n) \geq \chi_d(P_2\Box P_{n-1}) = \lfloor (n - 1)/2\rfloor + 3 = \lfloor n/2\rfloor + 3$.

In what follows, we deal with the case when n is even. First notice that by induction assumption, $\chi_d(P_2\Box P_{n-3}) \geq \lfloor (n-3)/2\rfloor + 3 = \lfloor n/2\rfloor + 1$ and $\chi_d(P_2\Box P_{n-1}) \geq \chi_d(P_2\Box P_{n-2}) = \lfloor (n-2)/2\rfloor + 3 = \lfloor n/2\rfloor + 2$. Suppose to the contrary that $\chi_d(P_2\Box P_n) \leq \lfloor n/2\rfloor + 2$.

Case 1. Neither $\{u_{n-2}\}$ nor $\{v_{n-2}\}$ is a singleton color class.

So the restriction of f on G_{n-3} is a proper dominator coloring. By inductive hypothesis, at least $\lfloor n/2\rfloor + 1$ colors are used to color the vertices of G_{n-3}.

1. If $\{u_n\}$ is a singleton color class, then $f(v_{n-2})$, $f(v_n)$, $f(v_{n-1})$ and $f(u_{n-1})$ are reused colors on G_{n-3}. In this case v_{n-1} cannot dominate any color class, a contradiction. By symmetry, we have that $\{v_n\}$ is not a singleton color class either.
2. If $\{u_{n-1}\}$ is a singleton color class, then $f(u_n)$, $f(v_n)$ and $f(v_{n-1})$ are reused colors on G_{n-3}. In this case v_n does not dominate any color class, a contradiction. By symmetry, we have that $\{v_{n-1}\}$ is not a singleton color class either.

Combining (1) and (2), we have that v_n dominates color class $\{u_n, v_{n-1}\}$ and u_n dominates color class $\{v_n, u_{n-1}\}$. That is, we need two more colors to color the vertices outside G_{n-3}, a contradiction.

Case 2. At least one of $\{u_{n-2}\}$ and $\{v_{n-2}\}$ is a singleton color class.

Then the restriction of f on G_{n-2} is a dominator coloring. By inductive hypothesis, at least $\lfloor n/2\rfloor + 2$ colors appear on G_{n-2}. Then $f(u_n)$, $f(v_n)$ and $f(u_{n-1})$ are reused colors on G_{n-2}. This implies that u_n does not dominate any color class, a contradiction. This completes the proof. □

4 Dominator Colorings for $P_2 \Box C_n$

In this section, we determine $\chi_d(P_2 \Box C_n)$ for all $n \geq 3$.

Lemma 10 *For the cycle C_3, we have $\chi_d(P_2 \Box C_3) = 3$.*

Proof A 3-dominator coloring of $P_2 \Box C_3$ is depicted in Fig. 2. On the other hand, $\chi_d(P_2 \Box C_3) \geq \chi(P_2 \Box C_3) = 3$. □

If $n \geq 4$ is even, then by Lemma 5 and the right side of inequality (1), it follows that

$$\chi_d(P_2 \Box C_n) \leq \begin{cases} 2\lfloor n/4 \rfloor + 2 & \text{if } n \equiv 0 \pmod 4; \\ 2\lfloor n/4 \rfloor + 4 & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

So to prove Theorem 2 when n is even, we only need to show the inverse direction. First, we give an important observation, which is applied frequently in the following proofs (see Figs. 3, 4).

Observation 1 *Let f be a dominator coloring of $P_2 \Box C_n$. For each i, let $T_i = \{u_i, u_{i+1}, u_{i+2}, u_{i+3}, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ (index mod n). We will show that there are at least four colors in the restriction of f on T_i. Indeed, if at most three colors appear on T_i, since $N[v_{i+3}]$ consumes at least one color, at most two colors appear on $N[u_{i+1}]$. Without loss of generality, let $f(u_{i+1}) = 1$ and $f(u_i) = f(u_{i+2}) = f(v_{i+1}) = 2$. By a similar argument, there are at most two colors on $N[v_{i+3}]$, one of which must be 1 or 2. If color 2 appears on $N[v_{i+3}]$, then we have $f(v_{i+3}) = 2$ and $f(v_{i+2}) = f(u_{i+3}) = f(v_{i+4}) = 3$. If color 1 appears on $N[v_{i+3}]$, then either $f(v_{i+3}) = 3$ and $f(v_{i+2}) = f(u_{i+3}) = f(v_{i+4}) = 1$ or $f(v_{i+3}) = 1$ and $f(v_{i+2}) = f(u_{i+3}) = f(v_{i+4}) = 3$. It is easy to check that either u_{i+2} or v_{i+2} does not dominate any color class in these cases.*

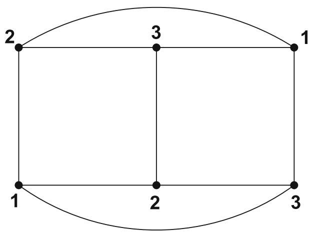

Fig. 2 A 3-dominator coloring of $P_2 \Box C_3$

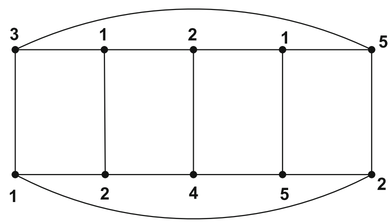

Fig. 3 A 5-dominator coloring of $P_2 \Box C_5$

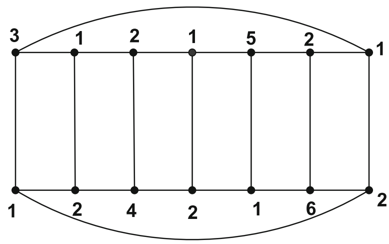

Fig. 4 A 6-dominator coloring of $P_2\Box C_7$

Lemma 11 *For* $n \equiv 0 \bmod 4$*, we have* $\chi_d(P_2\Box C_n) \geq 2\lfloor n/4\rfloor + 2$.

Proof By Observation 1, at least four colors appear on T_1, and when $n \geq 8$, $N[u_6] \cup N[v_8] \cup N[u_{10}] \cdots \cup N[v_n]$ consume at least $(n-4)/2$ colors. Hence at least $(n-4)/2 + 4 = 2\lfloor n/4\rfloor + 2$ colors appear on $P_2\Box C_n$. Then $\chi_d(P_2\Box C_n) \geq 2\lfloor n/4\rfloor + 2$. □

Lemma 12 *Let* f *be a dominator coloring of* $P_2\Box C_n$, $n \geq 5$*. If there is no singleton class under* f*, then* $f(u_1) \neq f(u_3)$.

Proof Assume to the contrary that $f(u_1) = f(u_3) = a$. Denote by $f(u_2) = b$. If $f(v_2) = a$, then v_3 does not dominate any color class; if $f(v_2) = c$, since v_1 does not dominate color class $\{f^{-1}(a)\}$, it must be $f(v_n) = c$. However, v_3 does not dominate $\{f^{-1}(a)\}$ or $\{f^{-1}(c)\}$, a contradiction. □

Lemma 13 *For* $n \equiv 1 \bmod 4$*, we have* $\chi_d(P_2\Box C_n) = 2\lfloor n/4\rfloor + 3$.

Proof First suppose to the contrary that $\chi_d(P_2\Box C_n) \leq 2\lfloor n/4\rfloor + 2$.

Fact 1. There is no singleton color class.

Otherwise, by symmetry we assume that $\{u_n\}$ is a singleton color class, and when $n \geq 9$, $N[u_6] \cup N[v_8] \cup \cdots \cup N[v_{n-1}]$ consume at least $(n-5)/2$ colors, then T_1 only has three colors available, a contradiction.

Therefore, any vertex dominates the color class consisting of at least two of its neighbors. By Lemma 12 and the above fact, we may assume that $f(u_2) = 1$, $f(v_2) = f(u_3) = 2$ and $f(u_1) = 3$. Since u_2 dominates color class $\{v_2, u_3\}$, color 2 cannot be used on other vertices any more. Furthermore, since when $n \geq 9$, $N[v_4] \cup N[u_6] \cup N[v_8] \cup \cdots \cup N[v_{n-1}]$ consume at least $(n-3)/2$ colors, we have $f(v_1) = f(u_n) = 1$. However, by Fact 1 v_2 does not dominate any color class, a contradiction. So $\chi_d(P_2\Box C_n) \geq 2\lfloor n/4\rfloor + 3$.

On the other hand, we give a $(2\lfloor n/4\rfloor + 3)$-dominator coloring for $P_2\Box C_n$ as follows.

$$f(u_i) = 2l + 3 \quad \text{for } i = 4l+1, l = 0, 1, \ldots, \lfloor n/4\rfloor,$$
$$f(u_i) = 2 \quad \text{for odd } i \text{ except } i = 4l+1, l = 0, 1, \ldots, \lfloor n/4\rfloor,$$
$$f(u_i) = 1 \quad \text{for even } i, 1 \leq i \leq n,$$
$$f(v_j) = 2l + 4 \quad \text{for } j = 4l+3, l = 0, 1, \ldots, \lfloor n/4\rfloor - 1,$$
$$f(v_j) = 1 \quad \text{for odd } j, j \leq n-2 \text{ except } j = 4l+3, l = 0, 1, \ldots, \lfloor n/4\rfloor - 1,$$

$$f(v_j) = 2 \quad \text{for even } j,\ 1 \le j \le n-3 \text{ or } j = n,$$
$$f(v_{n-1}) = 2 \quad \lfloor n/4 \rfloor + 3.$$

Hence $\chi_d(P_2 \square C_n) \le 2\lfloor n/4 \rfloor + 3$, and then $\chi_d(P_2 \square C_n) = 2\lfloor n/4 \rfloor + 3$. □

Lemma 14 *For* $n \equiv 2 \bmod 4$, *we have* $\chi_d(P_2 \square C_n) \ge 2\lfloor n/4 \rfloor + 4$.

Proof Suppose to the contrary that $\chi_d(P_2 \square C_n) \le 2\lfloor n/4 \rfloor + 3$.

Fact 2. There is no singleton color class.

Otherwise, by symmetry we assume that $\{u_n\}$ is a singleton color class with $f(u_n) = 2\lfloor n/4 \rfloor + 3$. By our observation, at least four colors appear on T_1, and when $n \ge 10$, $N[u_6] \cup N[v_8] \cup N[u_{10}] \cdots \cup N[v_{n-2}]$ consume at least $(n-6)/2$ colors. So $f(v_1)$, $f(v_n)$ and $f(u_{n-1})$ are reused colors. On the other hand, since $N[u_3] \cup N[v_5] \cup N[u_7] \cup \cdots \cup N[v_{n-1}]$ consume at least $(n-2)/2$ colors, we only have two colors to color vertices in set $\{u_1, v_1, v_2\}$. Without loss of generality, let $f(u_1) = f(v_2) = 1$ and $f(v_1) = 2$. Recall that $f(v_1)$ and $f(v_n)$ are reused colors, we have that v_1 dominates color class $\{f^{-1}(1)\}$. Assume that $N[v_5] \cup N[u_7] \cup \cdots \cup N[v_{n-1}]$ consumes colors $4, 5, \cdots, 2\lfloor n/4 \rfloor + 2$, we only have colors 2 and 3 to color $N[u_3]$. It is easy to check that v_2 does not dominate any color class, a contradiction.

Therefore, any vertex dominates the color class consisting of at least two of its neighbors. By Lemma 12 and Fact 2, we may assume that $f(u_2) = 1$, $f(v_2) = f(u_3) = 2$ and $f(u_1) = 3$. In this case, v_1 does not dominate color class $\{f^{-1}(2)\}$, so $f(v_n) = 3$. When $n \ge 10$, $N[v_4] \cup N[u_6] \cup N[v_8] \cdots \cup N[v_{n-2}]$ consume at least $(n-4)/2$ colors, say they are $\{4, 5, \ldots (n+2)/2\}$, then we only have two colors 1 and $2\lfloor n/4 \rfloor + 3$ to color vertices in $\{u_{n-1}, u_n, v_1\}$. So u_3 dominates color class $\{f^{-1}(4)\}$. That is $f(v_3) = f(u_4) = 4$. Since u_4 does not dominate color class $\{f^{-1}(2)\}$, we have $f(v_4) = f(u_5) = 2\lfloor n/4 \rfloor + 3$. So, if $n = 6$, then we have $f(v_5) = f(u_6) = f(v_1) = 1$. However u_5 does not dominate any color class. If $n \ge 10$, then u_4 does not dominate any color class, since color $2\lfloor n/4 \rfloor + 3$ appears in $\{u_{n-1}, u_n\}$, a contradiction. □

Lemma 15 *For* $n \equiv 3 \bmod 4$, *we have* $\chi_d(P_2 \square C_n) = 2\lfloor n/4 \rfloor + 4$.

Proof First suppose to the contrary that $\chi_d(P_2 \square C_n) \le 2\lfloor n/4 \rfloor + 3$.

Fact 3. There is no singleton color class.

Assume to the contrary that $\{u_n\}$ is a singleton color class with $f(u_n) = 2\lfloor n/4 \rfloor + 3$. By our observation, at least four colors appear on T_1, and when $n \ge 11$, $N[u_6] \cup N[v_8] \cup N[u_{10}] \cup \cdots \cup N[v_{n-3}]$ consume at least $(n-7)/2$ colors, so there is no new color on other vertices, then $f(v_n)$ is a reused color in $P_2 \square C_n$. This implies that $N[v_{n-1}] \cup N[v_1]$ consume at least two colors. Since when $n \ge 11$, $N[u_3] \cup N[v_5] \cup N[u_7] \cup \cdots \cup N[v_{n-6}]$ consume at least $(n-7)/2$ colors, there are at most two colors to color vertices in vertex set $\{u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, v_{n-4}, v_{n-3}\}$. Without loss of generality, let $f(u_{n-5}) = f(v_{n-4}) = f(u_{n-3}) = 1$ and $f(u_{n-4}) = f(v_{n-3}) = f(u_{n-2}) = 2$. Since v_{n-3} does not dominate color class $\{f^{-1}(1)\}$ or $\{f^{-1}(2)\}$, we have that $\{v_{n-2}\}$ is a single color class. Denote $T_{v_n} = \{v_n, v_1, v_2, v_3, u_1, u_2, u_3, u_4\}$. By our observation and symmetry, we know that there are at least four colors on T_{v_n}. Furthermore, when $n \ge 11$, $N[v_5] \cup N[u_7] \cup N[v_9] \cup \cdots \cup N[v_{n-2}]$ consume at least

$(n-5)/2$ colors. Now there are $4+(n-5)/2+1 = 2\lfloor n/4\rfloor + 4$ colors on $P_2\Box C_n$, a contradiction.

Therefore, any vertex dominates the color class consisting of at least two of its neighbors. By Lemma 12 and Fact 3, we may assume that $f(u_2) = 1$, $f(v_2) = f(u_3) = 2$ and $f(u_1) = 3$. In this case v_1 does not dominate color class $\{f^{-1}(2)\}$, so $f(v_n) = 3$ and v_1 dominates color class $\{f^{-1}(3)\}$. Since $N[v_4]\cup N[u_6]\cup N[v_8]\cup\cdots\cup N[u_{n-1}]$ consume at least $(n-3)/2$ colors, say colors $4, 5, \ldots, (n+3)/2$, respectively, then we have $f(v_1) = 1$ and each of $N[v_4], N[u_6], N[v_8], \ldots, N[u_{n-1}]$ consumes exactly one color. Now u_3 does not dominate color class $\{f^{-1}(1)\}$, so it dominates color class $\{v_3, u_4\}$. Since color class $\{f^{-1}(3)\}$ is dominated by v_1 and color class $\{f^{-1}(2)\}$ is dominated by u_2 , we have $f(v_3) = f(u_4) = 4$. And since $N[v_4]$ consumes only one color, we have $f(v_4) \in \{1, 2, 3\}$, and so $f(v_4) = 1$. However, in this case v_2 does not dominate any color class, a contradiction. So $\chi_d(P_2\Box C_n) \geq 2\lfloor n/4\rfloor + 4$

On the other hand, we give a $(2\lfloor n/4\rfloor + 4)$-dominator coloring for $P_2\Box C_n$ as follows.

$$
\begin{aligned}
f(u_i) &= 2l+3 \quad \text{for } i = 4l+1, l = 0, 1, \ldots, \lfloor n/4\rfloor,\\
f(u_i) &= 2 \quad \text{for odd } i, i \leq n-2 \text{ except } i = 4l+1, l = 0, 1, \ldots, \lfloor n/4\rfloor,\\
f(u_i) &= 1 \quad \text{for even } i, 1 \leq i \leq n-2,\\
f(u_{n-1}) &= 2 \quad \text{and } f(u_n) = 1,\\
f(v_j) &= 2l+4 \quad \text{for } j = 4l+3, l = 0, 1, \ldots, \lfloor n/4\rfloor - 1,\\
f(v_j) &= 1 \quad \text{for odd } j, j \leq n-2 \text{ except } j = 4l+3, l = 0, 1, \ldots, \lfloor n/4\rfloor - 1,\\
f(v_j) &= 2 \quad \text{for even } j, 1 \leq j \leq n-3 or j = n,\\
f(v_{n-1}) &= 2 \quad \lfloor n/4\rfloor + 4.
\end{aligned}
$$

Hence $\chi_d(P_2\Box C_n) \leq 2\lfloor n/4\rfloor + 4$, and then $\chi_d(P_2\Box C_n) = 2\lfloor n/4\rfloor + 4$. □

5 Conclusion

In [4] Gera proposed the question that for what graphs does $\chi_d(G) = \gamma(G) + \chi(G)$. By our results, we can see that for $G = P_2\Box P_n$ with $n = 3$ or $n \geq 5$ and $G = P_2\Box C_n$ with even n the equality holds.

References

1. Arumugam, S., Bagga, J., Chandrasekar, K.R.: On dominator colorings in graphs. Proc. Math. Sci. **122**(4), 561–571 (2012)
2. Chellali, M., Maffray, F.: Dominator colorings in some classes of graphs. Graphs Comb. **28**(1), 97–107 (2012)
3. Gera, R., Rasmussen, C., Horton, S.: Dominator colorings and safe clique partitions. Congressus Numerantium **181**, 19–23 (2006)
4. Gera, R.: On dominator colorings in graphs. Graph Theory Notes N. Y. **LII**, 25–30 (2007)
5. Gera, R.: On the dominator colorings in bipartite graphs. Inf. Technol. New Gen. **ITNG'07**, 947–952 (2007)

6. Jacobson, M.S., Kinch, L.F.: On the domination number of products of graphs I. Ars Combinatoria **18**, 33–44 (1984)
7. Merouane, H.B., Chellali, M.: On the dominator colorings in trees. Discussiones Mathematicae Graph Theory **32**(4), 677–683 (2012)
8. Nandi, M., Parui, S., Adhikari, A.: The domination numbers of cylindrical grid graphs. Appl. Math. Comput. **217**(10), 4879–4889 (2011)