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ORIGINAL PAPER

P_{11}-Coloring of Oriented Graphs

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Abstract We show that, every oriented graph with maximum average degree less than 28/9 admits a homomorphism into the Paley tournament with 11 vertices.

Keywords Oriented graph · Planar graph · Homomorphism · Homomorphism bound · Paley tournament

Mathematics Subject Classification Primary 05C15 · Secondary 05C25

1 Introduction

This paper is concerned with *oriented* graphs, that is digraphs without loops or opposite arcs. If G is an oriented graph, then we write $G = (V, A)$, where $V = V(G)$ and $A = A(G)$ are respectively the vertex set and arc set of G. For an unoriented graph G, $E = E(G)$ will denote the edge set of G. For an (oriented) graph G, $|G|$ and $||G||$ will denote respectively $|V(G)|$ and $|E(G)|$ ($|A(G)|$). A k-path in an oriented graph is a path with k edges. We will refer to a 2-path $v_0v_1v_2$ as *directed* if there are arcs from v_0 to v_1 and v_1 to v_2 or from v_2 to v_1 and v_1 to v_0, *non-directed* otherwise. All graphs will be finite in this paper.

A *homomorphism* from $G_1 = (V_1, A_1)$ to $G_2 = (V_2, A_2)$ is a function $\phi : V_1 \to V_2$ such that, if $(u, v) \in A_1$, then $(\phi(u), \phi(v)) \in A_2$. A homomorphism from G to H is also referred to as an *H-coloring* of G, and the vertices of H as colors. If such a homomorphism exists we say that G is *H-colorable*, this terminology being suggested by the fact that, for an unoriented graph, a proper vertex coloring with n colors is

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equivalent to a homomorphism to the complete graph K_n. An oriented graph is *H-critical* if it has no H-coloring, but every proper subgraph of it has. We say that a configuration is *H-reducible* if no H-critical graph can contain it.

The *average degree*, $\mathrm{ad}(G)$, of a graph G is defined by $\mathrm{ad}(G) = 2||G||/|G|$. From this we define the *maximum average degree*, $\mathrm{mad}(G)$, of G as the maximum of $\mathrm{ad}(H)$ taken over all subgraphs H of G.

Much work has has been done showing that the members of a certain class of oriented graphs (e.g. planar graphs, graphs with bounded maximum average degree, bounded treewidth, etc.) are all H-colorable for a certain graph H. Borodin et al. [1] have proved (among several similar results) that every oriented graph G with $\mathrm{mad}(G) < 3$ is P_{11}-colorable, where P_{11} is the Paley tournament on 11 vertices (see below). Our main theorem is a strengthening of this result.

Theorem 1 *Every oriented graph G with $\mathrm{mad}(G) < 28/9$ is P_{11}-colorable. This bound is sharp.*

A *forbidden graph* is an oriented graph with the same underlying unoriented graph as the graph shown in Fig. 1, oriented such that the 2-paths a_1va_2 and each $v_ix_iv_{i+1}$ (subscripts modulo 3) are all directed in either direction, and all the arcs between the vertex sets $\{a_1, a_2\}$ and $\{v_1, v_2, v_3\}$ are in the same direction. Figure 1 shows one such orientation, and there are three more (up to isomorphism). These can be obtained from the graph in the figure by reversing all the arcs between $\{a_1, a_2\}$ and $\{v_1, v_2, v_3\}$ and/or

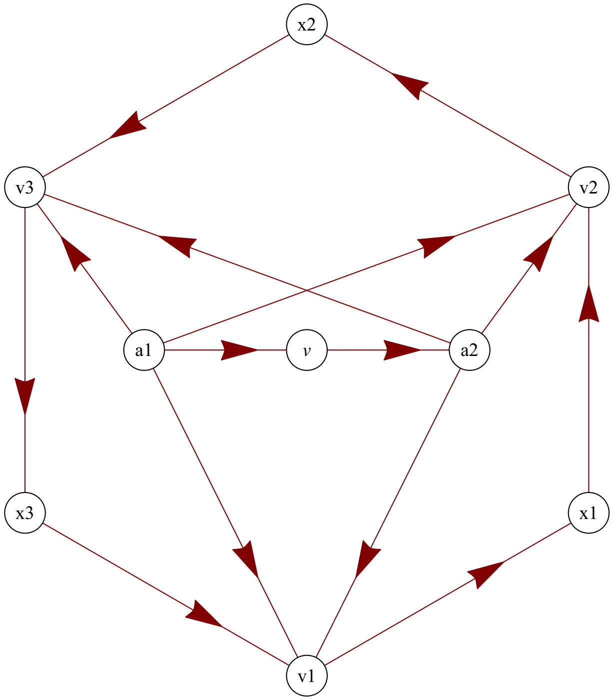

Fig. 1 A forbidden graph

reversing one of the directed paths linking two v_is. We will show that the forbidden graphs are not P_{11}-colorable (Corollary 16 below); since they have maximum average degree $28/9$, this gives the sharpness in Theorem 1. If we exclude the forbidden graphs, we get the following stronger result.

Theorem 2 *Every oriented graph G with* $\mathrm{mad}(G) < 22/7$ *is P_{11}-colorable if and only if it contains no forbidden subgraph.*

The key lemma used in the proof is the following.

Lemma 3 *Every cycle of vertices of degree at most 3 is P_{11}-reducible.*

This lemma also has an immediate application. For this purpose we first state it in a slightly more explicit version, in which the cycle is assumed to be induced.

Lemma 4 *If G is an oriented graph, C is an induced cycle of vertices of degree at most 3 in G and there is a P_{11}-coloring of $G - C$, then this can be extended to a P_{11}-coloring of G.*

Every graph which contains a cycle of vertices of degree at most 3 must contain an induced such cycle (choose one of minimum length), so that Lemma 4 immediately gives Lemma 3.

We now apply Lemma 4 to grid coloring. Fertin et al. [2] have shown that every grid is P_{11}-colorable. Since every cylindrical grid (the Cartesian product of a path and a cycle) contains an induced cycle C of vertices of degree at most 3, for which $G - C$ is either a cylindrical grid or empty, induction using Lemma 4 gives the following generalization.

Theorem 5 *Every cylindrical grid is P_{11}-colorable.*

We conclude the section with some more notation. Given two vertices v, w in an oriented graph, we set $[v, w]$ to be 1, -1 or 0 according as there is an arc from v to w, from w to v, or neither. If G is an oriented graph, and $V_0 \subseteq V(G)$, $G[V_0]$ will represent the induced oriented graph with vertex set V_0. We then set $G - V_0 = G[V(G)\setminus V_0]$. The degree of a vertex v in G is denoted by $d_G(v)$, with the subscript omitted if the graph G is clear from context. A *k-vertex* (resp. *$^{\leq}k$-vertex*, x*$^{\geq}k$-vertex*) is a vertex of degree k (resp. $\leq k$, $\geq k$). A *k-neighbor* (resp. *$^{\leq}k$-neighbor*, *$^{\geq}k$-neighbor*) of a vertex v is a neighbor of v which is a k-vertex (resp. $^{\leq}k$-vertex, $^{\geq}k$-vertex). If $v \in V(G)$ then $N^1(v)$ and $N^{-1}(v)$ will represent respectively the sets of out-neighbors and in-neighbors of v, also written as $out(v)$ and $in(v)$ respectively. We refer to these sets collectively as *oriented neighborhoods* of v. If $S \subset V(G)$ and $a = \pm 1$, then $N^a(S) := \cup_{v \in S} N^a(v)$.

2 The Target Graph: P_{11}

If q is prime, $q \equiv 3 \bmod 4$, then the *Paley Tournament* P_q is defined to be the tournament with $V(P_q) = \mathbb{Z}_q$ and

$$A(P_q) = \{(a, b) \mid b - a \text{is a non-zero quadratic residue mod } q\}$$

(the condition $q \equiv 3 \bmod 4$ guarantees that exactly one of a and $-a$ is a quadratic residue for $a \not\equiv 0 \bmod q$, so P_q is indeed a tournament). The automorphism group, $\mathrm{Aut}(P_q)$ of P_q comprises the maps $n \to an + b$, where $a, b \in \mathbb{Z}_q$ and a is a non-zero quadratic residue; in particular, P_q is arc-transitive.

We derive some preliminary results about P_{11}. All the computations we do are feasible by hand, but in order to reduce tedious analyses of cases, we have always used a computer for routine calculations whenever this saves work, with minimum explanation of details.

The automorphism group of P_{11} induces a natural action on the (unordered) k-subsets of $V(P_{11})$, namely

$$\phi(\{v_1, v_2, \dots v_k\}) = \{\phi(v_1), \phi(v_2), \dots \phi(v_k)\} \tag{1}$$

As nearly every property of vertex sets which we consider is invariant under this action, it will be useful, for small k, to count the orbits, and to find a representative for each one. We know that the action is transitive for $k \le 2$. For $k = 3, 4$ we have

Lemma 6 *Modulo automorphism there are three* 3*-element subsets and six* 4*-element subsets of* $\mathbb{Z}_{11}$*. The classes of* 3*-sets have representatives*

$$\{0, 1, 2\}, \quad \{0, 1, 4\}, \quad \{0, 1, 5\}. \tag{2}$$

The first of these induces a circuit, and the other two induce transitive triangles. The classes of 4*-sets have representatives*

$$\{0, 1, 2, 3\}, \quad \{0, 1, 2, 4\}, \quad \{0, 1, 2, 5\}, \quad \{0, 1, 2, 6\}, \quad \{0, 1, 2, 8\}, \quad \{0, 1, 3, 4\}. \tag{3}$$

Lemma 7 *Each set of at least 8 vertices in* $V(P_{11})$ *contains an oriented neighborhood.*

Proof It suffices to show this for the complements of the sets listed in (2), and these sets all include *out*(5). □

Lemma 8 *If* S *is a set of vertices in* $V(P_{11})$*, and* $a = \pm 1$*, then*

$$\begin{aligned}
|S| = 1 &\implies |N^a(S)| = 5\\
|S| = 2 &\implies |N^a(S)| = 8\\
|S| = 3 &\implies |N^a(S)| \ge 9 \quad (\textit{strictly if } S \textit{ induces a circuit, or } S \subseteq N^{-a}(v) \textit{ for some } v.)\\
|S| = 4 &\implies |N^a(S)| \ge 10
\end{aligned} \tag{4}$$

Sketch of Proof Transitivity and arc-transitivity of P_{11} reduce the cases $|S| = 1$ and $|S| = 2$ to $S = \{0\}$ and $S = \{0, 1\}$ respectively. Lemma 6 can be used for the cases $|S| = 3$ and $|S| = 4$ (note that in the latter case we need only check the set $S = \{0, 1, 3, 4\}$, since $\{0, 1, 2\}$ induces a circuit). □

The following, which is also easily verified using Lemma 6, slightly generalizes the case $|S| = 4$ above.

Table 1 Oriented neighborhoods in P_{11}

n	Out (n)	In (n)
0	1,3,4,5,9	2,6,7,8,10
1	2,4,5,6,10	0,3,7,8,9
2	0,3,5,6,7	1,4,8,9,10
3	1,4,6,7,8	0,2,5,9,10
4	2,5,7,8,9	0,1,3,6,10
5	3,6,8,9,10	0,1,2,4,7
6	0,4,7,9,10	1,2,3,5,8
7	0,1,5,8,10	2,3,4,6,9
8	0,1,2,6,9	3,4,5,7,10
9	1,2,3,7,10	0,4,5,6,8
10	0,2,3,4,8	1,5,6,7,9

Lemma 9 *If $S \subseteq \mathbb{Z}_{11}$, $|S| = 4$ and $a = \pm 1$, then $|N^a(A)| \geq 10$ for at least two of the 3-element subsets A of S.*

The (arc-)transitivity of P_{11} makes the next two results easy to verify using Table 1. We may assume $v = 0$ in Lemma 10 and $(v, w) = (0, 1)$ in Lemma 12.

Lemma 10 *If $a = \pm 1$, $v \in \mathbb{Z}_{11}$ and S is a 4-element subset of $N^a(v)$, then $N^a(S) = \mathbb{Z}_{11} \backslash \{v\}$, and $N^{-a}(S) = \mathbb{Z}_{11}$.*

Corollary 11 *If $a = \pm 1$ and $v \in \mathbb{Z}_{11}$, then $N^a(N^a(v)) = \mathbb{Z}_{11} \backslash \{v\}$, and $N^{-a}(N^a(v)) = \mathbb{Z}_{11}$.*

Lemma 12 *If $v, w \in V(P_{11})$, and $w \in out(v)$ then, $|out(v) \cap out(w)| = |in(v) \cap in(w)| = |out(v) \cap in(w)| = 2$, and $|in(v) \cap out(w)| = 3$.*

Corollary 13 *The intersection of two oriented neighborhoods in P_{11} is either empty, or contains at least two members.*

Lemma 12 shows in particular that oriented neighborhoods of different vertices always have a non-empty intersection. We extend this observation slightly.

Lemma 14 *If $v \in V(P_{11})$, $S \subseteq V(P_{11})$, $|S| = 1, 2$, $a, b = \pm 1$, and either $a = b$ or $v \notin S$ then, $|N^a(S) \cap N^b(v)| \geq |S| + 1$.*

Proof Lemma 12 deals with $|S| = 1$; $|S| = 2$ is an easy calculation (we may assume that $S = \{0, 1\}$). □

Lemma 15 *If $u, v, v' \in V(P_{11})$ and $a = \pm 1$, then $N^a(u) \cap N^a(v) = N^a(u) \cap N^a(v')$ only if $v = v'$.*

Proof Another straightforward computation. We may assume $u = 0$. □

Corollary 16 *The forbidden graphs are not P_{11}-colorable.*

Proof With reference to Fig. 1, a_1 and a_2 must take different colors, since they are joined by a directed 2-path. Moreover the same set of colors is available at each of v_1, v_2 and v_3, and by Lemma 12, this set has two elements. But since any two of the vertices v_1, v_2 and v_3 are joined by a directed 2-path, they must take distinct colors, which is impossible. □

Lemma 17 *Let $x, y \in V(P_{11})$, $a, b, c \in \{-1, 1\}$, then either*

1. $$N^b(t) \cap N^c(y) \text{ induces a circuit} \tag{5}$$
   for are at least two values of $t \in N^a(x)$ or
2. $$|N^b(S) \cap N^c(y)| \geq |S| + 1 \tag{6}$$
   for all $S \subseteq N^a(x)$ with $|S| = 1, 2$, and with at most two exceptions for $|S| = 3$.

Proof We first show that it suffices to prove the lemma for the case $a = 1$. From the definition of Paley graphs (in particular from the fact that -1 is not a quadratic residue modulo 11), we have $N^a(-S) = -N^{-a}(S)$ (where $-S = \{-v \mid v \in S\}$). Thus, if $S \subseteq N^{-a}(x)$, then $-S \subseteq N^a(-x)$, and

$$N^b(S) \cap N^c(y) = -[N^{-b}(-S) \cap N^{-c}(-y)].$$

Since the cardinality of a set S, and the property of inducing a circuit are both preserved in passing from S to $-S$, the lemma holds for both values of a if it holds for either one. We may thus assume that $a = 1$.

By arc transitivity we may further assume that $x = 0$. and that $y \in \{0, 1, 2\}$ (since this set contains x and one representative from each of $N^+(x)$ and $N^-(x)$).

By Lemma 14, (6) holds for $|S| = 1, 2$ if either $y \notin N^a(x)$ (that is $y \neq 1$ with our normalization) or $b = c$. When $|S| = 3$, Lemma 8 shows that (6) holds if either $b = -a$ (in which case $S \subseteq N^{-b}(x)$) or $x \notin N^c(y)$ (in which case, either $b = -a$ or x is absent from both $N^b(S)$ and $N^c(y)$).

In view of our normalizing assumptions, $x = 0$, $a = 1$, $y \in \{0, 1, 2\}$, these conditions ensure that (6) holds for $1 \leq |S| \leq 3$ except in the following cases: $(b, c, y) = (-1, 1, 1), (1, -1, 1)$ or $(1, 1, 2)$. In the first case (5) holds by direct calculation, since $in(3) \cap out(1) = \{2, 5, 10\}$ and $in(9) \cap out(1) = \{4, 5, 6\}$ both induce circuits. In the second case the same is true of $out(4) \cap in(1) = \{7, 8, 9\}$ and $out(5) \cap in(1) = \{3, 8, 9\}$. In the last case again $y \notin N^a(x)$, so (6) holds for $|S| = 1, 2$. Direct calculation shows that it also holds for $|S| = 3$ except when $S = \{1, 3, 4\}$ or $S = \{3, 5, 9\}$. □

3 Colorings

Let G and H be oriented graphs, and let L be a function which assigns to each $v \in V(G)$ a subset of $V(H)$. We say that G is *L-choosable* if there is a homomorphism $\phi : G \to H$ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. We refer to such a homomorphism as a *list coloring* for the list assignment L, or more briefly as an L-coloring. The target graph H is not referred to explicitly in our notation; in this paper it is always P_{11}.

Lemma 18 *A cycle C is L-choosable, if each $L(v)$ is an oriented neighborhood of a vertex in $\mathbb{Z}_{11}$.*

Remark In the above, the hypothesis on $L(v)$ cannot be relaxed to $|L(v)| \geq 5$. For example, if C is a transitive triangle $v_1v_2v_3$, with source v_1 and sink v_3, and with list assignment $L(v_1) = \{1, 3, 4, 5, 6\}$, $L(v_2) = \{0, 1, 3, 5, 6\}$, $L(v_3) = \{0, 1, 2, 3, 5\}$, one can readily verify that C is not L-choosable. Lemma 18 immediately gives

Proof of Lemma 4 Suppose that G contains an induced cycle C of $^{\leq}3$-vertices. Color $G - C$ and apply Lemma 18.

Proof of Lemma 18 We first suppose that $|C| \geq 4$. Let $C = v_1, v_2, \ldots v_n$ and $L(v_i) = N^{a_i}(x_i)$. For $1 \leq i \leq n$, we define P_i to be the path $v_1v_2, \ldots v_i$, and for $k \in L(v_1)$, let

$$c_i(k) = \{\phi(v_i) \mid \phi \text{ is an } L\text{-coloring of } P_i \text{ such that } \phi(v_1) = k\}, \tag{7}$$

then $c_1(k) = \{k\}$, and

$$c_{i+1}(k) = N^{e_i}(c_i(k)) \cap N^{a_{i+1}}(x_{i+1}) \text{ where } e_i = [v_i, v_{i+1}]. \tag{8}$$

We say that i is *good* if $|c_i(k)| \geq 4$ for at least 2 values of k.

We will show that every $i \geq 4$ is good. If the first case of Lemma 17 holds for some i, with $x = x_i$, $y = x_{i+1}$, $a = a_i$, $b = e_i$ and $c = a_{i+1}$, then by reindexing we may suppose that $i = 1$. It follows that $c_2(k)$ induces a circuit for at least 2 values of $k \in L(v_1)$. Then by induction using (8) and Lemma 8, all $i \geq 3$ are good.

Otherwise for all i, the second case of Lemma 17 holds with $x = x_i$, $y = x_{i+1}$, $a = a_i$, $b = e_i$ and $c = a_{i+1}$, and we can apply (6) repeatedly to get in turn that $|c_2(k)| \geq 2$ and $|c_3(k)| \geq 3$ for all $k \in L(v_1)$, and that $|c_4(k)| \geq 4$ for at least three of these values. In particular $i = 4$ is good, and as before this remains true for all $i \geq 4$.

Since n is good, there are $k_1, k_2 \in L(v_1)$ such that $|c_n(k_1)|, |c_n(k_2)| \geq 4$. By Lemma 10 it follows that $N^{e_n}(c_n(k_1)) = N^{e_n}(c_n(k_2))$ contains either k_1 or k_2, hence there is an L-coloring of C, which takes this color at v_1. Only the case $k = 3$ remains, and the argument above can be adapted to cover this. Here there are only finitely many cases to consider, and we can prove the following slightly stronger result using a computer. □

Lemma 19 *The following are L-choosable:*

1. *A triangle $v_1v_2v_3$, where $|L(v_1)| \geq 4$, and each other $L(v)$ is an oriented neighborhood of a vertex in Z_{11}.*
2. *A 4-cycle $v_1v_2v_3v_4$, where $L(v_1) = \mathbb{Z}_{11}$, $|L(v)| \geq 4$ for each other v_i, and $L(v_2) \neq L(v_4)$.*

Clearly the condition $L(v_2) \neq L(v_4)$ is redundant if either of these sets contains more than four vertices, a fact we use in the following application.

Corollary 20 *If G comprises a 6-cycle $u_0u_1u_2u_3u_4u_5$, together with an edge between u_0 and u_3, and $L(u_i)$ are subsets of $V(\mathbb{Z}_{11})$ such that $|L(u_0)| \geq 5$, $L(u_1), L(u_5) = \mathbb{Z}_{11}$, $|L(u_2)|, |L(u_4)| \geq 4$ and $|L(u_3)| \geq 10$, then any orientation of G is L-colorable.*

Proof Let $c \in L(u_2)$. By Corollary 11, there is an L-coloring of the oriented G for which u_2 takes the color c, if there is an L'-coloring of the cycle $G[u_0, u_3, u_4, u_5]$, where $L'(u_3) = L(u_3) \cap N^{[u_2,u_3]}(c)$, $L'(u_0) = L(u_0)\backslash\{c\}$, and $L'(u_i) = L(u_i)$ for $i = 4, 5$. By Lemma 19 (2), such a coloring exists if $L'(u_0) \neq L'(u_4)$ or if $|L'(u_0)| \geq 5$. At least one of these conditions is met if we can choose $c \notin L(u_0)$ or $c \in L(u_4)$. We are thus done unless $L(u_2) \cap L(u_4) = \emptyset$, $L(u_2) \subseteq L(u_0)$ and (interchanging the roles of u_2 and u_4) $L(u_4) \subseteq L(u_0)$, but these three together conditions force $|L(u_0)| \geq 8$, so we are done in this case too. □

Lemma 21 *A path $v_1 \ldots v_n$ is L-choosable, if $|L(v_1)|, |L(v_n)| \geq 3$, and if $|L(v)| \geq 5$ for each interior vertex of P.*

Proof A straightforward induction on $|P|$, using Lemma 8. □

The following result resembles Lemma 18, and the proof is similar but easier.

Lemma 22 *A cycle $C = v_1v_2, \ldots v_n$ is L-choosable, if $|L(v_1)|, |L(v_3)| \geq 4$, $L(v_2) = \mathbb{Z}_{11}$, $|L(v_k)| \geq 5$ for all $k \geq 4$, and if $n \geq 4$, $L(v_n)$ is either an oriented neighborhood or contains at least 6 vertices.*

Proof If $n = 3$, then we can L-color v_1 and v_3 using Lemma 8, and this coloring then extends to C by Lemma 12. Now suppose that $n \geq 4$. As in the proof of Lemma 18, for $1 \leq i \leq n$, we define P_i to be the path $v_1, v_2, \ldots v_i$, and we define $c_i(k)$ by (7). Thus $c_1(k) = \{k\}$, and

$$c_{i+1}(k) = N^{e_i}(c_i(k)) \cap L(v_{i+1}) \quad \text{where} \quad e_i = [v_i, v_{i+1}]. \tag{9}$$

Thus $c_2(k) = N^{e_1}(k)$, whence by Corollary 11, $c_3(k) \supseteq L(v_3)\backslash\{k\}$, for each $k \in L(v_1)$. By Lemma 9, $|c_i(k)| \geq 4$ for $i = 4$ for at least two values of $k \in L(v_1)$, whence by induction using Lemma 8, this is true when $4 \leq i \leq n$. If $|L(v_n)| \geq 6$, then there are k_1 and k_2, such that $|c_n(k_1)|, |c_n(k_2)| \geq 5$. Since therefore $c_n(k_1) \cap c_n(k_2) \geq 4$, Lemma 8 gives that $N^{e_n}(c_n(k_1)) \cap N^{e_n}(c_n(k_2))$ omits at most one vertex, and so contains either k_1 or k_2. If $L(v_n)$ is an oriented neighborhood, then the same conclusion holds, using Lemma 10 as in the proof of Lemma 18. Again as in the proof of Lemma 18, we conclude that there is an L-coloring of C, which takes the color k_1 or k_2 at v_1. □

4 Reducible Configurations

Definition A *weak* vertex is a 4-vertex with two 2-neighbors.

Definition A *bad* configuration is a 6-cycle $C = v_1x_1v_2x_2v_3x_3$, where the vertices v_i and x_i are of degree 4 and 2 respectively, together with a 4-vertex y, adjacent to each of

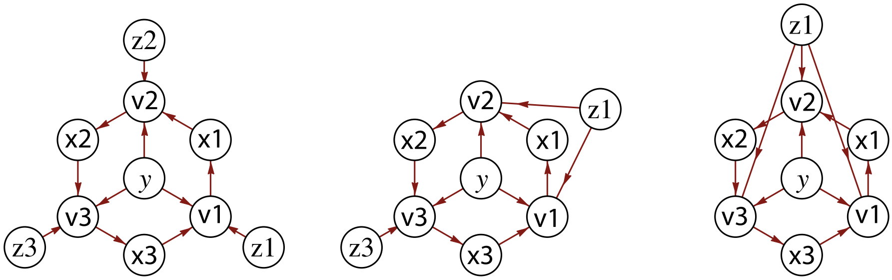

Fig. 2 Bad configurations

the v_i, and (not necessarily distinct) vertices z_1, z_2, z_3 with z_i adjacent to v_i, oriented so that that each of the 2-paths $v_i x_i v_{i+1}$ (counting subscripts modulo 3) is directed, and the arcs between $\{y, z_1, z_2, z_3\}$ and $\{v_1, v_2, v_3\}$ are all in the same direction (Fig. 2). Note that a bad configuration with $z_1 = z_2 = z_3$ is part of a forbidden subgraph (Fig. 1) if there is a directed 2-path between y and z_1.

Lemma 23 *The following configurations are P_{11}-reducible*

A $^{\leq}1-$ vertex. (10)

Two adjacent 2-vertices. (11)

A 3-vertex with a 2-neighbor. (12)

A non-directed 2-path with interior vertex of degree 2. (13)

Two vertices linked by two or more paths with interior vertices of degree 2. (14)

A 3-vertex with three $^{\leq}3$-neighbors. (15)

A weak vertex with two 3-neighbors. (16)

Two adjacent weak vertices. (17)

A path of 3-vertices with two weak neighbors. (18)

A triangle $v_1v_2v_3$ such that v_1 is a 4-vertex with a 2-neighbor, and each of v_2 and v_3 is a weak vertex or a 3-vertex. (19)

A 4-vertex with three or more 2-neighbors. (20)

A 4-vertex v with neighbors w, v_1, v_2 and v_3 such that $d(w) = 2$, each of v_1, v_2 and v_3 is a weak vertex or a $^{\leq}3$-vertex, and these three vertices are not the vertices v_1, v_2 and v_3 of a bad configuration (Fig. 2). (21)

A 5-vertex with four or more 2-neighbors. (22)

A 5-vertex with three 2-neighbors and two weak or 3-neighbors. (23)

A 6-vertex with six 2-neighbors. (24)

A 6-vertex with five 2-neighbors and a weak or 3-neighbor. (25)

A 7-vertex with seven 2-neighbors. (26)

Sketch Proof The proofs of reducibility follow a similar pattern for all cases. We suppose that G contains one of the listed configurations, and that every proper subgraph of G is P_{11}-colorable. We delete the vertices in the configuration (and possibly some others), P_{11}-color the remaining graph, and then extend this coloring back to G, using the list-coloring results of the previous section. We do a few cases in detail and sketch the rest.

(10) Trivial

(11) Similar to (and easier than) proof of (12) below

(12) Let u be a 3-vertex in G, with 2-neighbor v. Let w be the other neighbor of v. By assumption there is a P_{11}-coloring of $G - \{v\}$. In this graph $d(u) = 2$, so by Corollary 13, we can change the color of u if necessary, so that it differs from the color of w. We can now extend this coloring back to G, using Lemma 12.

(13) Delete the interior vertex. Color inductively. Extend using Lemma 14

(14) Let u and v be joined by two or more paths whose interior vertices are of degree 2. If any of these paths has length 3 or more or is a non-directed 2-path, then we have reducibility by (11) and (13) respectively, so we assume all the paths are directed 2-paths or arcs. Since at most one path is an arc, there must be a path of length 2. Delete its middle vertex. Color inductively. Since u and v are still joined by an arc or a directed 2-path, they take different colors. Extend using Lemma 12.

(15) Let v be a 3-vertex with three $^{\leq}3$-neighbors w_1, w_2 and w_3. We may assume these each of these vertices has degree 3, and that no two of them are adjacent, otherwise (10), (12) or Lemma 3 applies. Delete v, and inductively color $G - \{v\}$. Since w_i has degree 2 in $G - \{v\}$, Corollary 13 shows that we can choose independently one of at least two colors for each of these vertices. Lemma 8 then shows that for each i there at least 8 colors at v which are compatible with one or other of the possible colors at w_i. In other words (as we will phrase it from now on), each w_i deletes up to three colors at v; this leaves at least two colors available at v to extend the coloring of $G - \{v\}$ back to G.

(16) Let v be a weak vertex with two 3-neighbors w_1 and w_2, and hence two 2-neighbors x_1 and x_2. We may assume that neither w_1 nor w_2 is adjacent to x_1 or x_2, since (12) covers these cases. If w_1 and w_2 are not adjacent to each other, then delete v and color $G - \{v\}$ inductively. By Corollary 13 there are at least two colors available at each w_i, and at least five at each x_i (since each x_i has degree 1 in $G - \{v\}$). By Lemma 8, these together eliminate at most $3 + 3 + 1 + 1 = 8$ of the 11 colors available at v.

If w_1 and w_2 are adjacent, then inductively color $G - \{v, w_1, w_2\}$. There are 5 color choices available at each x_i, and hence, using Lemma 8, at least 10 at v. The set of colors available at each w_i is an oriented neighborhood in $\mathbb{Z}_{11}$. The coloring can thus be extended back to G using Lemmas 7 and 18.

(17) If the weak vertices u and v have a common 2-neighbor, then (14) applies, so we assume that u and v have distinct 2-neighbors. Let the neighbors of u be v, u_1, u_2 and u_3, where $d(u_1) = d(u_2) = 2$. Inductively color $G - \{u, v\}$. There are 5 choices of color available at each of u_1 and u_2. Thus, by Lemma 8, u_1 and u_2 each delete at most one color at u, while u_3 deletes 6. Thus we are left with at least 3 possible colors at u, and symmetrically at v also. (It is possible that u_3 is also a neighbor of v; the

argument still works in this case.) We can now extend the coloring back to G using Lemma 21.

(18) Let P be a path of 3-vertices with weak neighbors v_1 and v_2. In view of (17), (16) and (14), we may assume that v_1 and v_2 are not adjacent to each other, that neither is adjacent to more than one vertex of P, and that they have at most one common 2-neighbor. By shortening the path P if necessary, we may also assume that the neighbors of v_1 and v_2 in P are the two (not necessarily distinct) end vertices of P.

Suppose first that v_1 and v_2 have a common 2-neighbor x. We inductively color $G\backslash(V(P)\cup\{v_1, v_2, x\})$. If the neighbors of v_1 in $G\backslash(V(P)\cup\{x\})$ are y and z, with the latter a 2-vertex, then these vertices delete at most 6 colors and one color respectively from v_1. This leaves at least 4 colors at v_1, and symmetrically 4 colors at v_2. All 11 colors are available for x. Each vertex in P has a unique neighbor not in $V(P)\cup\{v_1, v_2\}$, so the set of colors available is an oriented neighborhood in Z_{11} The coloring can thus be extended back to G using Lemma 22.

If v_1 and v_2 have no common 2-neighbor, then we inductively color $G\backslash(V(P)\cup\{v_1, v_2\})$. We have at least three colors available at each of v_1 and v_2, and 5 at each vertex of P. We then extend the coloring back to G using Lemma 21.

(19) Let $T = v_1v_2v_3$, v_1 is a 4-vertex with a 2-neighbor, and v_2 and v_3 are weak or 3-vertices. If any two vertices of T have a common 2-neighbor, apply (14). Otherwise delete $V(T)$ and color inductively. This leaves at least 4 colors available at v_1 and the vertices of an oriented neighborhood at each of v_2 and v_3 (if v_i is weak, then we have available a set of at least 9 colors, which by Lemma 7 contains an oriented neighborhood.) Extend the coloring using Lemma 19 (1).

(20) Delete the 4-vertex. Inductively color. Extend.

(21) By (10), (16), (17) and (20), we may assume that w is the *only* $^{\leq}2$-neighbor of v. We may assume that none of the vertices v_1, v_2 and v_3 is adjacent to w, and that no two of them are adjacent to each other or have more than one common 2-neighbor, as these cases are covered by (14), (19) and (14) respectively.

Any two of the vertices v_1, v_2 and v_3 may have one common 2-neighbor, and we subdivide the proof according to how many of them there are.

Case 1: No common 2-neighbors Inductively color the graph obtained from G by deleting v and its neighbors. At least two colors are available at each v_i (at least two if v_i is a 3-vertex, at least 11-1-1-6 $= 3$ if v_i is weak); thus these vertices in turn each delete at most three colors from v; w deletes at most one more. Thus at least one color remains for v.

Case 2: One common 2-neighbor Suppose that v_1 and v_2 have a common 2-neighbor x. Using (12), we may assume that v_1 and v_2 are both weak. After inductive coloring of $G - \{v, w, v_1, v_2, v_3, x\}$, there are at least 2 colors available at v_3 and at least 6 at w, so these vertices in turn together delete at most 4 colors from v. There are thus at least 7 colors available at v. There at least 4 colors available at v_1 and v_2, and 11 at x. The coloring of G can then be completed by Lemma 22.

Case 3: Two common 2-neighbors We may assume that v_1 and v_2 have a common 2-neighbor x_1, and that v_2 and v_3 have a common 2-neighbor x_2. Using (12), we may assume that v_1, v_2 and v_3 are all weak. Inductively color $G - \{v, w, v_1, v_2, v_3, x_1, x_2\}$. As usual w deletes at most one color from v, so we have 10 colors available at v, at least 5 at v_2, at least 4 at each of v_1 and v_3, and all 11 at each of x_1 and x_2. The coloring of G can then be completed using Corollary 20.

Case 3: Three common 2-neighbors Again, using (12), we may assume that v_1, v_2 and v_3 are all weak. These vertices now alternate with 2-vertices in a 6-cycle $C = v_1x_1v_2x_2v_3x_3$. Let z_i be the neighbor of v_i not in $V(C) \cup \{v\}$. We may assume that each of the 2-paths in C from v_i to v_j is directed, otherwise we have reducibility by (13). Since $d(v) = 4$, and by the assumption that v and C do not form a bad configuration, there is some z_i such that the path vv_iz_i is directed. Suppose that vv_1z_1 is directed. Now inductively we have a coloring c of $G - V(C)$. There are at least 10 choices of color at v, so we recolor v if necessary, so that $[c(z_1), c(v)] = [v_1, z_1]$ (which eliminates 6 choices of color) and $c(v) \notin \{c(z_2), c(z_3)\}$ (which eliminates 2 more). We now extend c, first to the vertices v_i, then to the x_i. By Lemma 12, at least three colors are available at v_1, and at least two at each of v_2 and v_3. We can thus give v_1, v_2 and v_3 distinct colors. By Lemma 14, we can now color x_1, x_2 and x_3.

The remaining configurations are straightforward. We gives details only for (23); the others are similar and easier.

(23) Let v be 5-vertex with three 2-neighbors and two weak or 3-neighbors, v_1 and v_2. Using (14), we may assume that v_1 and v_2 are adjacent to none of the 2-neighbors of v, and have at most one common 2-neighbor. If v_1 and v_2 are neither adjacent nor have a common 2-neighbor, then the argument is similar to case 1 of the proof of (21). If v_1 and v_2 have a common 2-neighbor x, then by (14) they are not adjacent. Now we inductively color $G - \{v, v_1, v_2, x\}$, and apply Lemma 22. Finally suppose that v_1 and v_2 are adjacent. By (17), we may assume they are 3-vertices. Now inductively color $G - \{v, v_1, v_2\}$. There are at least 8 colors available at v, and the sets of colors available at v_1 and v_2 are oriented neighborhoods. Now apply Lemma 19 (1). □

5 Bad Configurations

To deal with bad configurations we use a slightly weaker notion of reducibility. Instead of considering graphs with a given property minimal under the subgraph relation, we use minimal *order* such graphs.

Lemma 24 *If H is a subgraph of a forbidden graph with $|H| \geq 3$, then $||H|| \leq 2|H| - 4$.*

Proof By Turan's theorem, triangle-free graphs of order 3,4,5, and 6, have at most 2,4,6 and 9 edges respectively, the upper bound in the last case being uniquely attained by the complete bipartite graph $K_{3,3}$. Since forbidden graphs are triangle-free, and clearly have no $K_{3,3}$ subgraph, this gives the lemma for $|H| \leq 6$. The remaining cases are easy. □

Lemma 25 *Let $\rho \leq 16/5$. If* $\text{mad}(G) < \rho$*, G is non-P_{11}-colorable, and contains no forbidden subgraph, and G is minimal order with these three properties, then G contains no bad configuration.*

Proof If $\rho \leq 2$, then $\text{mad}(G) < \rho$ implies that G is a forest, which is trivially P_{11}-colorable, so that the lemma holds vacuously in this case. We thus assume that $\rho > 2$.

Suppose for a contradiction that G is a minimal order non-P_{11}-colorable oriented graph G with $\mathrm{mad}(G) < \rho$ and no forbidden subgraph, but that G contains a bad configuration. We name the vertices of the bad configuration as in the definition at the start of Sect. 4 (and Fig. 2): that is $C = v_1x_1v_2x_2v_3x_3$ is a cycle, where the vertices v_i and x_i are of degree 4 and 2 respectively, and y is a 4-vertex adjacent to v_1, v_2 and v_3. Let e denote the fourth neighbor of y. Recall that z_1, z_2, and z_3 are (not necessarily distinct) vertices adjacent to v_1, v_2 and v_3 respectively. Let $W = V(C) \cup \{y\}$.

Case 1 $z_1 = z_2 = z_3$ Let z denote the common vertex z_i; y is not joined to z by either an arc (in which case $G[W \cup \{z\}]$ would be a subgraph of G with average degree $13/4 > 16/5 \geq \rho$) or a directed 2-path (in which case G would contain a forbidden subgraph). Thus $e \neq z$, and if e is adjacent to z, then we have $[z, e] = [y, e]$. Let G' be defined to be $G - W$, with an arc added between z and e, oriented so that $[z, e] = [y, e]$, if no such arc is already present in G. We will show that G' contains no forbidden subgraph, and that $\mathrm{mad}(G') < \rho$.

Suppose that M is a subgraph of G', which is either forbidden or has average degree $\geq \rho$. Since G contains no such subgraph, this implies that G' contains a new arc between z and e, and M includes this arc. Now let H be the subgraph of G induced by W and the vertices of M. We have $||H|| \geq ||M|| + 12$ (H contains all the arcs of M except one, together with 13 new arcs: 9 in W, 3 between the v_i and z and one between y and e), and $|H| = |M| + 7$. Thus, since H is a subgraph of G,

$$\rho > \mathrm{ad}(H) \geq 2\left(\frac{||M|| + 12}{|M| + 7}\right)$$

Since $2(12/7) > 16/5 \geq \rho$, this gives a contradiction when $\mathrm{ad}(M) \geq \rho$. If M is forbidden, then $||M|| = 14$ and $|M| = 9$, which again gives a contradiction. We conclude as required that G' contains no forbidden subgraph, and that $\mathrm{mad}(G') < \rho$.

Since $|G'| < |G|$, G' has a P_{11}-coloring by the minimality assumption. The restriction of this coloring to $G - W$, then extends readily back to G. Since e is the only vertex in G' adjacent to y, and $[y, e] = [z, e]$, we can first give y the same color as z, so that there are 5 colors available at v_1, v_2 and v_3. We can make any choice of distinct colors at these vertices, and then use Lemma 12 to extend the coloring to G. Thus G is P_{11}-colorable, and we have a contradiction.

Case 2: Exactly two of the z_i are the same We may suppose $z_1 = z_2 \neq z_3$. Define G' by adjoining to $G - W$ a new directed 2-path between z_1 and z_3, with the midpoint a new vertex x. Let M be a subgraph of G', which is either forbidden or has average degree $\geq \rho$, and in the latter case suppose that $\mathrm{ad}(M)$ is maximized. In either case M must have minimum degree at least two, in the latter case because if M had an isolated or pendant vertex, we could increase $\mathrm{ad}(M)$ by deleting it (using here the assumption that $\rho > 2$).

As in the previous case, M is not a subgraph of G, so must include the new vertex x, and by the minimum degree condition, both its neighbors z_1 and z_3 as well. Now let H be the subgraph of G induced by W and the vertices of M. We have $||H|| \geq ||M|| + 10$ (H contains all but two of the arcs of M, together with 9 arcs from W, 3 between the v_i and the z_i and possibly an arc between y and e), and $|H| = |M| + 6$

$(V(H) = W \cup V(M) \backslash \{x\})$. Thus

$$\rho > \mathrm{ad}(H) \geq 2\left(\frac{||M|| + 10}{|M| + 6}\right)$$

If $\mathrm{mad}(M) \geq \rho$ or M is forbidden, this gives a contradiction, as in the previous case.

As in the previous case there is a P_{11}-coloring of G', and in this case z_1 and z_3 take different colors. The restriction of this coloring to $G - W$, then extends readily back to G. Since y has at most one neighbor in G', at least five colors are available for y of which we choose one arbitrarily. Then there is a set of at least two colors available at each of v_1, v_2 and v_3, and by Lemma 15, these are not all the same; so one can choose distinct colors at these vertices. Thus G is P_{11}-colorable, and we have a contradiction.

Case 3 z_1, z_2 and z_3 are distinct For $i = 1, 2, 3$, define G_i by adjoining to $G - W$ a new directed 2-path between the vertices of $\{z_1, z_2, z_3\} \backslash \{z_i\}$, with the midpoint a new vertex x_i. Suppose that H_i is a subgraph of G_i which contains x_i, and both its neighbors. Given such graphs H_i and H_j, we define an induced subgraph of G, H_{ij} by

$$H_{ij} = G[V(H_i) \cup V(H_j) \cup W \backslash \{x_i, x_j\}] \tag{27}$$

We have

$$||H_{ij}|| \geq ||H_i \cup H_j|| + 8 \geq ||H_i|| + ||H_j|| - ||H_i \cap H_j|| + 8$$

[$A(H_{ij})$ includes all the arcs of $H_i \cup H_j$ except for the up to 4 arcs incident at x_i and x_j. Since z_1, z_2 and z_3 are all vertices of H, $A(H)$ also contains the 3 arcs between v_i and z_i, as well as 9 arcs from W, and possibly the arc between y and e.]

$$|H_{ij}| = |H_i \cup H_j| + 5 = |H_i| + |H_j| - |H_i \cap H_j| + 5$$

Thus

$$\mathrm{ad}(H_{ij}) \geq 2\left(\frac{||H_i|| + ||H_j|| - ||H_i \cap H_j|| + 8}{|H_i| + |H_j| - |H_i \cap H_j| + 5}\right) \tag{28}$$

Claim: $\mathrm{mad}(G_i) \geq \rho$ *for at most one* i, and G_i *contains a forbidden subgraph for at most one* i

To prove the claim suppose that–say–G_1 and G_2 both contain a forbidden subgraph, or both have maximum average degree $\geq \rho$; then each G_i has a subgraph H_i such that either H_1 and H_2 are both forbidden or $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$ are both at least ρ. In the latter case we choose H_i to maximize $\mathrm{ad}(H_i)$.

Arguing as in the previous case, each H_i must contain the vertex x_i and both its neighbors. Define H_{12} as in (27). Since $H_1 \cap H_2$ is a subgraph of G, $\mathrm{ad}(H_1 \cap H_2) < \rho$. Now, if $\mathrm{ad}(H_1), \mathrm{ad}(H_2) \geq \rho$, (28) gives $\mathrm{ad}(H_{12}) \geq \rho$, a contradiction, since H_{12} is a subgraph of G.

If H_1 and H_2 are both forbidden, then since $||H_i|| = 14$ and $|H_i| = 9$, (28) now gives

$$\mathrm{mad}(H_{12}) \geq 2\left(\frac{36 - ||H_1 \cap H_2||}{23 - |H_1 \cap H_2|}\right)$$

We have $x_1 \in V(H_1)\backslash V(H_2)$, $x_2 \in V(H_2)\backslash V(H_1)$ and $z_3 \in V(H_1) \cap V(H_2)$. Thus $1 \le |H_1 \cap H_2| \le 7$, and a simple calculation using Lemma 24 then shows that $\text{mad}(H_{12}) \ge 13/4 > 16/5 > \rho$, again a contradiction. This proves the claim.

The claim shows that, for some i, $\text{mad}(G_i) < \rho$ and G_i has no forbidden subgraph. Thus again, G_i is P_{11}-colorable. This restricts to a coloring of $G - W$ for which two of the z_i take different colors. We extend the coloring to G as in the previous case, and get a contradiction. □

Proof of Theorem 2 Suppose for a contradiction that there is an oriented graph G, with $\text{mad}(G) < 22/7$, which contains no forbidden subgraph, and is not P_{11}-colorable, and that G is chosen to be minimum order with these properties.

Clearly G is P_{11}-critical (since the property of containing no forbidden subgraph and the maximum average degree bound are both inherited by subgraphs), so that by Lemmas 3 and 23, G contains no cycle of $^{\le}3$-vertices and none of the reducible configurations (10)–(26). By Lemma 25 (with $\rho = 22/7$), G also contains no bad configuration.

Define a charge on each vertex v of G by $c(v) = 7d(v) - 22$, and move the charge according to the following rules

1. Each vertex sends a charge of 4 to each of its 2-neighbors.
2. Each $^{\ge}3$-vertex sends a charge of 1 to each of its weak neighbors.
3. Each non-weak $^{\ge}4$-vertex sends a charge of 1 to each of its 3-neighbors.

Let $c^*(v)$ denote the new charge on vertex v. By (10), there are no $^{\le}1$-vertices in G. By (11), $c^*(v) = 0$ when $d(v) = 2$. By Lemma 3 and (15) the 3-vertices of G induce a forest whose components are paths. The total initial charge on the vertices of a such a path P of n 3-vertices is $-n$. By (12) and (18), all of the $n + 2$ neighbors of the vertices in P are are $^{\ge}4$-vertices, of which at most one is weak. Thus the sum of $c^*(v)$ over the vertices of P is at least $-n + (n + 1) - 1 = 0$. Now suppose that $d(v) = 4$, then $c(v) = 6$, and v has at most two 2-neighbors by (20). In view of (17), if v is weak, then it loses 8 by the first rule, then gains 2 by the second; if v has a single 2-neighbor, then it loses at most 6 by (21) and the fact that G contains no bad configuration; if v has no 2-neighbor, then clearly it loses at most 4. Thus $c^*(v) \ge 0$ for every 4-vertex. Similar calculations, using (22)–(26), show that $c^*(v) \ge 0$ for $5 \le d(v) \le 7$. For $d(v) \ge 8$, $c^*(v) \ge 7d(v) - 22 - 4d(v) = 3d(v) - 22 \ge 0$. Thus the sum of $c^*(v)$ over all vertices of G is non-negative, which contradicts $\text{mad}(G) < 22/7$. □

6 Concluding Remarks

The bound of 22/7 in Theorem 2 can probably be improved. Some preliminary calculations suggest that a bound of at least 16/5 is possible, using much the same kind of proof, but with a larger set of reducible configurations and more elaborate discharging rules; there are however a huge number of cases to consider. Presumably, further increases still are possible if we exclude more graphs. After 28/9, the smallest value of $\text{mad}(G)$ taken by a P_{11}-critical graph appears to be 13/4, attained by the graphs obtained from a forbidden graph by replacing the path a_1va_2 (in the notation of Fig. 1) by a single arc. Let $\mathcal{S}_\rho$ denote the set of P_{11}-critical graphs (modulo isomorphism) with maximum average degree less than ρ. Then an oriented G with $\text{mad}(G) < \rho$ is P_{11}-colorable if and only if it has no subgraph in $\mathcal{S}_\rho$. We have proved that $\mathcal{S}_\rho$ is the set of forbidden graphs for $28/9 < \rho \leq 22/7$, and $\mathcal{S}_\rho = \emptyset$ for $\rho \leq 28/9$.

Problem 26 *What is the smallest value of ρ for which $\mathcal{S}_\rho$ is infinite?*

We prove below that this number is at most $65/19 = 3.42...$ by constructing infinitely many P_{11}-critical graphs G with $\text{mad}(G) < 65/19$. One way to construct such graphs is by subdivision.

Let G be an unoriented graph, and construct the *subdivision* $\tilde{G}$ of G, by replacing each edge of G with a path of length 2.

Lemma 27 *If G is a graph, then* $\text{mad}(\tilde{G}) = \frac{4\text{mad}(G)}{2+\text{mad}(G)}$

Proof First, if H is any graph, we compute

$$\text{ad}(\tilde{H}) = \frac{2||\tilde{H}||}{|\tilde{H}|} = \frac{4||H||}{|H| + ||H||} = \frac{4\text{ad}(H)}{2 + \text{ad}(H)} \tag{29}$$

Clearly we may assume that G is connected. If G is a tree, then so is $\tilde{G}$, and $\text{mad}(G) = \text{ad}(G)$, $\text{mad}(\tilde{G}) = \text{ad}(\tilde{G})$, so the lemma holds in this case. Otherwise $\tilde{G}$ contains a cycle. Let M be a subgraph of $\tilde{G}$ with $\text{ad}(M)$ maximal, and subject to this, $|M|$ minimal; $\text{ad}(M) \geq 2$, and M has no isolated or pendant vertices, since the removal of such a vertex would not decrease the average degree. Thus $M = \tilde{H}$ for some subgraph H of G, and the lemma follows from (29). □

We can make $\tilde{G}$ into an oriented graph by making each of the paths introduced in the construction of $\tilde{G}$ directed. It is clear that $\tilde{G}$ oriented in this way is P_{11}-colorable if and only if $\chi(G) \leq 11$. There exists a sequence H_n of 12-critical graphs with $\text{mad}(H_n) < 65/19$. To construct H_n take n disjoint copies $G_1, G_2, \ldots G_n$ of K_{11}. For each $1 \leq i \leq n$, let $v_i, w_i \in G_i$ $(1 \leq i \leq n)$, and add an edge between w_i and v_{i+1} $(1 \leq i \leq n-1)$. Finally add a new vertex v joined to all existing vertices except for v_i with $2 \leq i \leq n$ and w_i with $1 \leq i \leq n-1$. If we have an 11-coloring of H_n, then induction shows that $v, w_1, w_2, \ldots w_{n-1}$ must all take the same color, but then this color is not available to color G_n, a contradiction. It is routine to check that every

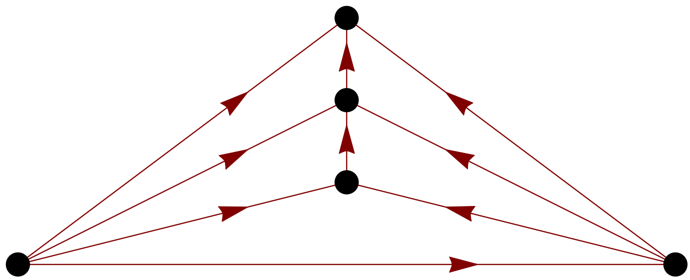

Fig. 3 A non-P_{11}-colorable planar graph

proper subgraph of H_n is 11-colorable. Hence each H_n is 12-critical, and so the $\tilde{H}_n$ are P_{11}-critical. We have $|H_n| = 11n+1$ and $||H_n|| = 55n+(n-1)+(9n+2) = 65n+1$, whence $\text{mad}(H_n) = \frac{2(65n+1)}{11n+1}$. This is increasing in n and has limiting value $130/11$, whence $\text{mad}(\tilde{H}_n) \to 65/19$.

Conjecture 28 *Every planar graph of girth at least 5 is P_{11}-colorable.*

Since $\text{mad}(G) < 10/3$, for every planar graph of girth at least 5 (see e.g. [4]), this result would follow if $\mathcal{S}_{10/3}$ contained no planar graphs. It is even possible that Conjecture 28 might hold with girth 4. On the other hand, it is known [3], that there are planar graphs with oriented chromatic number at least 18, so in particular there exist planar graphs which are not P_{11}-colorable. Figure 3 gives a simple example of such a graph (we leave the details as an exercise).

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