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Hamiltonian Cycles in Critical Graphs with Large Maximum Degree

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Abstract It is shown by Luo and Zhao (J Graph Theory 73:469–482, 2013) that an overfull Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. If Hilton's overfull subgraph conjecture (Chetwynd and Hilton 100:303–317, 1986) was proved to be true, then the above result could be said that any Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. Since the overfull subgraph conjecture is still open, the natural question is how to directly prove a Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. Luo and Zhao (J Graph Theory 73:469–482, 2013) show that a Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{6n}{7}$ is Hamiltonian. In this paper, by developing new lemmas for critical graphs, we show that if G is a Δ-critical graph with n vertices satisfying $\Delta \geq \frac{4n}{5}$, then G is Hamiltonian.

Keywords Edge colorings · Critical graphs · Hamiltonian cycles

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1 Introduction

Throughout this paper, let $G = (V(G), E(G))$ be a simple graph with n vertices, m edges, maximum degree $\Delta(G)$ (or Δ) and minimum degree $\delta(G)$ (or δ). A *k-vertex*, *($\geq k$)-vertex* or *($\leq k$)-vertex* is a vertex of degree k, at least k or at most k, respectively. For each vertex $x \in V(G)$, let $N_G(x)$ (or simply $N(x)$ if no confusion from the context) be the set of all neighbors of x and $d_G(x)$ (or $d(x)$) be the degree of x. For a subset $A \subseteq V(G)$, we use $G[A]$ to denote the subgraph of G induced by A. We call the length of a longest cycle in a graph G the *circumference* of G and denote it by $c(G)$. If a graph G has a cycle that includes every vertex in $V(G)$, then such a cycle is called a *Hamiltonian cycle*, and G is called a *Hamiltonian graph*.

An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *edge k-colorable*, if there is an edge coloring of the graph with colors from $\{1, \ldots, k\}$. A finite simple graph G of maximum degree Δ is *class one* if it is edge Δ-colorable. Otherwise, Vizing's Theorem [20] guarantees that it is edge $(\Delta + 1)$-colorable, in which case, it is said to be *class two*. An *edge chromatic critical graph* (or *critical graph* for short) is a connected graph G such that G is class two and $G - e$ is class one for each edge e of G. An edge chromatic critical graph of maximum degree Δ is called a *Δ-critical graph*. A graph G with maximum degree Δ is *overfull* if $m > \lfloor \frac{n}{2} \rfloor \Delta$.

Finding long cycles in graphs and finding nontrivial upper bounds and lower bounds for the circumference of graphs are important problems in graph theory and there are many papers dealing with these problems. But there are very few papers and results about long cycles of edge chromatic critical graphs and most of these existing results about long cycles of edge chromatic critical graphs were obtained more than thirty years ago. In 1965, Vizing started looking for lower bounds for the circumference of edge chromatic critical graphs and obtained the following result.

Theorem 1.1 (Vizing [20]) *Every Δ-critical graph has a cycle of length at least $\Delta+1$.*

The above result shows that $\Delta + 1$ is a lower bound for the circumference of any Δ-critical graph. Noticing that the above Vizing's result does not involve the order of the graph, in 1977, Fiorini and Wilson looked for the bounds for the circumference of edge chromatic critical graphs and they obtained the following result that gives a better lower bound than Vizing's result if the orders of the edge chromatic critical graphs are really big.

Theorem 1.2 (Fiorini and Wilson [13]) *If G is a Δ-critical graph of order n with minimum degree δ, then G contains a cycle whose length is at least $2 \times \frac{\log(n-1)(\Delta-2)-\log\delta}{\log(\Delta-1)}$.*

Furthermore, in [12], Fiorini studied upper bounds for the circumference of edge chromatic critical graphs and described an explicit construction which yields an infinite family of edge chromatic critical graphs whose circumference can be estimated. In fact, in the same paper, Fiorini obtained the following result about upper bounds for the circumference of Δ-critical graphs.

Theorem 1.3 (Fiorini [12]) *There exists an infinite family $\{G_k\}$ of Δ-critical graphs satisfying $c_k \leq 4(\Delta+1)\times 2^{q(k)}$, where c_k is the circumference of G_k, $n(k) = |V(G_k)|$, and $q(k) = \frac{\log n(k)-\log(\Delta-1)}{\log(2\Delta-2)}$.*

Clearly, the above Fiorini's result implies that there are many critical graphs that are not Hamiltonian. Thus, a natural question is that under what conditions, critical graphs will be Hamiltonian? In 2011, Luo and Zhao [17] gave an answer for the above question by providing a sufficient condition for edge chromatic critical graphs to be Hamiltonian.

Theorem 1.4 (Luo and Zhao [17]) *Let G be a Δ-critical graph of order n with $\Delta \geq \frac{6n}{7}$. Then G is Hamiltonian.*

In the same paper, they also proved that an overfull Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. If Hilton's Overfull Subgraph Conjecture [8] was proved to be true, then the above result could be said that any Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. Since the Overfull Subgraph Conjecture is still open, the natural question is how to directly prove a Δ-critical graph with n vertices that satisfies $\Delta \geq \frac{n}{2}$ is Hamiltonian. In this paper, we consider the above question and by applying our newly developed lemmas about critical graphs, we show that if G is a Δ-critical graph with n vertices satisfying $\Delta \geq \frac{4n}{5}$, then G is Hamiltonian.

2 Lemmas

The following simple result about Δ-critical graphs is from [21].

Lemma 2.1 *Let G be a Δ-critical graphs. Then G is 2-connected and the degree sum of each pair of adjacent vertices is at least $\Delta + 2$.*

Lemma 2.2 (Vizing's Adjacency Lemma or VAL [20]) *Let x be a vertex of a Δ-critical graph and y be a vertex adjacent to x. Then x is adjacent to at least* $\max\{\Delta - d(y) + 1, 2\}$ *Δ-vertices.*

Lemma 2.3 (Luo and Zhao [17]) *Every edge chromatic critical graph with at most 10 vertices is Hamiltonian.*

Remark Even though all critical graphs of order at most ten are Hamiltonian, not all critical graphs of order at least eleven are Hamiltonian. The graph with 11 vertices and a degree sequence 23^{10} given by Fiorini in [11] which is overfull but is not Hamiltonian.

The next lemma summarizes the results in [1,2,5–7].

Lemma 2.4 *(i) ([1,6,7]) There are no critical graphs of even order at most 14;*
(ii) ([5]) There are only two critical graphs of order 11 with size at most 5Δ, both of which are 3-critical;
(iii) ([2])There are only three critical graphs of order 13 with size at most 6Δ, which are 3-critical.

Lemma 2.5 (Sanders and Zhao [18] and Steibitz, Scheide, Toft, and Favrholdt [19]) *Let G be a Δ-critical graph. Let x be a j-vertex that is adjacent to a k-vertex y. If $j < \Delta$, then x is adjacent to at least $\Delta - k + 1$ vertices z satisfying the following: $z \neq y$; z is adjacent to at least $2\Delta - j - k$ vertices different from x of degree at least $2\Delta - j - k + 2$; and if z is not adjacent to y, then z is adjacent to at least $2\Delta - j - k + 1$ vertices different from x of degree at least $2\Delta - j - k + 2$.*

Lemma 2.5 was first proved by Sanders and Zhao [18] with the requirement $d(y) < \Delta$. Steibitz, Scheide, Toft, and Favrholdt [19] gave a proof without the condition $d(y) < \Delta$.

Let G be a graph, $v \in V(G)$ and f be an edge coloring of G. We define $c_f(v) = \{f(uv)|u \in N(v)\}$ and $A \oplus B = (A - B) \cup (B - A)$, where A, B are sets. For convenience, we use *adjacent* for the conventional terms adjacent and incident. Let the edges of a graph be colored with colors from $\{1, \ldots, k\}$ and let $u \in V(G)$. If an edge incident with u is colored i, then we say that u *sees* i. Otherwise, we say that u *misses* i. For $i, j \in \{1, \ldots, k\}$, an i–j *edge chain* is a chain of edges colored alternately i and j. Clearly, each maximal i–j edge chain is either a path or a cycle of even length and any two maximal i–j edge chains are either disjoint or identical. We denote a maximal i–j edge chain containing u by $L_{i,j}(u)$. If u does not see i or j, then $L_{i,j}(u)$ is a path where u is one of its endvertices.

The proofs of the following two lemmas can be found in [16].

Lemma 2.6 (Luo and Zhao [16]) *Let G be a Δ-critical graph and $xy \in E(G)$. Let $u \neq x$ be a neighbor of y and $v \notin \{x, y\}$ be a neighbor of u. Let f be an edge Δ-coloring of $G - xy$ with $f(uy) = k$ and $f(uv) = l$. If $k \notin c_f(x)$, then we have the following facts:*

(1) u sees every color in $c_f(x) \oplus c_f(y)$;
(2) If $l \in c_f(x) \oplus c_f(y)$, then
(2-a) v sees every color in $c_f(x) \oplus c_f(y)$, if $|c_f(x) - c_f(y)| \geq 2$;
(2-b) Otherwise, v misses at most one color in $c_f(y) \oplus c_f(x)$;
(3) Let $z \neq x \in N(y)$ and assume that z misses a color $q \in c_f(y) - c_f(x)$ and yz is colored $p \in c_f(y) \cap c_f(x)$. If $d(z) \leq 2\Delta - d(x) - d(y) + 1$, then $L_{p,q}(z)$ must end at x.

Lemma 2.7 (Luo and Zhao [16]) *Let G be a Δ-critical graph. Let $xy \in E(G)$ with $4 \leq d = d(x) \leq \Delta - 2$ and $d(y) = \Delta$. If for an integer $k \geq 0$, y is adjacent to $d - 2 - k$ $(\leq \Delta - d + 1)$-vertices $u \neq x$, where $d - 2 - k \geq 1$, then there exist at least $\Delta - d + 1$ neighbors $y' \neq x$ of y such that y' is adjacent to at least $\Delta - k - 2$ vertices distinct from y of degree at least $\Delta - k - 1$.*

Let G be a Δ-critical graph and $xy \in E$ with $4 \leq d(x) = d \leq \Delta - 2$ and $d(y) = \Delta$. Since G is Δ-critical, $G - xy$ is Δ-colorable and let f be a Δ-edge coloring of $G - xy$. Assume that $N(x) = \{x_1, \ldots, x_{d-1}, x_d\}$, $N(y) = \{y_1, y_2, \ldots, y_\Delta\}$, where $x_d = y$, $y_1 = x$, and $f(xx_i) = i$ and $f(yy_j) = j$ for $i \in \{1, \ldots, d - 1\}$ and $j \in \{2, \ldots, \Delta\}$. Furthermore, we assume that y is adjacent to exactly $d - k - 2 > 0$ vertices distinct from x with degree at most $\Delta - d + 1$. By Lemma 2.6, without loss of generality, we can assume that $y_2, \ldots, y_{d-k-1}$ are these $d - k - 2$ vertices distinct from x with degree at most $\Delta - d + 1$ and $d(y_j) \geq \Delta - d + 2$ for each $j \geq d - k$. Let $Y_1 = \{y_2, \ldots, y_{d-k-1}\}$, $Y_2 = \{y_{d-k}, \ldots, y_{d-1}\}$ and $Y_3 = \{y_d, \ldots, y_\Delta\}$. Then by Claims 7 and 8 of Lemma 2.12 in [16], we have the following lemma.

Lemma 2.8 *Let $u \in Y_3$ and $v \in N(u) - \{x, y\}$. If $f(uv) \in \{1, 2, \ldots, d - k - 1, d, \ldots, \Delta\}$, then $d(v) \geq \Delta - k - 1 \geq \Delta - d + 2$.*

The proof of the following lemma can be found in [17].

Lemma 2.9 (Luo and Zhao [17]) *Let G be a Δ-critical graph and x be a d-vertex. Then there does not exist a vertex subset U of $d-1$ vertices such that (1) $x \notin U$; (2) the degree of each vertex in U is at most $\frac{\Delta-d+1}{2}$; and (3) there are $d-1$ distinct neighbors of x, each of which is adjacent to a distinct vertex in U.*

Theorem 2.10 (Bondy [3]) *If $P = v_1 \ldots v_l$ is a longest path in a 2-connected graph G, then $c(G) \geq \min\{n(G), d(v_1) + d(v_l)\}$.*

Let G be a graph. The *closure* of G, denoted by $C(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$, until no such pair remains. The following theorem is due to Bondy and Chvátal [4].

Theorem 2.11 (Bondy and Chvátal [4]) *A graph G is Hamiltonian if and only if its closure is Hamiltonian.*

Theorem 2.12 (Chvátal [9]) *Let G be a graph with $n \geq 3$ vertices and vertex degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. If for each $i < \frac{n}{2}$, either $d_i > i$ or $d_{n-i} \geq n-i$, then G is Hamiltonian.*

Lemma 2.13 *Let G be a Δ-critical graph and A be an independent set. If A does not contain Δ-vertices, then $|N_G(A)| > |A|$.*

Proof Denote $B = N_G(A)$. Let $y \in B$ and $x \in A$ be a neighbor of y. By VAL, y is adjacent to at most $d(x) - 1$ vertices of degree less than Δ. Since A does not contain Δ-vertices, y is adjacent to at most $d(x) - 1$ vertices in A.

Let H be the bipartite graph induced by the edges with one endvertex in A and the other in B. Then for each $y \in B$, we have that $d_H(y) \leq d_H(x) - 1$ for any neighbor x of y in H. For each $y \in B$, let $\delta(y) = \min\{d_H(x) : x \in N_H(y)\}$. Then $d_H(y) \leq \delta(y) - 1$.

For each edge $yx \in E(H)$ with $x \in A$ and $y \in B$, we define $M(xy) = \frac{1}{d_H(x)}$. Thus we have

$$\sum_{xy \in E(H)} \frac{1}{M(xy)} = \sum_{x \in A} \sum_{y \in N_H(x)} \frac{1}{M(xy)} = \sum_{x \in A} \sum_{y \in N_H(x)} \frac{1}{d_H(x)} = \sum_{x \in A} 1 = |A|.$$

On the other hand, we also have

$$\begin{aligned} \sum_{xy \in E(H)} \frac{1}{M(xy)} &= \sum_{y \in B} \sum_{x \in N_H(y)} \frac{1}{M(xy)} = \sum_{y \in B} \sum_{x \in N_H(y)} \frac{1}{d_H(x)} \leq \sum_{y \in B} \sum_{x \in N_H(y)} \frac{1}{\delta(y)} \\ &= \sum_{y \in B} \frac{d_H(y)}{\delta(y)} \leq \sum_{y \in B} \frac{\delta(y) - 1}{\delta(y)} < \sum_{y \in B} 1 = |B|. \end{aligned}$$

Therefore $|A| < |B| = |N_G(A)|$. □

Lemma 2.14 *Let $G = (V, E)$ be a Δ-critical graph with n vertices such that $\Delta \geq \frac{4n}{5}$. If $\delta \geq \frac{\Delta}{4}$ and G is non-Hamiltonian, then $\delta \geq \frac{3\Delta}{8} + 2$.*

Proof Suppose to the contrary $\delta < \frac{3\Delta}{8} + 2$. Since $\delta \geq \frac{\Delta}{4}$, $\Delta + \delta \geq \frac{5\Delta}{4} \geq n$. Thus in the closure of G, $C(G)$, each vertex u is adjacent to every Δ-vertex in G. By VAL, G has at least $\Delta - \delta + 2$ Δ-vertices. Therefore,

$$d_{C(G)}(u) \geq \Delta - \delta + 2 \geq \frac{5\Delta}{8} \geq \frac{n}{2}.$$

By Dirac' s Theorem [10], $C(G)$ is Hamiltonian, so is G by Theorem 2.11. This contradicts the hypothesis that G is non-Hamiltonian and thus we have proved the above lemma. □

Lemma 2.15 *Let $G = (V, E)$ be a Δ-critical graph with n vertices such that $\Delta \geq \frac{4n}{5}$ and let $A = \{u|d(u) \leq \frac{\Delta}{2}\}$ satisfying $|A| \geq \max\{\delta, \frac{\Delta}{3} + 1\}$. If $\delta \geq \frac{\Delta}{6}$ and G is non-Hamiltonian, then $\delta \geq \frac{\Delta}{2}$.*

Proof Suppose to the contrary $\delta < \frac{\Delta}{2}$. Since $|A| \geq \max\{\delta, \frac{\Delta}{3} + 1\}$, by Lemma 2.14, we have the following claim.

Claim 2.1 *Either $\frac{\Delta}{6} \leq \delta < \frac{\Delta}{4}$ and $|A| \geq \frac{\Delta}{3} + 1$ or $|A| \geq \delta \geq \frac{3\Delta}{8} + 2$.*

Since G is a critical graph, by Lemma 2.1, A is an independent set. Let x be a δ-vertex that is adjacent to a Δ-vertex y. Consider $G - xy$. Since G is critical, $G - xy$ is Δ-colorable. Let f be a Δ-coloring of $G - xy$. Assume that $N(x) = \{x_1, \ldots, x_{\delta-1}, x_\delta\}$, $N(y) = \{y_1, y_2, \ldots, y_\Delta\}$, where $x_\delta = y$, $y_1 = x$, and $f(xx_i) = i$ and $f(yy_j) = j$ for $i \in \{1, \ldots, \delta - 1\}$ and $j \in \{2, \ldots, \Delta\}$. Furthermore, we assume that y is adjacent to exactly $\delta - k - 2$ vertices distinct from x with degree at most $\Delta - \delta + 1$. By Lemma 2.6, without loss of generality, we can assume that $y_2, \ldots, y_{\delta-k-1}$ are these $\delta - k - 2$ vertices distinct from x with degree at most $\Delta - \delta + 1$ and $d(y_j) \geq \Delta - \delta + 2$ for each $j \geq \delta - k$. Let $Y_1 = \{y_2, \ldots, y_{\delta-k-1}\}$, $Y_2 = \{y_{\delta-k}, \ldots, y_{\delta-1}\}$ and $Y_3 = \{y_\delta, \ldots, y_\Delta\}$.

Since $d(u) \leq \frac{\Delta}{2}$ for each $u \in A$ and $d(v) \geq \Delta - \delta + 2 > \frac{\Delta}{2} + 2$ for each $v \in Y_3 \cup Y_2$, we have $A \cap (Y_2 \cup Y_3) = \emptyset$ and thus $A \cap (Y_1 \cup Y_2 \cup Y_3) = A \cap Y_1$.

Since $n \geq |A \cup Y_1 \cup Y_2 \cup Y_3|$, we have the following inequality.

$$|A \cap Y_1| \geq |A| + |Y_1| + |Y_2| + |Y_3| - n \geq |A| + \Delta - 1 - \frac{5\Delta}{4} = |A| - \frac{\Delta}{4} - 1. \quad (1)$$

Since $|A \cap Y_1| + |Y_2| + |Y_3| \leq \Delta - 1$, by Eq. (1), we have the following inequality.

$$|Y_2| \leq \Delta - 1 - |A \cap Y_1| - |Y_3| \leq \Delta - 1 - \left(|A| - \frac{\Delta}{4} - 1\right) - (\Delta - \delta + 1) = \delta + \frac{\Delta}{4} - |A| - 1. \quad (2)$$

Let $B = V - (Y_3 \cup A)$. Since $|A| \geq \delta$, we have

$$|B| = n - |Y_3| - |A| \leq \frac{5\Delta}{4} - (\Delta - \delta + 1) - |A| = \frac{\Delta}{4} + \delta - |A| - 1 < \frac{\Delta}{4}. \quad (3)$$

Claim 2.2 *For each* $x \in Y_1 \cap A$, $N(x) \subseteq Y_3 \cup B$ *and*

$$|N(x) \cap Y_3| \geq |A| - \frac{\Delta}{4} + 1 > \frac{|Y_2|}{2}.$$

Proof Since A is an independent set, x is not adjacent to any vertex in A. Thus $N(x) \subseteq Y_3 \cup B$. Therefore by Eq. (3), we have

$$|N(x) \cap Y_3| \geq d(x) - |B| \geq \delta - \left(\frac{\Delta}{4} + \delta - |A| - 1\right) = |A| - \frac{\Delta}{4} + 1.$$

To prove the right inequality, by Eq. (2), it is sufficient to show that $3|A| > \delta + \frac{3\Delta}{4} - 3$. By Claim 2.1 we only need to consider the following two cases.

Case 1: $\delta < \frac{\Delta}{4}$ and $|A| \geq \frac{\Delta}{3} + 1$. Thus

$$3|A| = \frac{\Delta}{4} + 3|A| - \frac{\Delta}{4} > \delta + \Delta + 3 - \frac{\Delta}{4} > \delta + \frac{3\Delta}{4} - 3.$$

Case 2: $|A| \geq \delta \geq \frac{3\Delta}{8} + 2$. Hence

$$3|A| \geq \delta + 2|A| \geq \delta + \frac{3\Delta}{4} + 4 > \delta + \frac{3\Delta}{4} - 3.$$

□

Claim 2.3 *For an edge* uv*, where* $u \in Y_3$ *and* $v \in Y_1$*, we have* $f(vu) \in \{\delta - k, \ldots, \delta - 1\}$.

Proof This is a direct consequence of Lemma 2.8 since $d(v) \leq \Delta - \delta + 1$. □

Claim 2.4 $|A \cap Y_1| \geq 2$.

Proof Suppose to the contrary $|A \cap Y_1| \leq 1$. Then by Eq. (1), we have $|A| \leq \frac{\Delta}{4} + 2 < \frac{3\Delta}{8} + 2$. By Claim 2.1, we have $\delta < \frac{\Delta}{4}$ and $|A| \geq \frac{\Delta}{3} + 1$. Thus $\frac{\Delta}{3} + 1 \leq \frac{\Delta}{4} + 2$. This implies $\Delta \leq 12$ and $n \leq 15$. By Lemma 2.4, we have $n = 15$ and $\Delta = 12$. Thus $|A| \geq 5$ and $\delta = 2$ since $\delta < \frac{\Delta}{4} = 3$. By VAL, G has at least $\Delta - \delta + 2 = 12$ vertices of degree Δ. This implies that $n \geq 12 + 5 = 17 > 15$, a contradiction. □

The final step. By Claim 2.4, let y_p, y_q be two distinct vertices in $A \cap Y_1$. Then by Claim 2.2, $|N(y_p) \cap Y_3| + |N(y_q) \cap Y_3)| > |Y_2| = |\{\delta - k, \ldots, \delta - 1\}|$. By Claim 2.3, there exist two vertices $u \in N(y_p) \cap Y_3$ and $v \in N(y_q) \cap Y_3$ such that $f(y_p u) = f(y_q v) = i \in \{\delta - k, \ldots, \delta - 1\}$.

Since by Eq. (3), $|N(y_p) \cap B| + |N(y_q) \cap B| \leq 2|B| < \frac{\Delta}{2} \leq \Delta - \delta + 1 = |\{\delta, \ldots, \Delta\}|$, y_p and y_q both miss a color $j \in \{\delta, \ldots, \Delta\}$.

Consider $L_{i,j}(y_p)$. Since $d(y_p) \leq \Delta - d(x) + 1 = \Delta - \delta + 1$, by Lemma 2.6-(3), $L_{i,j}(y_p)$ ends at x. Similarly, one can show that $L_{i,j}(y_q)$ ends at x. But on the other hand, $L_{i,j}(y_p) \cap L_{i,j}(y_q) = \emptyset$, a contradiction. This completes the proof of the lemma. □

3 Hamiltonicity of Critical Graphs with Large Maximum Degree

Theorem 3.1 *Let G be a Δ-critical graph of order n with $\Delta \geq \frac{4n}{5}$. Then G is Hamiltonian.*

Proof By Lemma 2.3, we can assume that $n \geq 11$ and thus $\Delta \geq 9$. Suppose that G has no Hamiltonian cycles. Let $P = x_1 \dots x_t$ be a longest path such that $d(x_1) \leq d(x_t)$ and $d(x_1) + d(x_t)$ is as large as possible. Let $d = d(x_1)$.

Then by Theorem 2.10, $d + d(x_t) \leq n - 1 \leq \frac{5\Delta}{4} - 1$, so that

$$d \leq d(x_t) \leq \frac{5\Delta}{4} - d - 1, \quad \text{and} \quad d < \frac{5\Delta}{8}.$$

Since P is a longest path and G has no Hamiltonian cycles, x_1 and x_t are not adjacent.

Let $I_1 = \{i : x_i \in N(x_1)\}$. Then $1, t \notin I_1$ since x_1 is not adjacent to itself and x_1 is not adjacent to x_t. By the assumption about P, $d(x_{i-1}) \leq d$ for $i \in I_1$. Hence there are at least d vertices of degree at most d.

Claim 3.1 $d \geq \delta \geq \frac{\Delta}{2}$.

Proof Suppose that $d < \frac{\Delta}{2}$. First we prove that $d > \frac{\Delta}{3}$. Suppose that $d \leq \frac{\Delta}{3}$. Then $d \leq \frac{\Delta-d+1}{2}$. Let $U = \{x_{i-1} | i \geq 3 \text{ and } i \in I_1\}$. Then U has the following properties: (i) $|U| = d - 1$; (ii) $x_1 \notin U$; (iii) $d(x_{i-1}) \leq d \leq \frac{\Delta-d+1}{2}$ for each $x_{i-1} \in U$; and (iv) for $i \geq 3$ and $i \in I_1$, each x_i of x_1 is adjacent to $x_{i-1} \in U$. The existence of such a set U contradicts Lemma 2.9. Hence $d > \frac{\Delta}{3}$.

Second we prove $\delta \geq \frac{\Delta}{6}$. Suppose that $\delta < \frac{\Delta}{6}$. Then $\Delta - \delta + 1 > \frac{5\Delta}{6} + 1 > \max\{d, \Delta - d + 1\}$ since $\frac{\Delta}{3} < d < \frac{\Delta}{2}$. Let x be a vertex with $d(x) = \delta$ and y be a Δ-neighbor of x. We further assume that y has exactly $\delta - k - 2$ neighbors of degree at most $\Delta - \delta + 1$ distinct from x, where $k \geq 0$. Then by Lemma 2.7, y has a neighbor y' such that (1) $d(y') \geq \Delta - k - 1$ and (2) y' is adjacent to at least $\Delta - k - 2$ vertices distinct from y of degree at least $\Delta - k - 1$. Thus including y and y', G has at least $\Delta - k$ vertices of degree at least $\Delta - k - 1$. Note that

$$\Delta - k - 1 \geq \Delta - \delta - 1 > \max\{d, \Delta - d + 1\}.$$

Since G has at least d vertices of degree at most d, we have

$$d + \Delta - k \leq n \leq \frac{5\Delta}{4}$$

On the other hand, y is adjacent to $\Delta - \delta + 1 + k$ vertices of degree greater than $\Delta - \delta + 1$. Therefore we have

$$d + \Delta - \delta + 1 + k \leq n \leq \frac{5\Delta}{4}.$$

Adding the above two equations together, we have

$$2\Delta + 2d - \delta + 1 \leq \frac{5\Delta}{2}.$$

Since $\delta < \frac{\Delta}{6}$ and $d > \frac{\Delta}{3}$, we have

$$2\Delta + 2d - \delta + 1 > 2\Delta + \frac{2\Delta}{3} - \frac{\Delta}{6} + 1 > \frac{5\Delta}{2},$$

a contradiction. This contradiction shows that the minimum degree $\delta \geq \frac{\Delta}{6}$.

Let $W = \{u | d(u) \leq \frac{\Delta}{2}\}$. Since $\frac{\Delta}{3} < d < \frac{\Delta}{2}$ and $|W| \geq d \geq \delta$, one can conclude that $|W| \geq \max\{\delta, \frac{\Delta}{3} + 1\}$. Hence by Lemma 2.15, we have that $d \geq \delta \geq \frac{\Delta}{2}$. □

Claim 3.2 *For each edge $xy \in E(G)$, $d(x) + d(y) \geq \frac{3\Delta}{4} + d + 1 > \frac{5\Delta}{4} \geq n$.*

Proof Otherwise, suppose that $d(x) + d(y) < \frac{3\Delta}{4} + d + 1$ for some edge xy. By Lemma 2.5, G has at least $2\Delta - d(x) - d(y) > 2\Delta - (\frac{3\Delta}{4} + d + 1) = \frac{5\Delta}{4} - d - 1$ vertices of degree at least $\frac{5\Delta}{4} - d - 1 + 2 = \frac{5\Delta}{4} - d + 1$. Since $d \leq d(x_t) \leq \frac{5\Delta}{4} - d - 1$, there are at least $d + 1$ vertices with degree at most $\frac{5\Delta}{4} - d - 1$. Hence we have that $n > \frac{5\Delta}{4} - d - 1 + d + 1 = \frac{5\Delta}{4} \geq n$, a contradiction. □

Claim 3.3 *For each vertex x, if $d(x) \geq \frac{5\Delta}{4} - d$, then $d(x) \geq \frac{3\Delta}{4}$.*

Proof Let x be a vertex with $d(x) \geq \frac{5\Delta}{4} - d$. Then $d(x) > d$ since $d < \frac{5\Delta}{4} - d - 1$. If x has a neighbor of degree at most d, then by Claim 3.2, $d(x) + d \geq \frac{3\Delta}{4} + d$ and we have $d(x) \geq \frac{3\Delta}{4}$. Now assume that x is not adjacent to any ($\leq d$)-vertex. Since there are at least d vertices of degree at most d, we have $n \geq d(x) + 1 + d \geq \frac{5\Delta}{4} - d + 1 + d = \frac{5\Delta}{4} + 1 > n$, a contradiction. Thus $d(x) \geq \frac{3\Delta}{4}$. □

Let $A = \{x | d(x) < \frac{5\Delta}{8}\}$, $B = \{x | \frac{5\Delta}{8} \leq d(x) < \frac{5\Delta}{4} - d\}$, and $C = \{x | d(x) \geq \frac{5\Delta}{4} - d\}$. Then by Claim 3.3, $C = \{x | d(x) \geq \frac{3\Delta}{4}\}$. By Claim 3.2, A is an independent set.

Claim 3.4 *$C(G)$ is Hamiltonian.*

Proof Let $H = C(G)$. List the vertices of H as $v_1, \ldots, v_n$ such that $d_1 \leq \cdots \leq d_n$ where $d_i = d_H(v_i)$. By Theorem 2.12, it is sufficient to show that for each $i < \frac{n}{2}$, either $d_i \geq i + 1$ or $d_{n-i} \geq n - i$. Suppose to the contrary. Let $i < \frac{n}{2}$ be the smallest integer such that $d_i \leq i$ and $d_{n-i} < n - i$. Since $\delta(H) \geq \delta(G) \geq \frac{\Delta}{2}$ by Claim 3.1 and $\Delta \geq 9$, we have $i \geq 5$. By the minimality of i, we have $d_i = i$. Let $A' = \{v_1, \ldots, v_i\}$. By the definition of set A, clearly $v_i \in A$ since $d_G(v_i) \leq d_i = d_H(v_i) = i < \frac{n}{2} \leq \frac{5\Delta}{8}$. Let k be the least subscript such that v_k is adjacent to v_i.

(I) By the definition of the closure and Claim 3.2, for any two vertices $u, v \in V(H)$, $uv \in E(H)$ if and only if $d_H(u) + d_H(v) \geq n$.

By (I) we have the following.

(II) $v_j \in N_H(v_i)$ for each $j \geq k$ and $N_H(v_l) \subseteq N_H(v_i)$ for each $l \leq i$ since $d_i \geq d_l$. Therefore $N_H(A') = N_H(v_i)$.

(III) A' is an independent set in H and so is in G.

Since $v_i \in A$ and A is an independent set in G that consists of all vertices with degree less than $\frac{5\Delta}{8}$, it is clear that A' is an independent set in both H and G.

Since $d_G(v_i) \leq d_H(v_i) = i < \frac{n}{2}$, A' does not contain any Δ-vertex in G. By Lemma 2.13, $|N_G(A')| > |A'|$. Therefore $|N_H(A')| \geq |N_G(A')| > |A'|$. By (II) we have $d_i = d_H(v_i) > i$, a contradiction to the choice of i.

This shows that the degree sequence of H satisfying the degree condition in Theorem 2.12 and thus $C(G) = H$ has a Hamiltonian cycle. □

By Claim 3.4 and Theorem 2.11, G is Hamiltonian, a contradiction to the assumption that G is not Hamiltonian. This contradiction completes the proof of Theorem 3.1. □

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