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ORIGINAL PAPER

Convex Pentagons for Edge-to-Edge Tiling, III

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Abstract In previous papers, the cases in which there is a possibility that a convex pentagon generates an edge-to-edge tiling were sorted, and the remaining 42 cases in which there is uncertainty about whether a convex pentagon can generate an edge-to-edge tiling were shown. In this paper, the latter 42 cases are investigated using a computer. As a result, we find that convex pentagons that can generate edge-to-edge monohedral tiling of the plane can be sorted into eight types.

Keywords Convex pentagon · Tiling · Tile · Monohedral tiling · Edge-to-edge tiling

1 Introduction

A tiling by congruent tiles is called *monohedral*, and a polygon that can generate a monohedral tiling is called a *polygonal tile*. At present, essentially 14 different types of convex pentagonal tiles are known, but whether the list of 14 types is perfect or not remains unknown [1–12].

The tiling by convex pentagonal tiles can be divided into two kinds: edge-to-edge tiling and non-edge-to-edge tiling. In edge-to-edge tiling by convex pentagons, any two pentagons are either disjoint or share one vertex or one entire edge in common [8,11]. We have conducted research aiming for the perfect list (complete list) of types of convex pentagonal tiles that can generate an edge-to-edge tiling (such tiles are hereinafter called *EE convex pentagonal tiles*).

In [8] and [10], we produced and sorted patterns of pentagons based on the properties (Bagina's Proposition [1,8,10], etc.) as a convex pentagon satisfies if it is an EE convex

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pentagonal tile. By means of the sorting, we confirmed that convex pentagons that can generate an edge-to-edge monohedral tiling belong to one of the eight types in Table 1, and we obtained 42 cases for which it is not known whether the convex pentagons can generate an edge-to-edge monohedral tiling (see Table 1 in [10]). We have since researched the remaining 42 uncertain cases individually with a computer and have thereby arrived at the following results:

Theorem *If a convex pentagon can generate an edge-to-edge monohedral tiling, then it belongs to one of eight types in Table 1.*

Remark These eight types are not necessarily 'disjoint'. Some of them can contain the same convex pentagon (see Appendix A.1).

In this paper, the investigation methods and results of the 42 uncertain cases are introduced. In Sect. 2, we prepare for the investigations in the next sessions. Section 3 explains the investigation of the convex pentagons with one degree of freedom (DOF) from among the 42 uncertain cases. (The DOFs column of Table 1 in [10] lists the DOFs except for the size of each pentagon.) Section 4 explains the investigations of the convex pentagons with two and three DOFs from among the 42 uncertain cases. Section 5 concludes the investigations in this paper.

2 Preparation

In this study, a vertex of an edge-to-edge tiling is called a *node*, and the number of polygons meeting at the node is called the *valence* of the node. In addition, we call the multiset of vertices of polygons a *spot* if the sum of the interior angles at the vertices in the multiset is equal to 360° [8]. Let $G = ABCDE$ be an EE convex pentagonal tile candidate. Then, a spot of G that is supposed to become a node of the supposed edge-to-edge tiling is called a *tentative node* [8]. Hereafter, unless noted otherwise, an edge-to-edge monohedral tiling is written simply as 'an edge-to-edge tiling'.

Let $\Im_p$ be an edge-to-edge tiling by a convex pentagonal tile. As mentioned in Subsection 2.1 in [10] (or Subsection 2.2 in [11]), the average valence of nodes in $\Im_p$ is $\frac{10}{3} = 3.\dot{3}$. (Note that, in Subsection 2.2 in [10], a vertex of pentagon is called a corner, and a vertex of tiling is called a vertex.) Therefore, in $\Im_p$, there must be 3-valent nodes and there are no tilings with all nodes of the same valence.

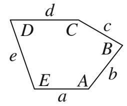
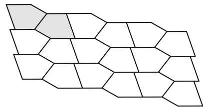
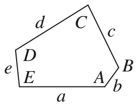
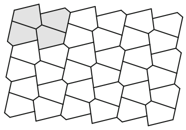
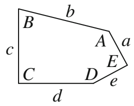
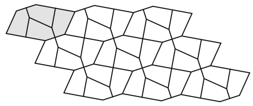
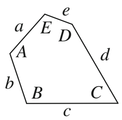
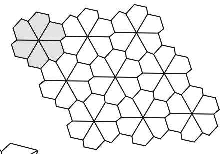
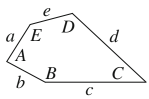
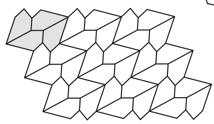
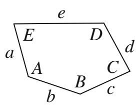
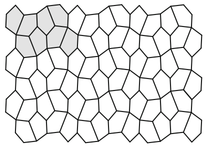
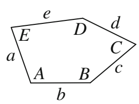
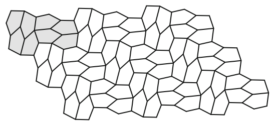
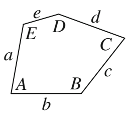
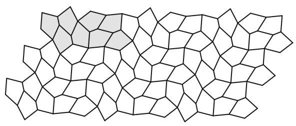
From Corollary 1 in [10] and the Extension Theorem in [4] (see Subsection 3.8 in [4]), we obtain the following corollary.

Corollary *If a convex pentagon with exactly three vertices that can simultaneously belong to tentative* 3*-valent nodes is an EE convex pentagonal tile, then an edge-to-edge monohedral tiling by the tile is a tiling of the plane with only* 3*- and* 4*-valent nodes.*

3 Investigation into 29 Cases of One DOF among 42 Uncertain Cases

Among the 42 uncertain cases, there are 29 cases of one DOF (see Table 1 in [10]). Since each convex pentagon in the 29 cases has one parameter corresponding to the

Table 1 Eight types of convex pentagonal tiles that can generate edge-to-edge tilings

Type [a]	Shape of convex pentagon	Conditions of convex pentagon [b]	Representative tiling [c]
type 1		$A+B+C=360^\circ$. [d]	
type 2		$A+B+D=360^\circ$, [e] $a=d$.	
type 4		$C=E=90^\circ$, $a=e, c=d$.	
type 5		$A=120^\circ, C=60^\circ$, $a=b, c=d$.	
type 6		$A+B+D=360^\circ$, $A=2C$, $a=b=e, c=d$.	
type 7		$2B+C=360^\circ$, $2D+A=360^\circ$, $a=b=c=d$.	
type 8		$2A+B=360^\circ$, $2D+C=360^\circ$, $a=b=c=d$.	
type 9		$2E+B=360^\circ$, $2D+C=360^\circ$, $a=b=c=d$.	

[a] A classification system of tiles is modeled on that used in Grünbaum and Shephard (1987). Properties of each type are essentially different, but these types are not necessarily disjoint since some degrees of freedom remain in the conditions. The number of each type follows a general 14 types notation. Since a convex pentagonal tile belonging only to type 3 cannot generate an edge-to-edge tiling, type 3 is not in this list.
[b] For example, a convex pentagonal tile belonging to type 1 satisfies that the sum of three consecutive angles is equal to 360°. This condition of type 1 is expressed as $A+B+C=360^\circ$.
[c] The pale gray pentagons in each tiling indicate the fundamental region (the unit that can generate a periodic tiling by translation only).
[d] In order to make an edge-to-edge tiling, one should impose an additional condition "$a=d$" or "$A=C, b=c$."
[e] In order to make an edge-to-edge tiling, one should impose an additional condition "$c=e$."

one remaining DOF, the value of each internal angle of G can be calculated from the value of the parameter. Therefore, for each pentagon in the 29 cases of one DOF, cases such as those in which the sum of the values of some internal angles (e.g., $A+B+D$, $2D + C + E$, $5E + C$) is 360° can be obtained using a computer. Hence, we use a computer to calculate the 3-, 4-, 5-, and 6-valent spots formed by each pentagon in the 29 cases, and we investigate each pentagon of each parameter value from the information on the spots. This investigation is performed in the following three steps:

STEP 1 The properties of convex pentagons are researched.
STEP 2 By using a computer, for $m = 3, 4, 5, 6$, each sum of the values of m internal angles corresponding to the possible label sets of m-valent spots is calculated, and all of the m-valent spots formed at each value of the parameter are obtained.
STEP 3 We judge whether each spot obtained in STEP 2 is a tentative node, and we consider the properties of the G with the spot (i.e., whether the G with the spot belongs to a known type, represents a new type, or cannot generate an edge-to-edge tiling).

Remarks In order to determine whether or not G belongs to a known type, the information of $m = 3, 4, 5, 6$ is sufficient. The information of $m \geq 7$ will be needed when it cannot be determined whether G can generate an edge-to-edge tiling in STEP 3. In such a case, an investigation of whether G can generate an edge-to-edge tiling is processed individually. However, such a case did not exist. As mentioned in Subsection 3.1 in [8], the total number of possible label-sets of the 3-valent spots of a convex pentagon $G = ABCDE$ is 35. Since the sum of interior angles of a convex pentagon is 540°, the cases wherein four different angles are contained in 4-, 5-, or 6-valent spots (e.g., $A + B + C + D = 360°$, $A + B + C + D + E = 360°$, etc) are geometrically impossible (for example, $A + B + C + D = 360°$, the remaining fifth angle is equal to 180°). Therefore, although the repeated combination of five angles taken four at a time is ${}_5\mathrm{H}_4 = 70$, the total number of possible label-sets of 4-valent spots of G is 65 because $A+B+C+D = 360°$, $A+B+C+E = 360°$, $A+B+D+E = 360°$, $A + C + D + E = 360°$, and $B + C + D + E = 360°$ are excepted. Similarly, the total number of possible label-sets of 5- and 6-valent spots of G is 105 and 155, respectively.

These investigations are conducted with each value of the parameter of each convex pentagon. Let us now give an example.

Example 3.1 Case where v_1 is *AAB*-1, v_2 is *DDE*-2, and the cyclic-edge-type is [11112] (Conditions: $2A + B = 2D + E = 360°$, $a = b = c = e \neq d$).

STEP 1 Two isosceles triangles *ABC* and *ADE* in the convex pentagon $G = ABCDE$ are considered (see Fig. 1). When the base angles of isosceles triangle *ADE* are denoted α, the base angles of isosceles triangle *ABC* are $\beta = \cos^{-1}\left(\cos\alpha / \sin\alpha\right)$ by triangle *ACD* and the sine formula. The interior angles of the pentagon can be expressed as follows:

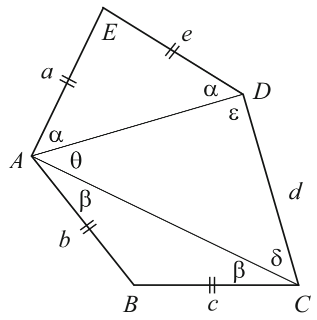


Fig. 1 Convex pentagon satisfying $2A + B = 2D + E = 360°$, $a = b = c = e \neq d$, and the triangles ABC, ACD and ADE in the pentagon

$$\begin{cases} A = \alpha + \theta + \beta = 90° + \beta = 90° + \cos^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right), \\ B = 180° - 2\beta = 180° - 2\cos^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right), \\ C = \beta + \delta = \beta + \alpha = \cos^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right) + \alpha, \\ D = \varepsilon + \alpha = 90° + \alpha, \\ E = 180° - 2\alpha, \end{cases} \tag{1}$$

where $45° < \alpha < 90°$ since $E < 180°$, and $\beta > 0°$.

This pentagon always has the tentative 4-valent node $\{B, C, C, E\}$; i.e., $2C + B + E = 360°$ (see [8] for the notation of spots and nodes). Even so, under the conditions, it cannot generate an edge-to-edge tiling by using only the tentative nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$. The reason for this is explained below. For the tentative 3-valent node $\{D, D, E\}$ (see Fig. 2), *DDE*-2 is used to meet the edge conditions.[1] Conversely, for the tentative 3-valent node $\{A, A, B\}$ (see Fig. 6 in [8]), *AAB*-1 or *AAB*-2 can be used. *DDE*-2 has a concentration of E and A (see Fig. 2), and there exists no concentration of E and A in the tentative nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$. Therefore, the convex pentagon that has only the tentative nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$ cannot generate an edge-to-edge tiling since it cannot use the node $\{D, D, E\}$ (i.e., the vertex D).

From the conditions, there are four vertices A, B, D, and E that can belong to tentative 3-valent nodes. If this convex pentagon has only the original relations between angles (i.e., the angle relation is always realized at all values of α, $2A+B = 2D+E = 360°$), it can have no more than three vertices belonging to tentative 3-valent nodes simultaneously. In this case, on basis of the considered combinations of possible geometrical arrangements of *AAB*-1, *AAB*-2, and *DDE*-2, there are 12 patterns for convex pentagons with three vertices belonging to 3-valent nodes simultaneously (see Fig. 3).

[1] *DDE*-2 represents the sub-cases of the tentative 3-valent node $\{D, D, E\}$. When convex pentagons are assembled around a tentative 3-valent node, there are eight possible ways to assemble the three pentagons. As a result, some sub-cases arise for edge fitting around the 3-valent node. For example, as shown in Fig. 2, $\{D, D, E\}$ has two sub-cases, *DDE*-1 and *DDE*-2. Notice that the equations following the colons for each label in the figure represent the conditions on edge-lengths. The convex pentagon of this example can use only *DDE*-2 (the two arrangement way of *DDE*-2) from the edge conditions. Refer to Section 4 in [8] for details.

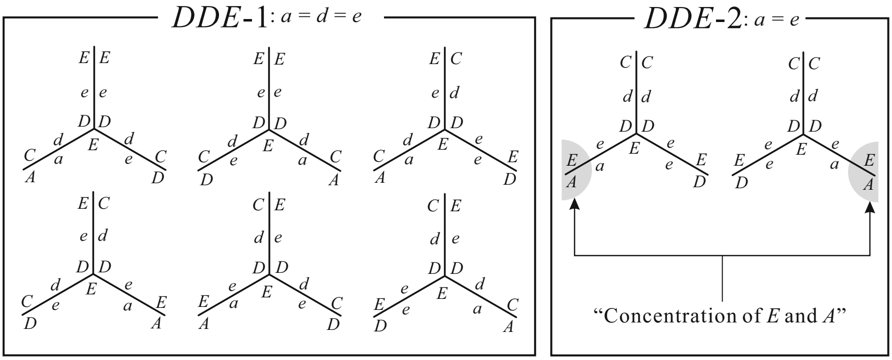


Fig. 2 Sub-cases of tentative 3-valent nodes $\{D, D, E\}$

Thus, for the 29 cases of one DOF, we have confirmed that there exists no convex pentagon with only the original relations between angles but with four or more vertices belonging to tentative 3-valent nodes simultaneously.

STEP 2 By using (1), we created a program that outputs spots with valences of 3, 4, 5, and 6. The program numerically calculates each internal angle and the sum of m $(= 3, 4, 5, 6)$ internal angles for several fixed values of the parameter α in the range $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ (i.e., $45° < \alpha < 90°$) and then outputs the m-valent spots together with the corresponding internal angles. Figure 4 shows a graph of the computational investigation. The curves $f_1(\alpha)$, $f_2(\alpha)$, and $f_3(\alpha)$ in Fig. 4 are functions of α, which completes a sum of m internal angles such as $A + B + D = 360° + \alpha - \cos^{-1}\left(\cos\alpha \big/ \sin\alpha\right)$. Although the values of the internal angles vary continuously with a geometric property, the calculation in the program is discrete. As shown in Fig. 4, our program calculates the values of internal angles at points $\alpha_{k-1}, \alpha_k, \alpha_{k+1}, \alpha_{k+2}, \alpha_{k+3}$, etc. (In the case shown, the step-size is $\Delta\alpha = 0.005$ rad $\approx 0.2865°$.) Our program outputs cases like $f_1(\alpha)$ and $f_2(\alpha)$ in Fig. 4, i.e., a case where $f_1(\alpha_k)$ is within the interval $[359°, 361°]$, as well as such cases as relation between $f_2(\alpha_{k+1}) > 361°$ and $f_2(\alpha_{k+2}) < 359°$ (or $f_2(\alpha_{k+1}) < 359°$ and $f_2(\alpha_{k+2}) > 361°$, i.e., 360° is sandwiched between adjoining points although each value of the adjoining points is not within the interval $[359°, 361°]$). However, a case like $f_3(\alpha)$ in Fig. 4, i.e., $f_3(\alpha_{k+2}) > 361°$ and $f_3(\alpha_{k+3}) > 361°$ (or $f_3(\alpha_{k+2}) < 359°$ and $f_3(\alpha_{k+3}) < 359°$) and $f_3(\alpha)$ crossing 360° in the range $\alpha_{k+2} < \alpha < \alpha_{k+3}$, is not output by our program. Since the program discretely calculates a continuous curve, it does not guarantee that a case like $f_3(\alpha)$ will not be overlooked, even if the step-size is simply made smaller. Consequently, in order to check the results of the program and find a case like $f_3(\alpha)$, the investigation that searches out forming spots is also performed using the mathematical software Maple.

STEP 3 This step investigates the properties of each convex pentagon from the results obtained by the computer. For example, by using the computer we newly find that when $\alpha \approx 47.059°$ the convex pentagon that satisfies (1) has 3-valent spots $\{A, A, D\}$, $\{B, D, E\}$, and $\{B, B, E\}$, 4-valent spots $\{A, A, C, C\}$ and $\{C, C, D, E\}$, and 5-valent spot $\{C, C, C, C, E\}$. However, the 3-valent spots $\{A, A, D\}$ and $\{B, D, E\}$ are not

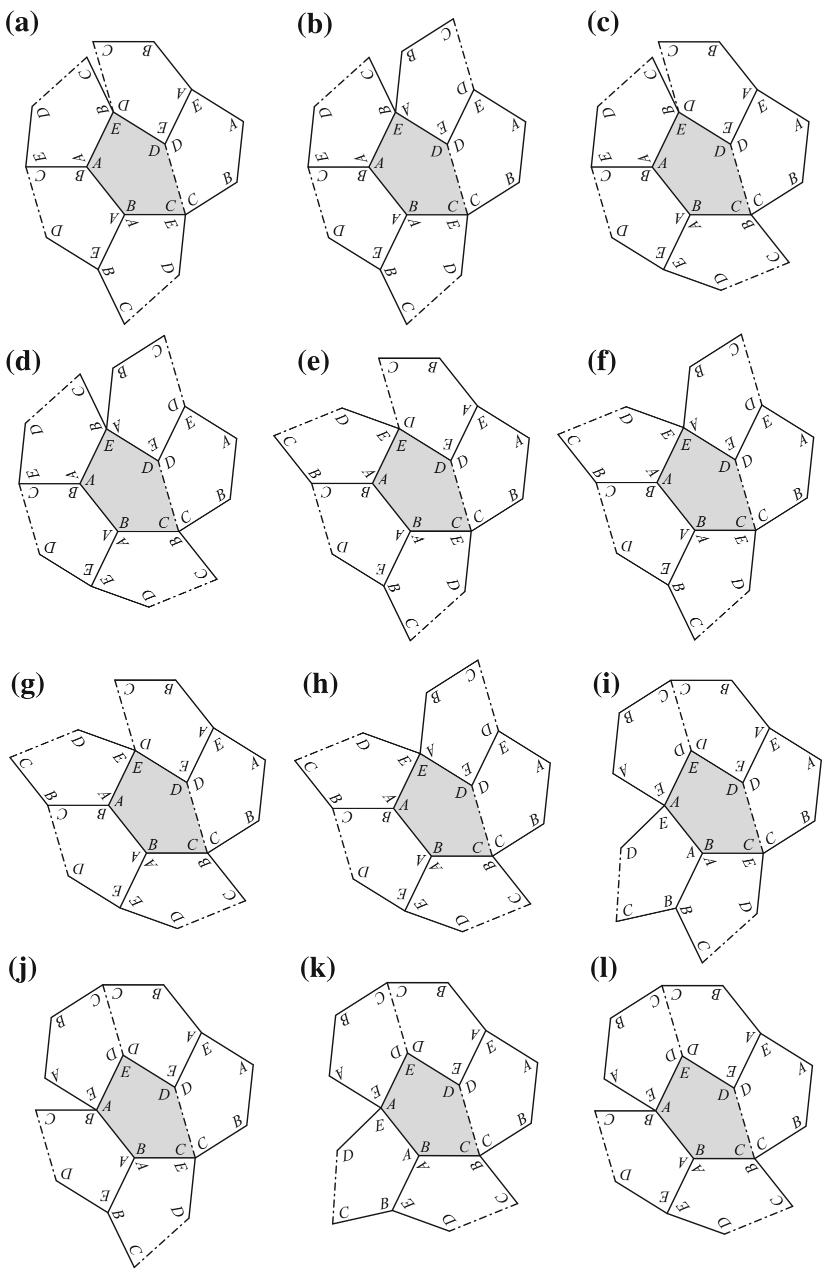


Fig. 3 In the case that v_1 is *AAB*-1, v_2 is *DDE*-2, and the cyclic edge type is [11112], there are 12 convex pentagon patterns (*gray pentagon*) with three vertices belonging simultaneously to tentative 3-valent nodes and using only *AAB*-1, *AAB*-2, or *DDE*-2. **a** *A* and *B* (of *gray pentagon*) are *AAB*-1, and *D* is *DDE*-2. **b** *A* and *B* are *AAB*-1, and *D* is *DDE*-2. **c** *A* is *AAB*-1, *B* is *AAB*-2, and *D* is *DDE*-2. **d** *A* is *AAB*-1, *B* is *AAB*-2, and *D* is *DDE*-2. **e** *A* is *AAB*-2, *B* is *AAB*-1, and *D* is *DDE*-2. **f** *A* is *AAB*-2, *B* is *AAB*-1, and *D* is *DDE*-2. **g** *A* and *B* are *AAB*-2, and *D* is *DDE*-2. **h** *A* and *B* are *AAB*-2, and *D* is *DDE*-2. **i–k** *B* is *AAB*-1, *D* and *E* are *DDE*-2. **l** *B* is *AAB*-2, *D* and *E* are *DDE*-2

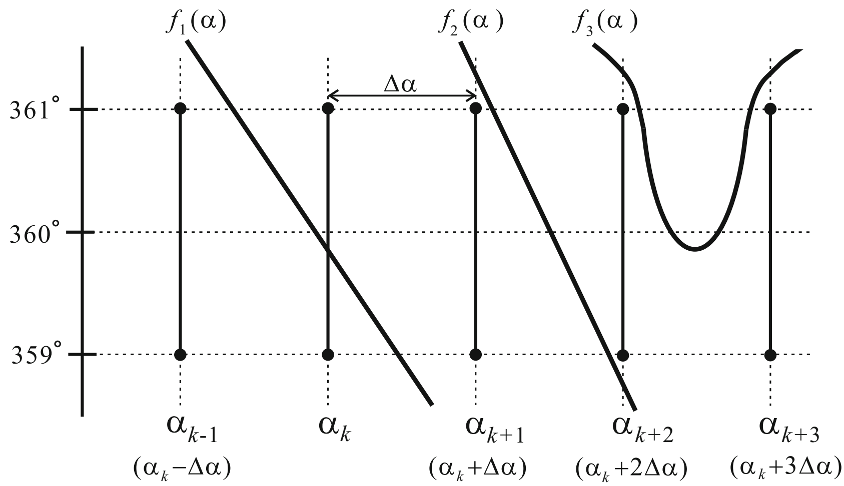


Fig. 4 Graph of the computational investigation

tentative 3-valent nodes under the edge conditions. Hence, let G(47.059°: ~~*AAD*~~, ~~*BDE*~~, *BBE, AACC, CCDE, CCCCE*) denote the convex pentagon of $\alpha \approx 47.059°$ that has tentative 3-valent nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, B, E\}$, tentative 4-valent nodes $\{B, C, C, E\}$, $\{A, A, C, C\}$, and $\{C, C, D, E\}$, tentative 5-valent node $\{C, C, C, C, E\}$, and 3-valent spots (not tentative 3-valent nodes) $\{A, A, D\}$ and $\{B, D, E\}$. Note that G(47.059°: ~~*AAD*~~, ~~*BDE*~~, *BBE, AACC, CCDE, CCCCE*) naturally also has the tentative nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$. The results of this case are briefly summarized below.

In the cases of G(45.542°: ~~*AAAC*~~, ~~*ACCCE*~~), G(45.879°: ~~*ACCD*~~, *CCCCCC*), G(46.039°: ~~*BBC*~~, ~~*AACE*~~, ~~*CCCEE*~~), G(54.000°: ~~*DEEE*~~, *EEEEE*), G(55.785°: ~~*BBBC*~~), G(56.999°: ~~*CCD*~~, ~~*BDEE*~~, *BBEEE*), G(58.837°: ~~*BBDE*~~, *BBBBE*), G(60.181°: ~~*BBBD*~~), and G(62.048°: ~~*CCC*~~, ~~*BBCEE*~~, *BBBEEE*), although these pentagons have 3- or 4-valent spots other than the original relations between angles, the spots are not tentative nodes. Therefore, these convex pentagons cannot generate an edge-to-edge tiling by using only the 3- and 4-valent nodes (i.e., $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$). On the other hand, the smallest angle of each pentagon is less than or equal to 72°; that is, tentative nodes with valences of five or more can exist. However, from the properties of the nodes for average valence in $\Im_p$ and Lemma 1 in [10], a tentative node with a valence of five or more (if one exists) is never a node in an edge-to-edge tiling in which the density of k-valent nodes for $k \geq 5$ is greater than zero because the tentative 3-valent nodes are only $\{A, A, B\}$ and $\{D, D, E\}$ (i.e., four or more vertices of a convex pentagon cannot belong to tentative 3-valent nodes simultaneously). Thus, from Corollary and the above considerations, these pentagons cannot generate an edge-to-edge tiling.

In the case of G(47.059°: ~~*AAD*~~, ~~*BDE*~~, *BBE, AACC, CCDE, CCCCE*), there are the relations $B + D + E = 360°$ and $a = c$. Therefore, this convex pentagon belongs to type 2. However, a representative tiling of type 2 by this pentagon is non-edge-to-edge.

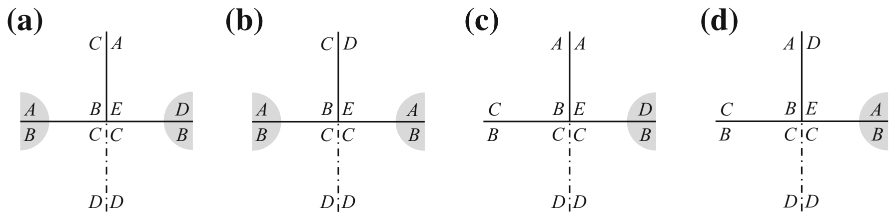


Fig. 5 Four sub-cases of tentative 4-valent node $\{B, C, C, E\}$

In the cases of G(48.039°: *CCCD*), G(48.486°: ~~*ACEE*~~), G(49.266°: ~~*ACCC*~~), G(49.525°: ~~*CDD*~~, ~~*BCCC*~~, ~~*BCEE*~~, *BEEE*), G(50.741°: ~~*BBD*~~), and G(53.460°: ~~*AAC*~~, *BBBE*, ~~*BBCE*~~, ~~*CCCE*~~), the angles are all greater than 72°; i.e., there are no spots with valences of five or more. For all tentative nodes in each case, there exists no concentration of E and A such as DDE-2 needs (i.e., the tentative node $\{D, D, E\}$ cannot be used). Therefore, these pentagons cannot generate an edge-to-edge tiling.

In the case of G(50.976°: *AEEE*), the angles are all greater than 72°. There are four sub-cases of the tentative 4-valent node $\{B, C, C, E\}$ under the edge conditions (see Fig. 5). The sub-cases of the tentative node $\{B, C, C, E\}$ have a concentration of either A and B or B and D. Since there exists no concentration of B and D in the tentative 3- and 4-valent nodes of this pentagon, the cases of the tentative node $\{B, C, C, E\}$ with a concentration of A and B only (i.e., Fig. 5b, d) are chosen. However, those with a concentration of A and B need to have a concentration of C and D (Fig. 5b) or a concentration of A and D (Fig. 5d), so the desired combinations do not exist in the tentative 3- and 4-valent nodes. Therefore, the tentative node $\{B, C, C, E\}$ cannot be used. That is, in this case, the vertex C cannot be used because the remaining tentative nodes $\{A, A, B\}$, $\{D, D, E\}$, and $\{A, E, E, E\}$ do not contain C. Thus, this pentagon cannot generate an edge-to-edge tiling.

In the case of G(49.107°: *AAA*, *ABB*, *BBB*, *ACCE*), there are the relations $2B + A = 2D + E = 360°$, and $a = b = c = e$. Therefore, this convex pentagon belongs to type 8. (If the labels are exchanged such that $A \to B$, $B \to A$, $C \to E$, $D \to D$, and $E \to C$, then the conditions are the same as those of type 8 in Table 1).

In the case of G(51.827°: *ACD*, *BBEE*, *CCCC*), there are the relations $C + D + A = 360°$, $a = c$, and $b = e$. Therefore, this convex pentagon belongs to type 2.

In the cases of G(54.736°: *BBBB*), G(55.735°: *ABEE*, *BEEEE*) and, G(59.554°: *ABBB*, *BBBBB*), the smallest angle is less than 72°. Then the only tentative 3-valent nodes are $\{A, A, B\}$ and $\{D, D, E\}$. Therefore, from the properties of the nodes for average valence in $\Im_p$ and Lemma 1 in [10], a tentative node with a valence of five or more cannot be used in an edge-to-edge tiling in which the density of k-valent nodes for $k \geq 5$ is greater than zero. In addition, as with the case of G(50.976°:*AEEE*), the tentative 4-valent node $\{B, C, C, E\}$ (the vertex C) cannot be used. Thus, from Corollary and the above considerations, these pentagons cannot generate an edge-to-edge tiling.

In the case of G(58.002°: *ACC*, *ABBE*, *BBBEE*), there are the relations $2C + A = 2D + E = 360°$, and $a = b = c = e$. Therefore, this convex pentagon belongs to type 9. (If the labels are exchanged such that $A \to B$, $B \to A$, $C \to E$, $D \to D$ and $E \to C$, then the conditions are the same as those of type 9 in Table 1).

In the other cases, for example, $G(63.819°:AEEEE)$, $G(70.146°:ABEEEE)$, etc., as well as G that has 7- or more-valent spots, if it exists, the convex pentagons do not have 3- or 4-valent spots except for $\{A, A, B\}$, $\{D, D, E\}$, and $\{B, C, C, E\}$. That is, these convex pentagons cannot generate an edge-to-edge tiling by using only the 3- and 4-valent nodes. Since four or more vertices of a convex pentagon cannot simultaneously belong to tentative 3-valent nodes, tentative nodes with valences of five or more, for example, $\{A, E, E, E, E\}$, $\{A, B, E, E, E, E\}$, etc., are never nodes in edge-to-edge tiling in which the density of k-valent nodes for $k \geq 5$ is greater than zero. Thus, from the properties of nodes for average valence in $\Im_p$, Lemmas 1 and 2 in [10], and Corollary, the convex pentagons in the other cases cannot generate an edge-to-edge tiling.

By analogy with Example 3.1, we consider each of the 29 uncertain cases of one DOF, and in each of those 29 cases we find that if a convex pentagon can generate an edge-to-edge tiling, it belongs to one of the eight types in Table 1.

4 Investigation into 13 Cases of Two or Three DOFs among 42 Uncertain Cases

In Table 1 of [9], there are 12 cases of two DOFs and 1 case of three DOFs. For these cases, the DOFs are lowered by imposing extra angular constraints, and then further consideration is possible. As for the 13 cases of two and three DOFs, we confirmed that if a convex pentagon has only the original relations between angles, it cannot have four or more tentative 3-valent nodes simultaneously. That is, there are some convex pentagons with three vertices belonging to tentative 3-valent nodes simultaneously, like the convex pentagon of Example 3.1, but there is no convex pentagon with four or more vertices belonging to tentative 3-valent nodes simultaneously.

First, on the basis of the patterns of convex pentagons with three vertices simultaneously belonging to tentative 3-valent nodes by using only the original relations between angles (the patterns are called the first-stage patterns), we consider cases where such a pentagon generates an edge-to-edge tiling with only 3- and 4-valent nodes. Hereafter, the case is called the first stage. If a convex pentagon has only the original relations with the 3-valent nodes and is an EE convex pentagonal tile, the pentagon can generate an edge-to-edge tiling with only 3- and 4-valent nodes from Corollary. From Bagina's Proposition, etc., if a convex pentagon in the 13 cases of two and three DOFs can generate an edge-to-edge tiling, except for the first stage, there then exists a pattern that is not in the first-stage patterns of a pentagon with three or more vertices simultaneously belonging to 3-valent nodes in the tiling. In order to admit a pattern that is not in the first-stage patterns, the pentagon needs to have tentative 3-valent nodes other than the original tentative 3-valent nodes. Therefore, we next consider the pentagon by imposing sub-cases of tentative 3-valent nodes other than the original tentative 3-valent nodes (see Tables 1–4 in [8] for the sub-cases of tentative 3-valent nodes). Hereafter, this case is called the second stage. If a convex pentagon in the 13 cases has a property in which four or more vertices can simultaneously belong to tentative 3-valent nodes, the pentagon is then contained within the pentagons that are considered part of the second stage. From Lemma 1 in [10], in the second stage, we will consider such a convex pentagon that can generate an edge-to-edge tiling in

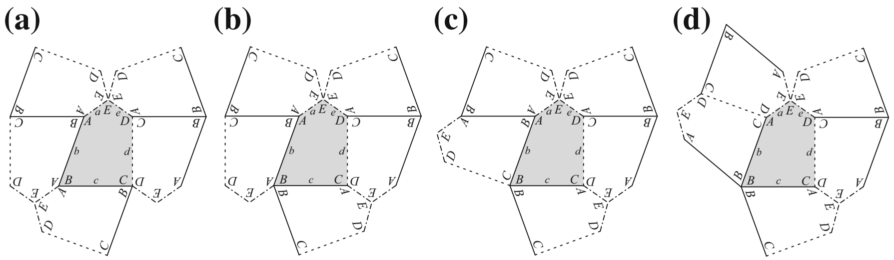


Fig. 6 In the case that v_1 is *AAB*-2, v_2 is *CDA*-1, and the cyclic-edge-type is [11223], there are four convex pentagon patterns (*gray pentagon*) with three vertices belonging simultaneously to tentative 3-valent nodes and using only *AAB*-2 or *CDA*-1

which the density of k-valent nodes for $k \geq 5$ is greater than zero. (Naturally, we also consider whether convex pentagons in the second stage can generate an edge-to-edge tiling with 4-valent nodes.) Thus, all cases are considered: the case where a convex pentagon can generate an edge-to-edge tiling with only 3- and 4-valent nodes and the other cases where a convex pentagon can generate an edge-to-edge tiling with k-valent nodes for $k \geq 5$. We show an exemplary investigation of the case of two DOFs below.

Example 4.1 Case where v_1 is *AAB*-2, v_2 is *CDA*-1, and the cyclic-edge-type is [11223] (Conditions: $2A + B = C + D + A = 360°$, $d \neq a = e \neq b = c \neq d$).

From the conditions, there are four vertices, A, B, C, and D, that can belong to tentative 3-valent nodes. However, four or more vertices of a convex pentagon cannot simultaneously belong to tentative 3-valent nodes if the pentagon has only $2A + B = C + D + A = 360°$ for its angles. For the convex pentagon with three vertices belonging to tentative 3-valent nodes simultaneously, there are four patterns as shown in Fig. 6 (i.e., these patterns are the first-stage patterns of this case).

First, on the basis of the four patterns in Fig. 6, the case where a convex pentagon generates an edge-to-edge tiling with only 4-valent nodes and the original 3-valent nodes is considered (i.e., this is the first stage). The remaining vertices of the gray pentagons of the four patterns in Fig. 6 must be tentative 4-valent nodes (i.e., the vertices C and E in Fig. 6a and the vertices B and E in Fig. 6b–d). However, from the edge conditions, the only tentative 4-valent node that can be formed by each E of the four patterns in Fig. 6 is $\{E, E, E, E\}$. Therefore, the convex pentagon with a tentative node $\{E, E, E, E\}$ belongs to type 4, as $B = E = 90°$, $a = e$, $b = c$ because $4E = 360°$ and $B + E = 180°$.

Next, the cases imposing sub-cases of tentative 3-valent nodes other than original tentative 3-valent nodes are considered (i.e., this is the second stage). Therefore, the cases where tentative 3-valent nodes other than *AAB*-2 and *CDA*-1 are admitted are considered. The cases contain such cases where a convex pentagon can generate an edge-to-edge tiling with k-valent nodes for $k \geq 5$.The sub-cases of tentative 3-valent nodes that can be formed are obtained from the edge conditions and Tables 1–4 in [8], whereupon *AAE*-1, *CCB*-1, *DDE*-2, *BBB*, and *EEE* are admitted.

If *AAE*-1 is admitted, then the convex pentagon belongs to type 4, as $B = E = 90°$, $a = e$, $b = c$ because $2A + B = 2A + E = 360°$ and $B + E = 180°$.

If *CCB*-1 is admitted, then the convex pentagon belongs to type 1, as $A+B+C=360°$ because $2A+B=2C+B=360°$; i.e., $A=C$.

If *DDE*-2 is admitted, the pentagon has the relations $A=90°+\frac{E}{2}$, $B=180°-E$, $C=90°$, $D=180°-\frac{E}{2}$ where $0°<E<180°$. That is, this is a pentagon with one DOF except for size. Therefore, as with the cases of one DOF in Table 1 of [10], this pentagon is investigated using a computer. Note that if the only tentative 3-valent nodes of this pentagon are *AAB*-1, *CDA*-1, and *DDE*-2, there does not exist a pentagon with four or more vertices belonging simultaneously to tentative 3-valent nodes. As a result, we find that if this pentagon can generate an edge-to-edge tiling, it belongs to one of the eight types in Table 1.

If *BBB* is admitted, then the pentagon belongs to type 5, as $B=120°, E=60°, a=e, b=c$ because $3B=360°$ and $B+E=180°$.

If *EEE* is admitted, then the pentagon belongs to type 5, as $B=60°, E=120°, a=e, b=c$.

Thus, if the convex pentagon that satisfies $2A+B=C+D+A=360°, d\neq a=e\neq b=c\neq d$ can generate an edge-to-edge tiling, it belongs to one of the eight types in Table 1.

By considering each of the 12 uncertain cases of two DOFs in analogy with the above, we find that if the convex pentagons in the 12 cases are EE convex pentagonal tiles, they belong to one of the eight types in Table 1.

Finally, we show an investigation of the case of three DOFs below.

Example 4.2 Case where v_1 is *ABD*-1, v_2 is *ABD*-1, and the cyclic-edge-type is [11223] (Conditions: $A+B+D=360°, b\neq a=e\neq c=d\neq b$).

This convex pentagon with three vertices belonging to tentative 3-valent nodes simultaneously is unique, as shown in Fig. 9 of [8], provided that it has only the original relations between angles.

First, on the base of the unique pattern of Fig. 9 in [8], the case where a convex pentagon generates an edge-to-edge tiling with only 4-valent nodes and the original 3-valent nodes is considered. Since the remaining vertices C and E of the gray pentagons in Fig. 9 of [8] must be tentative nodes with valence of 4, the vertex C forms a tentative node $\{C, C, C, C\}$ and the vertex E forms a tentative node $\{E, E, E, E\}$. Therefore, the convex pentagon with tentative nodes $\{C, C, C, C\}$ and $\{E, E, E, E\}$ belongs to type 4, as $C=E=90°, a=e, c=d$.

Next, the case where tentative 3-valent nodes other than *ABD*-1 are admitted is considered. The sub-cases of tentative 3-valent nodes that can be formed are obtained from the edge conditions and Tables 1–4 in [8], whereupon *AAE*-1, *BBC*-1, *DDC*-1, *DDE*-2, *CCC*, and *EEE* are admitted.

If *AAE*-1 is admitted, then the case is the same as that in Table 1 of [10] where v_1 is *AAB*-2, v_2 is *EAC*-1, and the cyclic edge type is [11223]. (If the labels are exchanged such that $A\rightarrow A$, $B\rightarrow E$, $C\rightarrow D$, $D\rightarrow C$, and $E\rightarrow B$, then we find that the expressions of the conditions are equivalent).

If *BBC*-2 is admitted, then the case is the same as that in Table 1 of [10] where v_1 is *AAB*-2, v_2 is *EAC*-1, and the cyclic-edge-type is [11223]. (If the labels are exchanged such that $A \to E$, $B \to A$, $C \to B$, $D \to C$, and $E \to D$, then we find that the expressions of conditions are equivalent).

If *DDC*-1 or *DDE*-2 is admitted, then the both cases are the same as that in Table 1 of [10] where v_1 is *AAB*-2, v_2 is *CDA*-1, and the cyclic-edge-type is [11223]. (If the labels are exchanged such that $A \to D$, $B \to C$, $C \to B$, $D \to A$, and $E \to E$, or $A \to C$, $B \to D$, $C \to E$, $D \to A$, and $E \to B$, then we find that the expressions of conditions are equivalent).

If *CCC* is admitted, then the pentagon belongs to type 5, as $C = 120°$, $E = 60°$, $a = e$, $c = d$ because $3C = 360°$ and $C + E = 180°$.

If *EEE* is admitted, then the pentagon belongs to type 5, as $E = 120°$, $C = 60°$, $a = e$, $c = d$.

Thus, if the convex pentagon that satisfies $A + B + D = 360°$, $b \neq a = e \neq c = d \neq b$ is an EE convex pentagonal tile, it belongs to one of the eight types in Table 1.

5 Conclusion

We have known that the same result of Theorem was obtained by Bagina in 2011 after we derived Theorem in 2012 [2,9]. Now, we think that a proof of Theorem will be possible even without using a computer. The new analytical method improves upon the present method but does not eliminate the need to investigate many patterns.

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Appendix A.1

Let T_x be a set of convex pentagonal tiles that belongs to a type x for x = 1, 2, 4, 5, 6, 7, 8, 9. Figure 7 shows examples of the Venn diagram of T_x. Figure 8 shows examples of the convex pentagonal tiles that are contained in each intersection of Fig. 7.

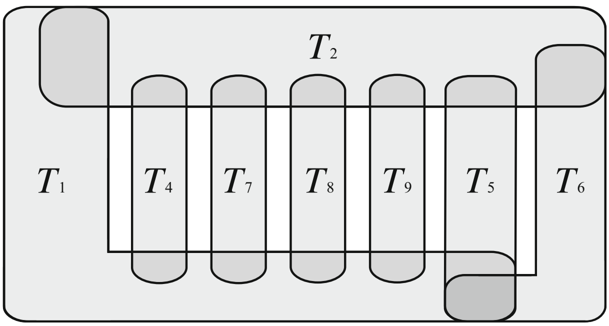

Fig. 7 Venn diagram of T_x

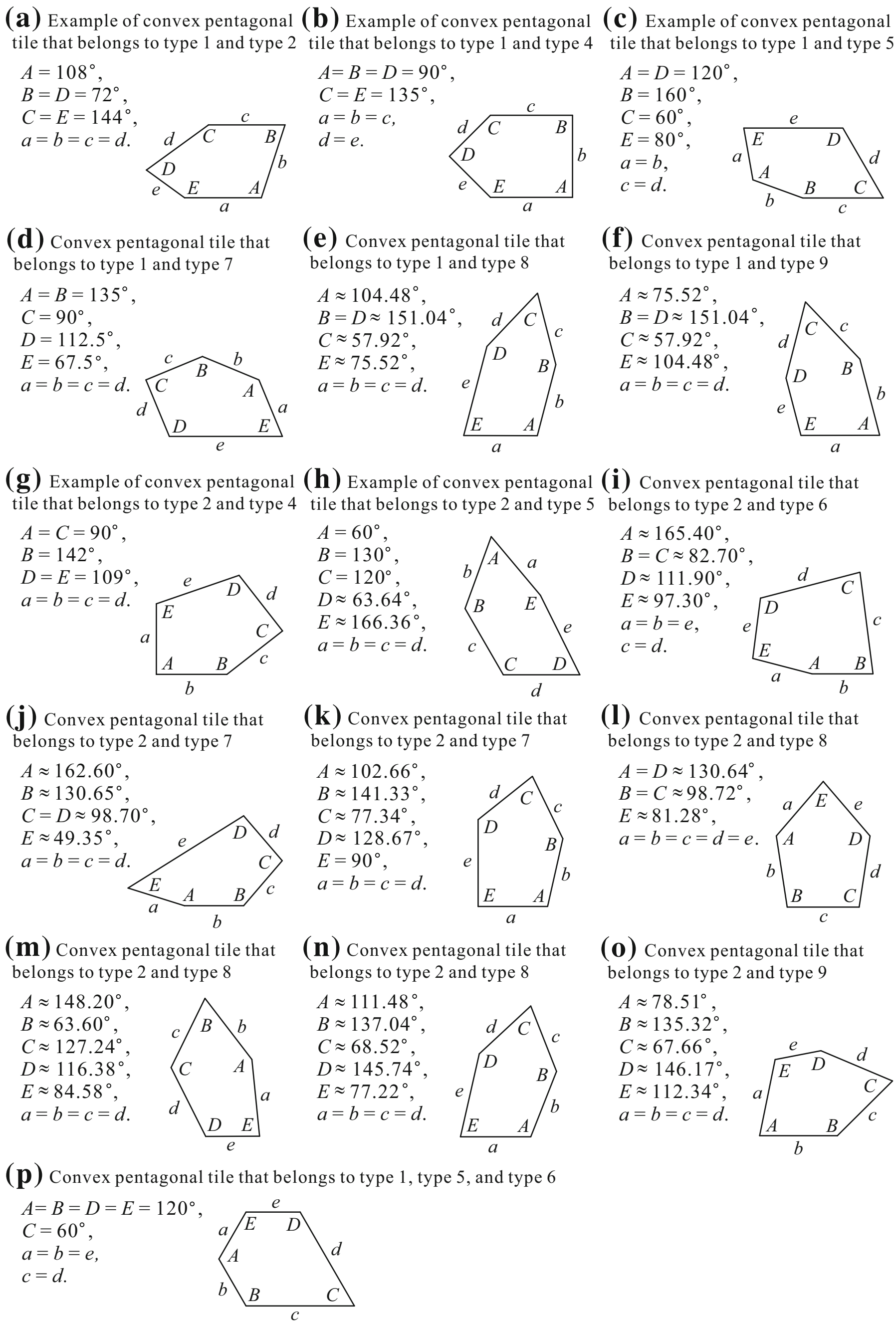

Fig. 8 Convex pentagonal tiles that are contained in each intersection of Fig. 7. Representative tilings of type 2 by convex pentagonal tiles of **a**, **i**, **j**, **m**, and **o** are always non-edge-to-edge. Representative tilings of type 1 by convex pentagonal tiles of **c**, **e**, **f**, and **p** are always non-edge-to-edge

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