Graphs and Combinatorics (2016) 32:773–776
DOI 10.1007/s00373-015-1580-z

ORIGINAL PAPER

Cubic Graphs with Large Ratio of Independent Domination Number to Domination Number

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Received: 18 July 2014 / Revised: 6 February 2015 / Published online: 30 April 2015

Abstract A *dominating set* in a graph G is a set S of vertices such that every vertex outside S has a neighbor in S; the *domination number* $\gamma(G)$ is the minimum size of such a set. The *independent domination number*, written $i(G)$, is the minimum size of a dominating set that also induces no edges. Henning and Southey conjectured $i(G)/\gamma(G) \le 6/5$ for every cubic (3-regular) graph G with sufficiently many vertices. We provide an infinite family of counterexamples, giving for each positive integer k a 2-connected cubic graph H_k with $14k$ vertices such that $i(H_k) = 5k$ and $\gamma(H_k) = 4k$.

Keywords Independent domination number · Domination number · Cubic graph · 3-regular

1 Introduction

A *dominating set* in a graph G is a vertex subset S such that every vertex outside S has a neighbor in S. The *domination number* of G, written $\gamma(G)$, is the minimum size of such a set. An *independent dominating set* in G is a dominating set of vertices

Research supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

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that induces no edges. The *independent domination number* of G, written $i(G)$, is the minimum size of such a set. An independent set of vertices is also a dominating set if and only if it is a maximal independent set, so $i(G)$ is the minimum size of a maximal independent set in G.

Favaron [3] and Gimbel and Vestergaard [4] proved that if G is an n-vertex graph with no isolated vertices, then $i(G) \leq n + 2 - 2\sqrt{n}$. However, this bound is not sharp for cubic (3-regular) graphs: Lam, Shiu, and Sun [8] proved that if G is an n-vertex connected cubic graph, then $i(G) \leq 2n/5$ except for $K_{3,3}$. Possibly the bound can be strengthened by excluding finitely many other examples.

The independent domination number and domination number of a graph may differ greatly: note that $i(K_{m,m}) = m$ and $\gamma(K_{m,m}) = 2$. Barefoot, Harary, and Jones [1] suggested studying the difference between $i(G)$ and $\gamma(G)$ in cubic graphs (see also [2]). They showed that the difference can be about $n/20$ for 2-connected cubic graphs and conjectured that it is bounded for 3-connected cubic graphs. Kostochka [7] disproved that by constructing 3-connected cubic graphs with $130k$ vertices where the difference is at least k.

The definition of $i(G)$ yields $\gamma(G) \leq i(G) \leq \alpha(G)$, where $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices. It is easy to show that if G is regular, then $\alpha(G) \leq n/2$, with equality only when G is bipartite. Also, note that $\gamma(G) \geq n/(r+1)$ for an n-vertex r-regular graph. Thus the difference between $\gamma(G)$ and $i(G)$ is at most $\frac{r-1}{2r+2}n$ for r-regular graphs. Goddard, Henning, Lyle, and Southey [5] asked whether a stronger bound than the resulting $n/4$ holds when $r = 3$ and a connectivity condition is imposed.

Question 1.1 [5] *Does $i(G) - \gamma(G) \leq n/16$ hold whenever G is an n-vertex 3-connected cubic graph with $n \geq 12$?*

Equality is known to hold on two infinite families of examples [5]. Later, a conjecture was posed for the ratio $i(G)/\gamma(G)$.

Conjecture 1.2 (Henning and Southey [6]) *If G is a connected cubic graph with sufficiently many vertices, then $i(G)/\gamma(G) \leq 6/5$.*

In [5] it was shown that $i(G)/\gamma(G) \leq 3/2$ for connected cubic graphs G, with equality if and only if $G = K_{3,3}$. In [6], it was shown that $i(G)/\gamma(G) \leq 4/3$ for connected cubic graphs other than $K_{3,3}$, with equality if and only if G is the cartesian product of C_5 and K_2.

In this note we provide an infinite family of counterexamples to the conjecture of [6]. For $k \geq 1$, we construct a 2-connected cubic graph H_k with $14k$ vertices such that $i(H_k) = 5k$ and $\gamma(H_k) = 4k$. These graphs also show that Question 1.1 requires 3-connectedness. However, the first graph H_1 is 3-connected, showing that a positive answer to Question 1.1 at least requires restricting to $n \geq 16$. The family also suggests the question of finding the sharpest bound on $i(G)/\gamma(G)$ for cubic graphs that has only finitely many exceptions.

Question 1.3 *Does $i(G)/\gamma(G) \leq 5/4$ hold whenever G is an n-vertex cubic graph with sufficiently many vertices?*

2 Counterexamples

We first describe our construction.

Construction 2.1 Construct a graph F from the 14-cycle on vertices $x, a^1, \ldots, a^6, y, b^6, \ldots, b^1$ in order by adding the chords $a^j b^j$ for $j \in \{1, 2, 5, 6\}$ and $\{a^4 b^3, a^3 b^4\}$ (see Fig. 1). Given k disjoint copies $F_1, \ldots, F_k$ of F, with x_i and y_i being the copies of x and y in F_i, form H_k by adding the edges of the form $y_{i-1} x_i$ (with indices taken modulo k).

When $k = 1$, the indices $i - 1$ and i are congruent modulo k, and the construction just adds the edge yx to F. The resulting graph H_1 is 3-connected. For $k \geq 2$, H_k has connectivity 2. Always H_k has $14k$ vertices and is cubic.

Theorem 2.2 *$i(H_k) = 5k$ and $\gamma(H_k) = 4k$.*

Proof First, we prove $\gamma(H_k) = 4k$. Since $\{a^1, b^3, b^4, a^6\}$ is a dominating set in F, we have $\gamma(H_k) \leq 4k$. If $\gamma(H_k) < 4k$, then H_k has a dominating set S such that $|S \cap V(F_i)| \leq 3$ for some i. Since each vertex dominates only four vertices, both x_i and y_i are dominated by vertices of S outside F_i (which do not exist when $k = 1$), and each vertex of S in F_i dominates four vertices not in $\{x_i, y_i\}$. This requires using the copies of a^2, b^2, a^5, b^5 in F_i to dominate the copies of a^1, b^1, a^6, b^6, which contradicts $|S \cap V(F_i)| \leq 3$.

Next, we prove $i(H_k) = 5k$. Since $\{a^1, a^4, a^6, b^2, b^4\}$ is an independent dominating set in F, we have $i(H_k) \leq 5k$. If $i(H_k) < 5k$, then H_k has an independent dominating set S such that $|S \cap V(F_i)| \leq 4$ for some i. Within the copy F_i of F, let X be the set of copies of $\{x, a^1, a^2, a^3, b^1, b^2, b^3\}$, and let Y be the set of copies of $\{a^4, a^5, a^6, b^4, b^5, b^6, y\}$. Indeed, for clarity we keep these names as in Fig. 1, without subscripts.

The only vertices of X that can be dominated by vertices outside X are x, a^3, and b^3. Hence if $|X \cap S| \leq 1$, then one vertex must dominate $\{a^1, a^2, b^1, b^2\}$, which is impossible. Since the subgraphs of F_i induced by X and Y are isomorphic, we conclude $|X \cap S| = |Y \cap S| = 2$.

Since S cannot contain $\{a^2, b^2\}$ or $\{a^5, b^5\}$, it must contain a vertex in $\{a^3, b^3, a^4, b^4\}$. By symmetry, we may assume $a^4 \in S$, and then $a^3, b^3 \notin S$. Dominating b^4 now requires b^4 or b^5 in S, which leaves no vertex available to dominate a^6, since $|Y \cap S| = 2$. □

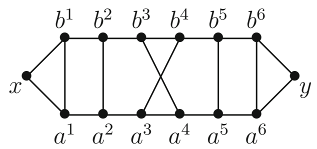

Fig. 1 The graph F

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