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ORIGINAL PAPER

# Long Paths and Cycles Passing Through Specified Vertices Under the Average Degree Condition

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**Abstract** Let $G$ be a $k$-connected graph with $k \geq 2$. In this paper we first prove that: For two distinct vertices $x$ and $z$ in $G$, it contains a path connecting $x$ and $z$ which passes through its any $k-2$ specified vertices with length at least the average degree of the vertices other than $x$ and $z$. Further, with this result, we prove that: If $G$ has $n$ vertices and $m$ edges, then it contains a cycle of length at least $2m/(n-1)$ passing through its any $k-1$ specified vertices. Our results generalize a theorem of Fan on the existence of long paths and a classical theorem of Erdős and Gallai on the existence of long cycles under the average degree condition.

**Keywords** Long path · Long cycle · Average degree

## 1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple graphs only.

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Let $G$ be a graph and $H$ a subgraph of $G$. We use $V(H)$ and $E(H)$ to denote the set of vertices and edges of $H$, respectively, and use $e(H)$ for the number of the edges of $H$. For a vertex $v \in V(G)$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of neighbors of $v$ in $H$. We call $d_H(v)$ the *degree* of $v$ in $H$. Let $x$ and $z$ be two distinct vertices of $G$. A path connecting $x$ and $z$ is called an $(x, z)$-*path*. For a subset $Y$ of $V(G)$, an $(x, z)$-path passing through all the vertices in $Y$ is called an $(x, Y, z)$-*path*, and a cycle passing through all the vertices in $Y$ is called a $Y$-*cycle*. If $Y$ contains only one vertex $y$, an $(x, \{y\}, z)$-path and a $\{y\}$-cycle are simply denoted by an $(x, y, z)$-path and a $y$-cycle, respectively. The *distance* between $x$ and $z$ in $H$, denoted by $d_H(x, z)$, is the length of a shortest $(x, z)$-path with all its internal vertices in $H$. If no such a path exists, we define $d_H(x, z) = \infty$. The *codistance* between $x$ and $z$ in $H$, denoted by $d^*_H(x, z)$, is the length of a longest $(x, z)$-path with all its internal vertices in $H$. If no such a path exists, we define $d^*_H(x, z) = 0$. We remark that in the definitions of $d_H(x, z)$ and $d^*_H(x, z)$, the vertices $x$ and $z$ is not necessarily in $H$. When no confusion occurs, we use $N(v)$, $d(v)$, $d(x, z)$ and $d^*(x, z)$ instead of $N_G(v)$, $d_G(v)$, $d_G(x, z)$ and $d^*_G(x, z)$, respectively.

Long path and cycle problems are interesting and important in graph theory and have been deeply studied, see [1,7]. The following theorem by Erdős and Gallai [5] opened the study on long paths with specified end vertices.

**Theorem 1** (Erdős and Gallai [5]) *Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If $d(v) \geq d$ for every vertex $v \in V(G) \backslash \{x, z\}$, then $G$ contains an $(x, z)$-path of length at least $d$.*

Theorem 1 has a stronger extension due to Enomoto [4].

**Theorem 2** (Enomoto [4]) *Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If $d(v) \geq d$ for every vertex $v \in V(G) \backslash \{x, z\}$, then for every given vertex $y \in V(G) \backslash \{x, z\}$, $G$ contains an $(x, y, z)$-path of length at least $d$.*

Another direction of extending Theorem 1 is to weaken the minimum degree condition to the average degree condition. Fan [6] finished this work as follows.

**Theorem 3** (Fan [6]) *Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If the average degree of the vertices other than $x$ and $z$ is at least $r$, then $G$ contains an $(x, z)$-path of length at least $r$.*

The following graph shows that one cannot replace the minimum degree condition in Theorem 2 by the average degree condition. Let $H$ be the complete graph on $n - 1$ vertices and $x, z \in V(H)$, and $G$ be the graph obtained from $H$ by adding a new vertex $y$ and two edges $xy$, $yz$. Then the length of the longest $(x, y, z)$-path in $G$ is 2, less than the average degree of the vertices other than $x$ and $z$ when $n \geq 5$.

Our first result in this paper is a generalization of Theorem 3.

**Theorem 4** *Let $G$ be a $k$-connected graph with $k \geq 2$, and $x$ and $z$ be two distinct vertices of $G$. If the average degree of the vertices other than $x$ and $z$ is at least $r$, then for any subset $Y$ of $V(G)$ with $|Y| = k - 2$, $G$ contains an $(x, Y, z)$-path of length at least $r$.*

We remark here that the size of $Y$ cannot be replaced by $k-1$. Let $H$ be a complete graph on $n-k+1$ vertices with $n>3k$ and $u_1=x, u_2, \ldots, u_k=z$ be $k$ vertices of $H$, and $Y=\{y_1, y_2, \ldots, y_{k-1}\}$ be a set of vertices not in $V(H)$. We construct a graph $G$ with $V(G)=V(H)\cup Y$ and $E(G)=E(H)\cup\{u_i y_j : 1\le i\le k, 1\le j\le k-1\}$. Then $G$ is a $k$-connected graph and the longest $(x, Y, z)$-path has length $2k-1$, which is less than

$$\frac{\sum_{v\in V(G)\setminus\{x,z\}} d(v)}{n-2} = \frac{(k-1)k+(k-2)(n-1)+(n-2k+1)(n-k)}{n-2} = \frac{n^2-2kn+n+3k^2-3k}{n-2}.$$

Besides, the complete graph $K_n$ with $n\ge k+1$ shows that the bound $r$ on the length of the $(x, Y, z)$-path is sharp.

There also exist results on long cycles passing through specified vertices in graphs. Theorem 5 shows the existence of long cycles in 2-connected graph under the minimum degree condition, and Theorem 6 extends Theorem 5 to graphs with higher connectivity.

**Theorem 5** (Locke [8]) *Let $G$ be a 2-connected graph. If the minimum degree of $G$ is at least $d$, then for any two vertices $y_1$ and $y_2$ of $G$, $G$ contains either a $\{y_1, y_2\}$-cycle of length at least $2d$ or a Hamilton cycle.*

**Theorem 6** (Egawa et al. [3]) *Let $G$ be a $k$-connected graph with $k\ge 2$. If the minimum degree of $G$ is at least $d$, then for any subset $Y$ of $V(G)$ with $|Y|=k$, $G$ contains either a $Y$-cycle of length at least $2d$ or a Hamilton cycle.*

On the existence of long cycles in graphs with a given number of edges, Erdős and Gallai [5] gave the following result.

**Theorem 7** (Erdős and Gallai [5]) *Let $G$ be a 2-edge-connected graph on $n$ vertices. Then $G$ contains a cycle of length at least $2e(G)/(n-1)$.*

In this paper, as an application of Theorem 4, we give the following theorem on long cycles passing through specified vertices of graphs with a given number of vertices and edges.

**Theorem 8** *Let $G$ be a $k$-connected graph on $n$ vertices with $k\ge 2$. Then for any subset $Y$ of $V(G)$ with $|Y|=k-1$, $G$ contains a $Y$-cycle of length at least $2e(G)/(n-1)$.*

In Theorem 8, one cannot expect a cycle passing through $k$ specified vertices of length at least $2e(G)/(n-1)$. Let $H$ be a complete graph on $n-k$ vertices with $n>3k$ and $u_1, u_2, \ldots, u_k$ be $k$ vertices of $H$, and $Y=\{v_1, v_2, \ldots, v_k\}$ be a set of vertices not in $V(H)$. We construct a graph $G$ with $V(G)=V(H)\cup Y$ and $E(G)=E(H)\cup\{u_i v_j : 1\le i, j\le k\}$. Then $G$ is a $k$-connected graph and the longest $Y$-cycle has length $2k$, which is less than

$$\frac{2e(G)}{n-1} = \frac{(n-k)(n-k-1)+2k^2}{n-1}.$$

In the following section we will give some further notations and preliminary results that will be used later. The proofs of Theorems 4 and 8 are given in Sects. 3 and 4, respectively.

## 2 Preliminaries

Let $G$ be a graph and $P$, $H$ two disjoint subgraphs of $G$. We use $E(P, H)$ to denote the set, and $e(P, H)$ the number, of edges with one vertex in $P$ and the other in $H$. If $E(P, H) \neq \emptyset$, then we call $P$ and $H$ are *joined*. We use $N_P(H)$ to denote the set of vertices in $P$ which are joined to $H$. If $x$ is a vertex in $G - P$, we say that $x$ is *locally k-connected* to $P$ (in $G$) if there are $k$ paths connecting $x$ and vertices in $P$ such that any two of them have only the vertex $x$ in common. We say that $H$ is *locally k-connected* to $P$ (in $G$) if for every vertex $x \in V(H)$, $x$ is locally $k$-connected to $P$. Note that if $H$ is locally $k$-connected to $P$, then $H$ is locally $l$-connected to $P$ for all $l$, $0 \leq l \leq k$; and, if $G$ is $k$-connected and $|V(P)| \geq k$, then $H$ is locally $k$-connected to $P$ in $G$.

The following propositions on local $k$-connectedness are proved in [6].

**Proposition 1** (Fan [6]) *Let $H$ and $P$ be two disjoint subgraphs of a graph $G$. If $H$ is locally $k$-connected to $P$ in the subgraph induced by $V(H) \cup V(P)$, then $E(P, H)$ contains an independent set of $t$ edges, where $t \geq \min\{k, |V(H)|\}$.*

**Proposition 2** (Fan [6]) *Let $H$ and $P$ be two disjoint subgraphs of a graph $G$. Let $u \in N_P(H)$ and $G'$ be the graph obtained from $G$ by deleting all edges from $u$ to $H$. If $H$ is locally $k$-connected to $P$ in $G$, then $H$ is locally $(k-1)$-connected to $P$ in $G'$.*

**Proposition 3** (Fan [6]) *Let $H$ and $P$ be two disjoint subgraphs of a graph $G$, and $B$ a block of $H$. Let $H'$ be the subgraph obtained from $H$ by contracting $B$. If $H$ is locally $k$-connected to $P$ in $G$, then $H'$ is also locally $k$-connected to $P$ in the resulting graph.*

Next we introduce the concept of local maximality for paths.

Let $P$ be a path of a graph $G$, and $u, v \in V(P)$. We use $P[u, v]$ to denote the segment of $P$ from $u$ to $v$, and $P(u, v)$ the segment obtained from $P[u, v]$ by deleting the two end vertices $u$ and $v$. Let $H$ be a component of $G - P$. We say that $P$ is a *locally longest path with respect to $H$* if we cannot obtain a longer path than $P$ by replacing the segment $P[u, v]$ by a $(u, v)$-path with all its internal vertices in $H$ for any $u, v \in V(G)$. In other words, $P$ is locally longest with respect to $H$ if, for any $u, v \in V(P)$,

$$e(P[u, v]) \geq d^*_H(u, v).$$

If $P$ is an $(x, Y, z)$-path of $G$, where $x, z \in V(G)$ and $Y \subset V(G)$, then we say that $P$ is a *locally longest $(x, Y, z)$-path with respect to $H$* if we cannot obtain an $(x, Y, z)$-path longer than $P$ by replacing the segment $P[u, v]$ with $Y \cap V(P(u, v)) = \emptyset$ by a $(u, v)$-path with all its internal vertices in $H$. Note that if $P$ is a longest path [longest $(x, Y, z)$-path] in a graph $G$, then, of course, $P$ is a locally longest path [locally longest $(x, Y, z)$-path] with respect to any component of $G - P$. If two vertices $u$ and $u'$ in $V(P)$ are joined to $H$ by two independent edges, then we call $\{u, u'\}$ a *strong attached pair* of $H$ to $P$. A *strong attachment* of $H$ to $P$ (in $G$) is a subset $T = \{u_1, u_2, \dots, u_t\} \subset N_P(H)$, where $u_i$, $1 \le i \le t$, are in order along $P$, such that each ordered pair $\{u_i, u_{i+1}\}$, $1 \le i \le t-1$, is a strong attached pair of $H$ to $P$. A strong attachment $T$ of $H$ to $P$ is *maximum* if $T$ has maximum cardinality over all strong attachments of $H$ to $P$.

The following result due to Fan is useful in our proofs.

**Lemma 1** (Fan [6]) *Let $G$ be a graph and $P$ an $(x, z)$-path of $G$. Suppose that $H$ is a component of $G - P$ and $T = \{u_1, u_2, \dots, u_t\}$ is a maximum strong attachment of $H$ to $P$. Set $S = N_P(H) \backslash T$. Then the following statements are true:*

(1) *Every vertex in $S$ is adjacent to exactly one vertex in $H$.*
(2) *For each segment $P[u_i, u_{i+1}]$, $1 \le i \le t-1$, suppose that*

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \dots, a_q, a_{q+1}\},$$

*where $a_0 = u_i$, $a_{q+1} = u_{i+1}$ and $a_j$, $0 \le j \le q+1$, are in order along $P$. Then there is a subscript $m$, $0 \le m \le q$, such that*

$$N_H(a_j) = N_H(a_0), \quad \textit{for } 0 \le j \le m,$$

*and*

$$N_H(a_j) = N_H(a_{q+1}), \quad \textit{for } m+1 \le j \le q+1.$$

*Besides, if*

$$N_P(H) \cap V(P[x, u_1]) = \{a_1, \dots, a_q, a_{q+1}\},$$

*where, $a_{q+1} = u_1$, then*

$$N_H(a_j) = N_H(a_{q+1}), \quad \textit{for } 1 \le j \le q+1;$$

*and if*

$$N_P(H) \cap V(P[u_t, z]) = \{a_0, a_1, \dots, a_q\},$$

*where, $a_0 = u_t$, then*

$$N_H(a_j) = N_H(a_0), \quad \textit{for } 0 \le j \le q.$$

(3) *If $H$ is locally $k$-connected to $P$ in $G$, then*

$$t \ge \min\{k, h + d_2\},$$

*where $h = |V(H)|$ and $d_2$ is the number of vertices in $N_P(H)$ each of which has at least two neighbors in $H$.*

Lemma 1 (2) is somewhat different from that in [6], but the proofs of them are similar.

For a path $P$, we use $l(P)$ to denote the length of $P$.

**Lemma 2** *Let $G$ be a graph, $P$ an $(x, Y, z)$-path of $G$, where $x, z \in V(G)$ and $Y \subset V(G)$, $H$ a component of $G - P$ and $T = \{u_1, u_2, \ldots, u_t\}$ a maximum strong attachment of $H$ to $P$. Set $S = N_P(H)\backslash T$ and $s = |S|$. Suppose that $P$ is a locally longest $(x, Y, z)$-path with respect to $H$, and $\theta = |\{x, z\} \cap N_P(H)|$. Set*

$$T_r = \{u_i \in T\backslash\{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\} \quad \text{and} \quad t_r = |T_r|.$$

*Then*

$$l(P) \geq \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta.$$

*Proof* If $t = 0$, then $s = 0$ and the statement is trivially true. Suppose now that $t \geq 1$.

Consider a segment $P[u_i, u_{i+1}]$, $1 \leq i \leq t - 1$. Suppose that

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \ldots, a_q, a_{q+1}\},$$

where $q = |S \cap V(P[u_i, u_{i+1}])|$, $a_0 = u_i$, $a_{q+1} = u_{i+1}$, and $a_j$, $0 \leq j \leq q + 1$, are in order along $P$.

If $Y \cap V(P(u_i, u_{i+1})) = \emptyset$, then by Lemma 1 (2), there is a subscript $m$, $0 \leq m \leq q$, such that

$$N_H(a_0) = N_H(a_m) \quad \text{and} \quad N_H(a_{q+1}) = N_H(a_{m+1}).$$

Therefore

$$d_H^*(a_m, a_{m+1}) = d_H^*(a_0, a_{q+1}) = d_H^*(u_i, u_{i+1}).$$

Since $P$ is a locally longest $(x, Y, z)$-path with respect to $H$, we have

$$\begin{aligned} l(P[u_i, u_{i+1}]) &\geq \sum_{j=0}^{q} d_H^*(a_j, a_{j+1}) = d_H^*(a_m, a_{m+1}) + \sum_{\substack{j=0 \\ j \neq m}}^{q} d_H^*(a_j, a_{j+1}) \\ &= d_H^*(u_i, u_{i+1}) + \sum_{\substack{j=0 \\ j \neq m}}^{q} d_H^*(a_j, a_{j+1}). \end{aligned}$$

Note that $d_H^*(a_j, a_{j+1}) \geq 2$, for every $j$, $0 \leq j \leq q$, we have

$$l(P[u_i, u_{i+1}]) \geq d_H^*(u_i, u_{i+1}) + 2q.$$

If $Y \cap V(P(u_i, u_{i+1})) \neq \emptyset$, then noting that $l(P[a_j, a_{j+1}]) \geq 2$, we have

$$l(P[u_i, u_{i+1}]) = \sum_{j=0}^{q} l(P[a_j, a_{j+1}]) \geq 2q + 2.$$

Besides, consider the two segments $P[x, u_1]$ and $P[u_t, z]$. Suppose that

$$N_P(H) \cap V(P[x, u_1]) = \{a_0, a_1, \ldots, a_m\}$$

and

$$N_P(H) \cap V(P[u_t, z]) = \{a_{m+1}, a_{m+2}, \ldots, a_{q+1}\},$$

where $m = |S \cap V(P[x, u_1])|, q - m = |S \cap V(P[u_t, z])|, a_m = u_1, a_{m+1} = u_t$, and $a_j, 0 \leq j \leq q+1$ are in order along $P$. Note that $l(P[x, a_0]) + l(P[a_{q+1}, z]) \geq 2 - \theta$ and $l(P[a_j, a_{j+1}]) \geq 2$, for every $0 \leq j \leq q$, and $j \neq m$, we have

$$l(P[x, u_1]) + l(P[u_t, z]) \geq 2q + 2 - \theta.$$

Thus summing over the lengths of all the segments, yields

$$\begin{aligned} l(P) &= l(P[x, u_1]) + \sum_{i=1}^{t-1} l(P[u_i, u_{i+1}]) + l(P[u_t, z]) \\ &\geq 2(|S \cap V(P[x, u_1])| + |S \cap V(P[u_t, z])|) + 2 - \theta \\ &\quad + \sum_{\substack{i=1 \\ u_i \in T_r}}^{t-1} (d_H^*(u_i, u_{i+1}) + 2|S \cap V(P[u_i, u_{i+1}])|) \\ &\quad + \sum_{\substack{i=1 \\ u_i \notin T_r}}^{t-1} (2|S \cap V(P[u_i, u_{i+1}])| + 2) \\ &= \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta. \end{aligned}$$

This ends the proof.

For a strong attachment $T = \{u_1, u_2, \ldots, u_t\}$, the pairs $\{u_j, u_{j+1}\}$, $1 \leq j \leq t-1$, are called strong attached pairs *supported by* $T$, and we call a strong attached pair $\{u_j, u_{j+1}\}$ of $H$ to $P$ *transitive* if $Y \cap V(P(u_j, u_{j+1})) = \emptyset$.

A connected graph is *separable* if it has at least one cut-vertex.

**Lemma 3** *Let $G$ be a graph and $P$ an $(x, z)$-path of $G$. Suppose that $H$ is a separable component of $G - P$, $B$ is an endblock of $H$, $b$ is the cut vertex of $H$ contained in $B$, and $M = B - b$. Let $T = \{u_1, u_2, \ldots, u_t\}$ be a maximum strong attachment of $H$ to $P$. If $H$ is locally $k$-connected to $P$, then*

(1) $|N_P(M) \cap T| \geq \min\{k-1, m+d_2'\}$*; and*
(2) *there exist at least* $\min\{k-1, m+d_2'\}$ *strong attached pairs supported by* $T$ *which are joined to* $M$*,*

*where* $m = |V(M)|$ *and* $d_2'$ *is the number of vertices in* $N_P(M)$ *each of which has at least two neighbors in* $H$*.*

*Proof* Since $H$ is locally $k$-connected to $P$, $|V(P)| \geq k$. It is easy to know that $M$ is locally $(k-1)$-connected to $P$ in the subgraph induced by $V(P) \cup V(M)$. By Proposition 1, there are $\min\{k-1, m\}$ independent edges in $E(P, M)$. Let $v_i w_i$, $1 \leq i \leq \min\{k-1, m\}$ be such edges, where $v_i \in V(P)$ and $w_i \in V(M)$.

If $v_i$ has at least two neighbors in $H$, then by Lemma 1 (1), $v_i \in T$. If $v_i$ has only one neighbor $w_i$ in $H$, then by Lemma 1 (2), there exists a vertex $v_i'$ (maybe equal to $v_i$) in $T$ which also has only one neighbor $w_i$ in $H$. This implies that $|N_P(M) \cap T| \geq \min\{k-1, m\}$.

Now, we prove (1) by induction on $d_2'$. If $d_2' = 0$, then by the analysis above, the assertion is true. Thus we assume that $d_2' \geq 1$.

Let $u_j$ be a vertex in $N_P(M)$ which has at least two neighbors in $H$ [$u_j$ is of course in $T$ by Lemma 1 (1)]. Let $G'$ be the graph obtained from $G$ by deleting all edges from $u_j$ to $H$. By Proposition 2, $H$ is locally $(k-1)$-connected to $P$ in $G'$.

If $u_j = u_1$ or $u_t$, or $\{u_{j-1}, u_{j+1}\}$ is joined to $H$ by two independent edges, then $T' = T \backslash \{u_j\}$ is a strong attachment of $H$ to $P$ in $G'$. Since $u_j$ is joined to at least two vertices of $H$ in $G$, any strong attachment of $H$ to $P$ in $G'$ together with $u_j$ is a strong attachment of $H$ to $P$ in $G$. Since $|T'| = t-1$, we see that $T'$ is a maximum strong attachment of $H$ to $P$ in $G'$. By the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k-2, m+d_2'-1\}.$$

Therefore

$$|N_P(M) \cap T| \geq \min\{k-1, m+d_2'\},$$

as required.

If $u_j \in \{u_2, \ldots, u_{t-1}\}$, and $\{u_{j-1}, u_{j+1}\}$ are not joined to $H$ by two independent edges, i.e.,

$$N_H(u_{j-1}) = N_H(u_{j+1}) = \{w\},$$

for some $w \in V(H)$, then

$$T' = T \backslash \{u_j, u_{j+1}\} = \{u_1, \ldots, u_{j-1}, u_{j+2}, \ldots, u_t\}$$

is a strong attachment of $H$ to $P$ in $G'$. We prove now that $T'$ is maximum by showing that any strong attachment of $H$ to $P$ in $G'$ has cardinality at most $t-2 = |T'|$.

Let $v_1, v_2$ ($\neq u_j$) be the two vertices in $N_P(H)$ which are closest to $u_j$ on $P$, say $v_1$ preceding, and $v_2$ following, $u_j$ on $P$ (but not necessarily adjacent to $u_j$ on $P$). Since $|N_H(u_j)| \geq 2$, it follows from Lemma 1 (2) that

$$N_H(v_1) = N_H(u_{j-1}) = \{w\} = N_H(u_{j+1}) = N_H(v_2).$$

By the choices of $v_1$ and $v_2$, for any maximum strong attachment $\{a_1, a_2, \ldots, a_p\}$ of $H$ to $P$ in $G'$, there is an integer $l$, $0 \le l \le p$, such that $v_1, v_2 \in V(P[a_l, a_{l+1}])$, where $a_0 = x$ and $a_{p+1} = z$. Since $N_H(v_1) = \{w\} = N_H(v_2)$, it follows from Lemma 1 (2) that either $N_H(a_l)$ or $N_H(a_{l+1}) = \{w\}$. The former implies a strong attachment $\{a_1, \ldots, a_l, u_j, v_2, a_{l+1}, \ldots, a_p\}$, the latter a strong attachment $\{a_1, \ldots, a_l, v_1, u_j, a_{l+1}, \ldots, a_p\}$, of $H$ to $P$ in $G$; in either case we have that $p+2 \le t$, that is, $p \le t - 2 = |T'|$. This shows that $T'$ is a maximum strong attachment of $H$ to $P$ in $G'$, as claimed. As before, by the induction hypothesis,

$$|N_P(M) \cap T'| \ge \min\{k-2, m+d_2'-1\}.$$

Consequently

$$|N_P(M) \cap T| \ge \min\{k-1, m+d_2'\},$$

which completes the proof of (1).

Now we prove (2). Clearly for every vertex $u_j \in N_P(M) \cap T\backslash\{u_t\}$, the strong attached pair $\{u_j, u_{j+1}\}$ supported by $T$ is joined to $M$. If $|N_P(M) \cap T\backslash\{u_t\}| \ge \min\{k-1, m+d_2'\}$, then the assertion is true. By (1), we assume that $|N_P(M) \cap T| = \min\{k-1, m+d_2'\}$ and $u_t \in N_P(M) \cap T$.

By Lemma 1 (3), $t \ge \min\{k, h+d_2\} \ge \min\{k-1, m+d_2'\}+1$. This implies that there exists at least one vertex in $T\backslash N_P(M)$. We choose a vertex $u_i \in T\backslash N_P(M)$ such that $u_{i+1} \in N_P(M) \cap T$. Then $\{u_i, u_{i+1}\}$ together with $\{u_j, u_{j+1}\}$ for $u_j \in N_P(M) \cup T\backslash\{u_t\}$ are $\min\{k-1, m+d_2'\}$ strong attached pairs supported by $T$ joined to $M$.

Let $P, H, B, M$ be defined as in Lemma 3. In the following, we call a strong attached pair which is joined to $M$ a *good pair* (with respect to $M$). Let $\{u_j, u_{j+1}\}$ be a strong attached pair. If one of the vertices in $\{u_j, u_{j+1}\}$ is joined to $M$, and the other to $H - M$, then we call it a *better pair* (with respect to $M$); and if one of the vertices in $\{u_j, u_{j+1}\}$ is joined to $M$, and the other to $H - B$, then we call it a *best pair* (with respect to $M$).

## 3 Proof of Theorem 4

If $k = 2$, then the assertion is Theorem 3. So we assume that $k \ge 3$. Since $G$ is $k$-connected and $|Y| = k-2$, $G$ contains an $(x, Y, z)$-path. In order to prove the theorem, we choose a longest $(x, Y, z)$-path $P$ in $G$. Clearly $|V(P)| \ge |Y| + 2 = k$. Moreover, by the $k$-connectedness of $G$, for each component $H$ of $G - P$, $H$ is locally $k$-connected to $P$, and $P$ is a locally longest $(x, Y, z)$-path with respect to $H$. So it is sufficient to prove that:

**Proposition 4** *Let $G$ be a graph, $P$ an $(x, Y, z)$-path of $G$, where $x, z \in V(G)$, $Y \subset V(G)$, and $|Y| = k-2$. Suppose that the average degree of vertices in $V(G)\backslash\{x, z\}$ is $r$. If for each component $H$ of $G - P$, $H$ is locally $k$-connected to $P$, and $P$ is a locally longest $(x, Y, z)$-path with respect to $H$, then $l(P) \ge r$.*

*Proof* We prove this proposition by induction on $|V(G-P)|$. If $V(G-P)=\emptyset$, noting that $r \le |V(G)|-1$, the result is trivially true. So we assume that $V(G-P)\neq\emptyset$. Let $H$ be a component of $G-P$.

Let $d=|N_P(H)|$, $\theta=|\{x,z\}\cap N_P(H)|$ and $N_P(H)=\{v_1,v_2,\ldots,v_d\}$, where $v_i$, $1\le i\le d$, are in order along $P$. Then, we have

$$l(P)=l(P[x,v_1])+\sum_{i=1}^{d-1} l(P[v_i,v_{i+1}])+l(P[v_d,z]).$$

It is easy to know that $l(P[x,v_1])+l(P[v_d,z])\ge 2-\theta$ and $l(P[v_i,v_{i+1}])\ge 2$ for $1\le i\le d-1$. Thus, we have

$$l(P)\ge 2d-\theta.$$

Note that $d\ge k$ by the local $k$-connectedness of $H$ to $P$ and clearly $\theta\le 2$. If $r\le 2k-2$, then we have $l(P)\ge 2k-2\ge r$, and the proof is complete. Thus we assume that

$$r>2k-2. \tag{1}$$

Besides, if $d\ge(r+\theta)/2$, then $l(P)\ge r$, and we complete the proof. Thus, we assume that

$$d<(r+\theta)/2. \tag{2}$$

Let $T=\{u_1,u_2,\ldots,u_t\}$ be a maximum strong attachment of $H$ to $P$. Set $S=N_P(H)\backslash T$ and $s=|S|$ (note that $s+t=d$). Let $T_r=\{u_i\in T\backslash\{u_t\}: Y\cap V(P(u_i,u_{i+1}))=\emptyset\}$ and $t_r=|T_r|$.

Clearly, for every transitive strong attached pair $\{u_j,u_{j+1}\}$, where $u_j\in T_r$, we have

$$d_H^*(u_j,u_{j+1})\ge 2. \tag{3}$$

We distinguish two cases:

*Case 1* $H$ is nonseparable.

Let $h=|V(H)|$ and $r'$ the average degree of vertices in $V(H)$. If $r'h+e(P-\{x,z\},H)\le rh$, then we consider the graph $G'$ obtained from $G$ by deleting the component $H$. Note that

$$\begin{aligned}\sum_{v\in V(G')\backslash\{x,z\}} d_{G'}(v) &= r(|V(G)|-2)-r'h-e(P-\{x,z\},H)\\ &\ge r(|V(G)|-2)-rh\\ &= r(|V(G')|-2).\end{aligned}$$

By the induction hypothesis, we have $l(P) \geq r$, and the proof is complete. Thus we assume that

$$r'h + e(P - \{x, z\}, H) > rh. \tag{4}$$

We use $d_1$ to denote the number of vertices in $N_P(H)$ which have only one neighbor in $V(H)$, $d_2 = d - d_1$, $\theta_1$ to denote the number of vertices in $\{x, z\}$ which have only one neighbor in $V(H)$ and $\theta_2 = \theta - \theta_1$.

Clearly,

$$r'h \leq h(h - 1 + d_2) + d_1 \quad \text{and} \quad e(P - \{x, z\}, H) \leq h(d_2 - \theta_2) + d_1 - \theta_1.$$

Thus, by (4), we have

$$h(h - 1 + 2d_2 - \theta_2) + 2d_1 - \theta_1 \geq r'h + e(P - \{x, z\}, H) > rh.$$

Note that $d_1 = d - d_2$ and $\theta_1 = \theta - \theta_2$, we have

$$h(h - 1 + 2d_2 - \theta_2) + 2d - 2d_2 - \theta + \theta_2 \geq rh.$$

By (2), we have

$$h(h - 1 + 2d_2 - \theta_2) + (r + \theta) - 2d_2 - \theta + \theta_2 > rh.$$

Thus,

$$(h - 1)(h + 2d_2 - r - \theta_2) > 0.$$

This implies that $h \geq 2$ and $h + 2d_2 > r + \theta_2 \geq r$, and then $2h + 2d_2 > r + 2$. By (1), we have $2h + 2d_2 > 2k$, that is

$$h + d_2 > k. \tag{5}$$

By (5) and Lemma 1 (3), $t \geq k$. Since $|Y| \leq k-2$, there exists at least one transitive strong attached pair $(u_p, u_{p+1})$ supported by $T$, where $u_p \in T_r$.

Let $G'$ be the subgraph induced by $V(H) \cup \{u_p, u_{p+1}\}$. If $u_p u_{p+1} \notin E(G)$, we add the edge $u_p u_{p+1}$ in $G'$. Thus $G'$ is 2-connected, and by (4),

$$\begin{aligned}\sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) &= \sum_{v \in V(H)} d(v) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H) \\ &= r'h - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H) \\ &\geq rh - e(P - \{x, z\}, H) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H).\end{aligned}$$

Note that

$$e(P-\{x,z\},H)\le(s+t-\theta)h,\quad \text{and}$$
$$e(N_P(H)\backslash\{u_p,u_{p+1}\},H)\le(s+t-2)h,$$

we have

$$\sum_{v\in V(G')\backslash\{u_p,u_{p+1}\}} d_{G'}(v)\ge rh-(s+t-\theta)h-(s+t-2)h$$
$$=(r-2s-2t+\theta+2)h.$$

By Theorem 3, $G'$ contains a $(u_p,u_{p+1})$-path of length at least $r-2s-2t+\theta+2$, which implies that

$$d^*_H(u_p,u_{p+1})\ge r-2s-2t+\theta+2. \tag{6}$$

Substituting (6) for $d^*_H(u_p,u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P)\ge(r-2s-2t+\theta+2)+2(t_r-1)+2(s+t-t_r)-\theta\ge r.$$

*Case 2* $H$ is separable.

Let $B$ be an endblock of $H$, $b$ the cut vertex of $H$ contained in $B$, $M=B-b$, $m=|V(M)|$, and $r''$ the average degree of the vertices in $V(M)$.

If $r''m+e(P-\{x,z\},M)+d_M(b)\le rm$, then we consider the graph $G'$ obtained from $G$ by contracting $B$. Let $H'$ be the component of $G'-P$ obtained from $H$ by contracting $B$. By Proposition 3, $H'$ is locally $k$-connected to $P$. Clearly $P$ is a locally longest $(x,Y,z)$-path with respect to $H'$, and

$$\sum_{v\in V(G')\backslash\{x,z\}} d_{G'}(v)\ge\sum_{v\in V(G)\backslash\{x,z\}} d(v)-r''m-e(P-\{x,z\},M)-d_M(b)$$
$$\ge r(|V(G)|-2)-rm$$
$$=r(|V(G')|-2).$$

By the induction hypothesis, $l(P)\ge r$, and the proof is complete. Thus we assume that

$$r''m+e(P-\{x,z\},M)+d_M(b)>rm. \tag{7}$$

Let $d'_0=|N_P(H)\backslash N_P(M)|$, $d'_1$ be the number of vertices in $N_P(M)$ which have only one neighbor in $V(H)$, $d'_2=d-d'_0-d'_1$; $\theta'_0=|\{x,z\}\cap N_P(H)\backslash N_P(M)|$, $\theta'_1$ be the number of vertices in $\{x,z\}\cap N_P(M)$ which have only one neighbor in $V(H)$ and $\theta'_2=\theta-\theta'_0-\theta'_1$.

Now we prove that

$$m + d'_2 \geq k - 1. \tag{8}$$

Let $B'$ be an endblock of $H$ other than $B$, $b'$ the cut vertex of $H$ contained in $B'$, $M' = B' - b'$ and $m' = |V(M')|$.

By the local $k$-connectedness of $H$ to $P$, $|N_P(M')| \geq k-1$. If $|N_P(M')\backslash N_P(M)| \leq m$, then $d'_2 \geq |N_P(M) \cap N_P(M')| \geq k - 1 - m$, and $m + d'_2 \geq k - 1$, and (8) holds. Thus we assume that $|N_P(M')\backslash N_P(M)| \geq m + 1$. So we have

$$d'_0 \geq m + 1. \tag{9}$$

Clearly,

$$\begin{aligned} &r''m \leq m(m + d'_2) + d'_1, \\ &e(P - \{x, z\}, M) \leq m(d'_2 - \theta'_2) + d'_1 - \theta'_1, \quad \text{and} \\ &d_M(b) \leq m. \end{aligned}$$

Thus, by (7),

$$m(m + 2d'_2 + 1 - \theta'_2) + 2d'_1 - \theta'_1 \geq r''m + e(P - \{x, z\}, M) + d_M(b) > rm.$$

Note that $d'_1 = d - d'_0 - d'_2$ and $\theta'_1 = \theta - \theta'_0 - \theta'_2$, we have

$$m(m + 2d'_2 + 1 - \theta'_2) + 2d - 2d'_0 - 2d'_2 - \theta + \theta'_0 + \theta'_2 > rm.$$

By (2) and (9), we have

$$m(m + 2d'_2 + 1 - \theta'_2) + (r + \theta) - 2(m + 1) - 2d'_2 - \theta + \theta'_0 + \theta'_2 > rm.$$

Thus,

$$(m - 1)(m + 2d'_2 - r - \theta'_2) > 2 - \theta'_0 \geq 0.$$

This implies that $m \geq 2$ and $m + 2d'_2 > r + \theta'_2 \geq r$, and then $2m + 2d'_2 > r + 2$. By (1), $2m + 2d'_2 > 2k$, that is $m + d'_2 > k$, and (8) holds.

By Lemma 3 (2), there exist at least $k - 1$ good pairs supported by $T$ with respect to $M$. Since $|Y| = k - 2$, there exists at least one transitive good pair $\{u_p, u_{p+1}\}$ with respect to $M$. Similarly there exists at least one transitive good pair $\{u_q, u_{q+1}\}$ with respect to $M'$.

First we assume that there is a transitive best pair supported by $T$ with respect to $M$ or $M'$. Without loss of generality, we assume that $\{u_p, u_{p+1}\}$ is a best pair, where $u_p \in N_P(M)$ and $u_{p+1} \in N_P(H - B)$. Consider the subgraph $G'$ induced by

$V(B)\cup\{u_p\}$. If $u_pb\notin E(G)$, we add the edge $u_pb$ in $G'$. Thus $G'$ is 2-connected, and by (7),

$$\begin{aligned}\sum_{v\in V(G')\backslash\{u_p,b\}} d_{G'}(v) &= \sum_{v\in V(M)} d(v) - e(N_P(H)\backslash\{u_p\},M)\\ &= r''m - e(N_P(H)\backslash\{u_p\},M)\\ &\ge rm - e(P-\{x,z\},M) - d_M(b) - e(N_P(H)\backslash\{u_p\},M).\end{aligned}$$

Note that

$$\begin{aligned}&e(P-\{x,z\},M)\le (s+t-\theta)m,\\ &d_M(b)\le m, \quad \text{and}\\ &e(N_P(H)\backslash\{u_p\},M)\le (s+t-1)m,\end{aligned}$$

we have

$$\begin{aligned}\sum_{v\in V(G')\backslash\{u_p,b\}} d_{G'}(v) &\ge rm-(s+t-\theta)m-m-(s+t-1)m\\ &=(r-2s-2t+\theta)m.\end{aligned}$$

By Theorem 3, $G'$ contains a $(u_p,b)$-path of length at least $r-2s-2t+\theta$. It is clear that there is a $(b,u_{p+1})$-path in $H-M$ of length at least 2, which implies that

$$d^*_H(u_p,u_{p+1})\ge r-2s-2t+\theta+2. \tag{10}$$

Substituting (10) for $d^*_H(u_p,u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P)\ge (r-2s-2t+\theta+2)+2(t_r-1)+2(s+t-t_r)-\theta\ge r,$$

as required.

So, we assume that there are no transitive best pairs supported by $T$ with respect to $M$ or $M'$.

Now we assume that there is a transitive better pair (but not best pair) supported by $T$ with respect to $M$ or $M'$. Without loss of generality, we assume that $\{u_p,u_{p+1}\}$ is a better pair, where $u_p\in N_P(M)$ and $u_{p+1}\in N_P(b)$. Consider the subgraph $G'$ induced by $V(B)\cup\{u_p\}$. If $u_pb\notin E(G)$, we add the edge $u_pb$ in $G'$. Thus $G'$ is 2-connected and

$$\sum_{v\in V(G')\backslash\{u_p,b\}} d_{G'}(v)\ge rm-e(P-\{x,z\},M)-d_M(b)-e(N_P(H)\backslash\{u_p\},M).$$

Note that

$$\begin{aligned}&e(P-\{x,z\},M)\le(s+t-\theta)m,\quad\text{and}\\&d_M(b)\le m,\end{aligned}$$

and since at least one vertex of $u_q$ and $u_{q+1}$ is not joined to $M$ [otherwise, $\{u_q, u_{q+1}\}$ will be a best pair], we have

$$e(N_P(H)\backslash\{u_p\},M)\le(s+t-2)m.$$

Thus we have

$$\begin{aligned}\sum_{v\in V(G')\backslash\{u_p,b\}} d_{G'}(v)&\ge rm-(s+t-\theta)m-m-(s+t-2)m\\&=(r-2s-2t+\theta+1)m.\end{aligned}$$

By Theorem 3, $G'$ contains a $(u_p, b)$-path of length at least $r-2s-2t+\theta+1$, and then, by $bu_{p+1}\in E(G)$,

$$d^*_H(u_p,u_{p+1})\ge r-2s-2t+\theta+2.$$

Thus we also have $l(P)\ge r$.

So, we assume that there are no transitive better pairs supported by $T$ with respect to $M$ or $M'$. Thus $\{u_p, u_{p+1}\}\cap\{u_q, u_{q+1}\}=\emptyset$, and $\{u_p, u_{p+1}\}$ and $\{u_q, u_{q+1}\}$ are two distinct strong attached pairs.

Note that $u_p$ and $u_{p+1}$ are joined to $M$ by two independent edges. Consider the subgraph $G'$ induced by $V(B)\cup\{u_p, u_{p+1}\}$. If $u_pu_{p+1}\notin E(G)$, we add the edge $u_pu_{p+1}$ in $G'$. Thus $G'$ is 2-connected, and by (7),

$$\begin{aligned}&\sum_{v\in V(G')\backslash\{u_p,u_{p+1}\}} d_{G'}(v)\\&=\sum_{v\in V(M)} d(v)-e(N_P(H)\backslash\{u_p,u_{p+1}\},M)+d_M(b)+|\{u_p,u_{p+1}\}\cap N_P(b)|\\&=r''m+d_M(b)-e(N_P(H)\backslash\{u_p,u_{p+1}\},M)+|\{u_p,u_{p+1}\}\cap N_P(b)|\\&\ge rm-e(P-\{x,z\},M)-e(N_P(H)\backslash\{u_p,u_{p+1}\},M).\end{aligned}$$

Note that

$$e(P-\{x,z\},M)\le(s+t-\theta)m,$$

and since neither $u_q$ and $u_{q+1}$ has neighbors in $M$ [otherwise $\{u_q, u_{q+1}\}$ will be a better pair], we have

$$e(N_P(H)\backslash\{u_p,u_{p+1}\},M)\le(s+t-4)m.$$

Thus, we have,

$$\sum_{v \in V(G')\backslash\{u_p, u_{p+1}\}} d_{G'}(v) \geq rm - (s+t-\theta)m - (s+t-4)m$$
$$= (r - 2s - 2t + \theta + 4)m.$$

By Theorem 3, $G'$ contains a $(u_p, u_{p+1})$-path of length at least $(r - 2s - 2t + \theta + 4)m/(1+m) \geq (r - 2s - 2t + \theta + 4)/2$ (note that $m \geq 1$), which implies that

$$d^*_H(u_p, u_{p+1}) \geq (r - 2s - 2t + \theta + 4)/2.$$

and similarly,

$$d^*_H(u_q, u_{q+1}) \geq (r - 2s - 2t + \theta + 4)/2.$$

Then,

$$d^*_H(u_p, u_{p+1}) + d^*_H(u_q, u_{q+1}) \geq r - 2s - 2t + \theta + 4.$$

Thus, by Lemma 2, we have

$$l(P) \geq (r - 2s - 2t + \theta + 4) + 2(t_r - 2) + 2(s + t - t_r) - \theta \geq r.$$

The proof is complete.

## 4 Proof of Theorem 8

By the $k$-connectedness of $G$, it contains a $Y$-cycle. If $2e(G)/(n-1) \leq 3$, then the result is trivially true. Thus we assume that $2e(G)/(n-1) > 3$.

We choose a vertex $y \in Y$, and construct a graph $G'$ such that $V(G') = V(G) \cup \{y'\}$, where $y' \notin V(G)$ and $E(G') = E(G) \cup \{vy' : v \in N_G(y)\}$. Clearly, $G'$ is $k$-connected. Besides, we have that

$$e(G') = e(G) + d_G(y) \quad \text{and} \quad d_{G'}(y) = d_{G'}(y') = d_G(y),$$

and the order of $G'$ is $n+1$. Now, by Theorem 4, there exists a $(y, Y\backslash\{y\}, y')$-path $P$ of length at least

$$\frac{2e(G') - d_{G'}(y) - d_{G'}(y')}{(n+1) - 2} = \frac{2(e(G) + d_G(y)) - 2d_G(y)}{n-1} = \frac{2e(G)}{n-1}.$$

Let $uy'$ be the last edge of $P$. Then $uy \in E(G)$ and $C = P[y, u]uy$ is a cycle of $G$ passing through all the vertices in $Y$ of length at least $2e(G)/(n-1)$, which completes the proof.

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