

Signed Roman k -Domination in Graphs

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Abstract Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A signed Roman k -dominating function (SRkDF) on a graph G is a function $f : V(G) \rightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$, where $N[v]$ is the closed neighborhood of v , and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an SRkDF f is $w(f) = \sum_{v \in V(G)} f(v)$. The signed Roman k -domination number $\gamma_{sR}^k(G)$ of G is the minimum weight of an SRkDF on G . In this paper we initiate the study of the signed Roman k -domination number of graphs, and we present different bounds on $\gamma_{sR}^k(G)$. In addition, we determine the signed Roman k -domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed Roman domination number $\gamma_{sR}(G) = \gamma_{sR}^1(G)$, introduced and investigated by Ahangar et al. (J Comb Optim doi:[10.1007/s10878-012-9500-0](https://doi.org/10.1007/s10878-012-9500-0), 2014).

Keywords Signed Roman k -dominating function · Signed Roman k -domination number

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1 Terminology and Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [6]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G , respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X . For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) = \bigcup_{v \in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. For disjoint subsets X and Y of vertices of a graph G , we denote by (X, Y) the set of edges between X and Y . The *complement* of a graph G is denoted by \overline{G} . For sets $A, B \subseteq V(G)$, we say that A *dominates* B if $B \subseteq N[A]$. Let K_n be the complete graph of order n and $K_{p,p}$ the complete bipartite graph of order $2p$.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T , the *parent* of v is the neighbor of v on the unique (r, v) -path, while a *child* of v is any other neighbor of v . A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u) -path contains v . We let $D(v)$ denote the set of descendants of v . A leaf of T is a vertex of degree 1, while a support vertex of T is a vertex adjacent to a leaf.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f: V(G) \rightarrow \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$. For a vertex v , we denote $f(N[v])$ by $f[v]$ for notational convenience.

If $k \geq 1$ is an integer, then the *signed Roman k -dominating function* (SRkDF) on a graph G is defined as a function $f: V(G) \rightarrow \{-1, 1, 2\}$ such that $f[v] \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$.

The weight of an SRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman k -domination number* $\gamma_{sR}^k(G)$ of G is the minimum weight of an SRkDF on G . The special case $k = 1$ was introduced and investigated by Ahangar et al. [1]. Sheikholeslami and Volkmann [8] studied the signed Roman domination number in digraphs.

A $\gamma_{sR}^k(G)$ -function is a signed Roman k -dominating function on G of weight $\gamma_{sR}^k(G)$. For an SRkDF f on G , let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed Roman k -dominating function $f: V(G) \rightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.

A *signed dominating function* (SDF) on a graph $G = (V, E)$ is a function $f: V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for every vertex $v \in V$. Thus a signed Roman k -dominating function combines the properties of both a Roman dominating function and a signed dominating function. The *signed domination number*, denoted $\gamma_s(G)$, is the minimum weight of an SDF in G . Signed domination in graphs is well studied in the literature, see for example, [2–4, 7, 9] and elsewhere.

The signed Roman k -domination number exists when $\delta \geq \frac{k}{2} - 1$. However, for investigations of the signed Roman k -dominating number it is reasonable to claim that $\delta(G) \geq k - 1$. Thus we assume throughout this paper that $\delta(G) \geq k - 1$. We present different bounds on $\gamma_{sR}^k(G)$. In addition, we determine the signed Roman k -domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed Roman domination number $\gamma_{sR}(G) = \gamma_{sR}^1(G)$, given by Ahangar et al. [1].

2 Preliminary Results

In this section we present basic properties of the signed Roman k -dominating functions and the signed Roman k -domination numbers. The definitions immediately lead to our first proposition.

Proposition 1 *If $f = (V_{-1}, V_1, V_2)$ is an SRkDF on a graph G of order n , then*

- (a) $|V_{-1}| + |V_1| + |V_2| = n$.
- (b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.
- (c) *Every vertex in V_{-1} is dominated by a vertex of V_2 .*
- (d) $V_1 \cup V_2$ is a dominating set of G .

Proposition 2 *Assume that $f = (V_{-1}, V_1, V_2)$ is an SRkDF on a graph G of order n . If $\delta(G) \geq k - 1$, then*

- (i) $(2\Delta + 2 - k)|V_2| + (\Delta + 1 - k)|V_1| \geq (\delta + k + 1)|V_{-1}|$.
- (ii) $(2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + k + 1)n$.
- (iii) $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|$.
- (iv) $\omega(f) \geq (\delta - 2\Delta + 2k - 1)n / (2\Delta + \delta + 3) + |V_2|$.

Proof (i) It follows from Proposition 1 (a) that

$$\begin{aligned} k(|V_{-1}| + |V_1| + |V_2|) &= kn \leq \sum_{v \in V(G)} f[v] = \sum_{v \in V(G)} (d_G(v) + 1) f(v) \\ &= \sum_{v \in V_2} 2(d_G(v) + 1) + \sum_{v \in V_1} (d_G(v) + 1) \\ &\quad - \sum_{v \in V_{-1}} (d_G(v) + 1) \\ &\leq 2(\Delta + 1)|V_2| + (\Delta + 1)|V_1| - (\delta + 1)|V_{-1}|. \end{aligned}$$

This inequality chain yields to the desired bound in (i).

- (ii) Proposition 1 (a) implies that $|V_{-1}| = n - |V_1| - |V_2|$. Using this identity and Part (i) of Proposition 2, we arrive at (ii).
- (iii) According to Proposition 1 and Part (ii) of Proposition 2, we obtain Part (iii) of Proposition 2 as follows

$$\begin{aligned}
 (\Delta + \delta + 2)\omega(f) &= (\Delta + \delta + 2)(2(|V_1| + |V_2|) - n + |V_2|) \\
 &\geq 2(\delta + k + 1)n - 2(\Delta + 1)|V_2| + (\Delta + \delta + 2)(|V_2| - n) \\
 &= (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|.
 \end{aligned}$$

(iv) The inequality chain in the proof of Part (i) and Proposition 1 (a) show that

$$\begin{aligned}
 kn &\leq 2(\Delta + 1)|V_1 \cup V_2| - (\delta + 1)|V_{-1}| \\
 &= 2(\Delta + 1)|V_1 \cup V_2| - (\delta + 1)(n - |V_1 \cup V_2|) \\
 &= (2\Delta + \delta + 3)|V_1 \cup V_2| - (\delta + 1)n
 \end{aligned}$$

and thus

$$|V_1 \cup V_2| \geq \frac{n(\delta + k + 1)}{2\Delta + \delta + 3}.$$

Using this inequality and Proposition 1, we obtain

$$\begin{aligned}
 \omega(f) &= 2|V_1 \cup V_2| - n + |V_2| \\
 &\geq \frac{n(2\delta + 2k + 2 - 2\Delta - \delta - 3)}{2\Delta + \delta + 3} + |V_2| \\
 &= \frac{n(\delta - 2\Delta + 2k - 1)}{2\Delta + \delta + 3} + |V_2|.
 \end{aligned}$$

This is the bound in Part (iv), and the proof is complete. □

3 Bounds on the Signed Roman k -Domination Number

We start with some simple upper bounds on the signed Roman k -domination number of a graph.

Theorem 3 *Let G be a graph of order n with $\delta(G) \geq k - 1$. Then $\gamma_{sR}^k(G) \leq n$. If $\delta(G) \geq k + 2t - 1$ for an integer $t \geq 1$, then $\gamma_{sR}^k(G) \leq n - t$.*

Proof Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ such that $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k - 1$, the function f is an SRkDF on G of weight n and thus $\gamma_{sR}^k(G) \leq n$.

Let now $\delta(G) \geq k + 2t - 1$ for an integer $t \geq 1$. Let $A = \{u_1, u_2, \dots, u_t\}$ be a set of t vertices in $V(G)$, and let $B = \{v_1, v_2, \dots, v_t\}$ be a set of t vertices such that $v_i \in V(G) - A$ and v_i is adjacent to u_i for $1 \leq i \leq t$. Define the function $g : V(G) \rightarrow \{-1, 1, 2\}$ such that $g(u_i) = -1$, $g(v_i) = 2$ for $1 \leq i \leq t$ and $g(x) = 1$ for $x \in V(G) - (A \cup B)$. Then $g[u_i] \geq k + 1$, $g[v_i] \geq k + 1$ for $1 \leq i \leq t$ and $g[x] \geq k$ for $x \in V(G) - (A \cup B)$. Therefore g is an SRkDF on G of weight $n - t$ and thus $\gamma_{sR}^k(G) \leq n - t$. □

As an application of Proposition 2 (iii), we obtain a lower bound on the signed Roman k -domination number for r -regular graphs.

Corollary 4 *If G is an r -regular graph of order n with $r \geq k - 1$, then*

$$\gamma_{sR}^k(G) \geq \frac{kn}{r + 1}.$$

Example 5 If H is a $(k - 1)$ -regular graph of order n , then it follows from Corollary 4 that $\gamma_{sR}^k(H) \geq n$ and thus $\gamma_{sR}^k(H) = n$, according to Theorem 3.

Example 5 shows that Theorem 3 and Corollary 4 are both sharp.

Theorem 6 *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$\gamma_{sR}^k(G) \geq k + 1 + \Delta(G) - n.$$

Proof Let $w \in V(G)$ be a vertex of maximum degree, and let f be a $\gamma_{sR}^k(G)$ -function. Then the definitions imply

$$\begin{aligned} \gamma_{sR}^k(G) &= \sum_{x \in V(G)} f(x) = \sum_{x \in N[w]} f(x) + \sum_{x \in V(G) - N[w]} f(x) \\ &\geq k + \sum_{x \in V(G) - N[w]} f(x) \geq k - (n - (\Delta(G) + 1)) = k + 1 + \Delta(G) - n, \end{aligned}$$

and the proof of the desired lower bound is complete. □

In Ahangar et al. [1] it was shown that $\gamma_{sR}(K_3) = 2$ and $\gamma_{sR}(K_n) = 1$ for $n \neq 3$. In the next example we determine $\gamma_{sR}^k(K_n)$ for $k \geq 2$.

Example 7 If $n \geq k \geq 2$ are integers, then $\gamma_{sR}^k(K_n) = k$.

Proof Corollary 4 implies that $\gamma_{sR}^k(K_n) \geq k$. If $n = k$, then it follows from Theorem 3 that $\gamma_{sR}^k(K_n) \leq n = k$ and thus the desired result in that case. Now let $n \geq k + 1$. We distinguish two cases.

Case 1. Assume that $n - k$ is even. Let the function $f : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to two vertices the value 2, to $(n + k - 6)/2$ vertices the value 1 and to $(n - k + 2)/2$ vertices the value -1 . Then f is an SRkDF on K_n of weight $\omega(f) = k$, and so $\gamma_{sR}^k(K_n) \leq k$. Consequently, $\gamma_{sR}^k(K_n) = k$ in that case.

Case 2. Assume that $n - k$ is odd. Let the function $f : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to one vertex the value 2, to $(n + k - 3)/2$ vertices the value 1 and to $(n - k + 1)/2$ vertices the value -1 . Then f is an SRkDF on K_n of weight $\omega(f) = k$, and so $\gamma_{sR}^k(K_n) \leq k$. Consequently, $\gamma_{sR}^k(K_n) = k$ in that case. □

Example 7 shows that Theorem 6 is sharp.

Corollary 8 *Let G be a graph of order n , minimum degree $\delta \geq k - 1$ and maximum degree Δ . If $\delta < \Delta$, then*

$$\gamma_{sR}^k(G) \geq \left(\frac{-2\Delta^2 + 2\Delta\delta + 3k\Delta - 2\Delta + 2\delta + 3k}{(\Delta + 1)(2\Delta + \delta + 3)} \right) n.$$

Proof Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound. \square

Example 9 Case 1. Assume that $k = 2t + 1$ for an integer $t \geq 1$.

Let $A_1, A_2, A_3, B_1, B_2, B_3$ be isomorphic to the complete graph K_{2t-1} , and let H_1, H_2 be isomorphic to the complete graph K_{2t} . Let $A = A_1 \cup A_2 \cup A_3$, $B = B_1 \cup B_2 \cup B_3$ and $H = H_1 \cup H_2$. Now let G be the disjoint union of H, A and B such that each vertex of A is adjacent to each vertex of H_1 , each vertex of B is adjacent to each vertex of H_2 , and $G[V(H)]$ is $(4t - 2)$ -regular. Then $\delta(G) = 4t - 2$, $\Delta(G) = 10t - 5$ and $n(G) = 16t - 6$.

Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in V(A \cup B)$ and $f(x) = 2$ for $x \in V(H)$. It is easy to verify that $f[x] = k$ for each vertex $x \in V(G)$. Therefore f is an SRkDF on G of weight $\omega(f) = 6 - 4t$, and so $\gamma_{SR}^k(G) \leq 6 - 4t$. Corollary 8 implies $\gamma_{SR}^k(G) \geq 6 - 4t$ and thus $\gamma_{SR}^k(G) = 6 - 4t$.

Case 2. Assume that $k = 4t$ for an integer $t \geq 1$.

Let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ be isomorphic to the complete graph K_{2t} , and let H_1, H_2 be isomorphic to the complete graph K_{3t} . Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$, $B = B_1 \cup B_2 \cup B_3 \cup B_4$ and $H = H_1 \cup H_2$. Now let G be the disjoint union of H, A and B such that each vertex of A is adjacent to each vertex of H_1 , each vertex of B is adjacent to each vertex of H_2 , and each vertex of H_1 is adjacent to each vertex of H_2 . Then $\delta(G) = 5t - 1$, $\Delta(G) = 14t - 1$ and $n(G) = 22t$.

Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in V(A \cup B)$ and $f(x) = 2$ for $x \in V(H)$. It is easy to verify that $f[x] = k$ for each vertex $x \in V(G)$. Therefore f is an SRkDF on G of weight $\omega(f) = -4t$, and so $\gamma_{SR}^k(G) \leq -4t$. Corollary 8 implies $\gamma_{SR}^k(G) \geq -4t$ and thus $\gamma_{SR}^k(G) = -4t$.

Case 3. Assume that $k = 4t + 2$ for an integer $t \geq 1$.

Let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ be isomorphic to the complete graph K_{2t} , and let H_1, H_2 be isomorphic to the complete graph K_{3t+1} . Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$, $B = B_1 \cup B_2 \cup B_3 \cup B_4$ and $H = H_1 \cup H_2$. Now let G be the disjoint union of H, A and B such that each vertex of A is adjacent to each vertex of H_1 , each vertex of B is adjacent to each vertex of H_2 , and $G[V(H)]$ is $(6t)$ -regular. Then $\delta(G) = 5t$, $\Delta(G) = 14t$ and $n(G) = 22t + 2$.

Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in V(A \cup B)$ and $f(x) = 2$ for $x \in V(H)$. It is easy to verify that $f[x] = k$ for each vertex $x \in V(G)$. Therefore f is an SRkDF on G of weight $\omega(f) = 4 - 4t$, and so $\gamma_{SR}^k(G) \leq 4 - 4t$. Corollary 8 implies $\gamma_{SR}^k(G) \geq 4 - 4t$ and thus $\gamma_{SR}^k(G) = 4 - 4t$.

Example 9 demonstrates that Corollary 8 is sharp for $k \geq 3$. The next example shows that Corollary 8 is sharp for $k = 1$ and $k = 2$ too.

Example 10 Case 1. Let $\{x_1, x_2, x_3, y_1, y_2, y_3, x, y\}$ be the vertex set of the graph G such that x_i is adjacent to x , y_i is adjacent to y for $1 \leq i \leq 3$ and x is adjacent to y .

Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x_i) = f(y_i) = -1$ for $1 \leq i \leq 3$ and $f(x) = f(y) = 2$. Then $f[u] = 1$ for each vertex $u \in V(G)$.

Therefore f is an SR1DF on G of weight $\omega(f) = -2$, and so $\gamma_{sR}^1(G) \leq -2$. Corollary 8 implies $\gamma_{sR}^1(G) \geq -2$ and thus $\gamma_{sR}^1(G) = \gamma_{sR}(G) = -2$.

Case 2. Let $A_1, A_2, A_3, B_1, B_2, B_3, H_1, H_2$ be isomorphic to the complete graph K_2 . Let $A = A_1 \cup A_2 \cup A_3, B = B_1 \cup B_2 \cup B_3$ and $H = H_1 \cup H_2$. Now let G be the disjoint union of H, A and B such that each vertex of A is adjacent to each vertex of H_1 , each vertex of B is adjacent to each vertex of H_2 , and each vertex of H_1 is adjacent to each vertex of H_2 . Then $\delta(G) = 3, \Delta(G) = 9$ and $n(G) = 16$.

Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in V(A \cup B)$ and $f(x) = 2$ for $x \in V(H)$. Then $f[x] = 2$ for each vertex $x \in V(G)$. Therefore f is an SR2DF on G of weight $\omega(f) = -4$, and so $\gamma_{sR}^2(G) \leq -4$. Corollary 8 implies $\gamma_{sR}^2(G) \geq -4$ and thus $\gamma_{sR}^2(G) = -4$.

The special case $k = 1$ of Proposition 2 and Corollaries 4, 8 can be found in Ahangar et al. [1].

A set $S \subseteq V(G)$ is a 2-packing of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2-packing number $\rho(G)$ of G is defined by

$$\rho(G) = \max \{|S| : S \text{ is a } 2\text{-packing of } G\}.$$

Theorem 11 *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$\gamma_{sR}^k(G) \geq \rho(G) (k + \delta(G) + 1) - n.$$

Proof Let $\{v_1, v_2, \dots, v_{\rho(G)}\}$ be a 2-packing of G , and let f be a $\gamma_{sR}^k(G)$ -function. If we define the set $A = \cup_{i=1}^{\rho(G)} N[v_i]$, then since $\{v_1, v_2, \dots, v_{\rho(G)}\}$ is a 2-packing, we have that

$$|A| = \sum_{i=1}^{\rho(G)} (d(v_i) + 1) \geq \rho(G) (\delta(G) + 1).$$

It follows that

$$\begin{aligned} \gamma_{sR}^k(G) &= \sum_{x \in V(G)} f(x) = \sum_{i=1}^{\rho(G)} f[v_i] + \sum_{x \in V(G)-A} f(x) \\ &\geq k\rho(G) + \sum_{x \in V(G)-A} f(x) \geq k\rho(G) - n + |A| \\ &\geq k\rho(G) - n + \rho(G) (\delta(G) + 1) = \rho(G) (k + \delta(G) + 1) - n. \end{aligned}$$

□

We show next that the bound in Theorem 11 is sharp. For integers $s \geq k \geq 2$ and $t \geq 1$, let G be a graph of order st obtained as follows: Let F be an arbitrary graph of order t and for each vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s and identify the vertex v with one vertex of the added complete graph. Let G

denote the resulting graph. Further, let G_1, G_2, \dots, G_t be the added copies of K_s . For $i = 1, 2, \dots, t$, let v_i be the vertex of G_i that is identified with a vertex of F . Thus, $G[\{v_1, v_2, \dots, v_t\}] \cong F$. We now construct a signed Roman k -dominating function on G as follows. For each $i = 1, 2, \dots, t$, let $f_i : V(G_i) \rightarrow \{-1, 1, 2\}$ be the SRkDF on the complete graph $G_i \cong K_s$ defined as in Example 7. We note that the function f_i assigns to at least one vertex of G_i the value 2. We choose v_i to be one such vertex of G_i , and so $f_i(v_i) = 2$. As shown in Example 7, we have $w(f_i) = k$. Now let $f : V(G) \rightarrow \{-1, 1, 2\}$ be the function defined by $f(v) = f_i(v)$ for each vertex $v \in V(G_i)$, and so $f = \bigcup_{i=1}^t f_i$. If $v = v_i$ for some $i, 1 \leq i \leq t$, then $f[v] \geq f_i[v]$ with strict inequality if the vertex corresponding to v_i is not isolated in F . If $v \neq v_i$ for any $i, 1 \leq i \leq t$, then $f[v] = f_i[v]$. Hence, the function f is an SRkDF of G , and so $\gamma_{sR}^k(G) \leq w(f) = \sum_{i=1}^t w(f_i) = tk$. However by Theorem 11, and noting that here $\delta(G) = s - 1, \rho(G) = t$, and $n(G) = st$, we have $\gamma_{sR}^k(G) \geq \rho(G)(k + \delta(G) + 1) - n(G) = tk$. Consequently, $\gamma_{sR}^k(G) = \rho(G)(k + \delta(G) + 1) - n(G) = tk$. Thus there is an infinite family of graphs achieving equality in Theorem 11.

Corollary 12 *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$\gamma_{sR}^k(G) \geq \left(1 + \left\lfloor \frac{\text{diam}(G)}{3} \right\rfloor \right) (k + \delta(G) + 1) - n.$$

Proof Let $d = \text{diam}(G) = 3t + r$ with integers $t \geq 0$ and $0 \leq r \leq 2$, and let $x_0x_1 \dots x_d$ be a diametral path. Then $A = \{x_0, x_3, \dots, x_{3t}\}$ is a 2-packing of G such that $|A| = 1 + \lfloor d/3 \rfloor$. Since $\rho(G) \geq |A|$, Theorem 11 implies that

$$\gamma_{sR}^k(G) \geq \rho(G)(k + \delta(G) + 1) - n \geq \left(1 + \left\lfloor \frac{\text{diam}(G)}{3} \right\rfloor \right) (k + \delta(G) + 1) - n,$$

and the proof is complete. □

Next we present a so called Nordhaus-Gaddum type inequality for the Roman k -domination number of regular graphs.

Theorem 13 *If G is an r -regular graph of order n such that $r \geq k - 1$ and $n - r - 1 \geq k - 1$, then*

$$\gamma_{sR}^k(G) + \gamma_{sR}^k(\overline{G}) \geq \frac{4kn}{n + 1}.$$

If n is even, then $\gamma_{sR}^k(G) + \gamma_{sR}^k(\overline{G}) \geq 4k(n + 1)/(n + 2)$.

Proof Since G is r -regular, the complement \overline{G} is $(n - r - 1)$ -regular. Therefore it follows from Corollary 4 that

$$\gamma_{sR}^k(G) + \gamma_{sR}^k(\overline{G}) \geq kn \left(\frac{1}{r + 1} + \frac{1}{n - r} \right).$$

The conditions $r \geq k - 1$ and $n - r - 1 \geq k - 1$ imply that $k - 1 \leq r \leq n - k$. As the function $g(x) = 1/(x + 1) + 1/(n - x)$ has its minimum for $x = (n - 1)/2$ when $k - 1 \leq x \leq n - k$, we obtain

$$\gamma_{sR}^k(G) + \gamma_{sR}^k(\bar{G}) \geq kn \left(\frac{1}{r + 1} + \frac{1}{n - r} \right) \geq kn \left(\frac{2}{n + 1} + \frac{2}{n + 1} \right) = \frac{4kn}{n + 1},$$

and this is the desired bound. If n is even, then the function g has its minimum for $r = x = (n - 2)/2$ or $r = x = n/2$, since r is an integer. Hence this case leads to

$$\gamma_{sR}^k(G) + \gamma_{sR}^k(\bar{G}) \geq kn \left(\frac{1}{r + 1} + \frac{1}{n - r} \right) \geq kn \left(\frac{2}{n} + \frac{2}{n + 2} \right) = \frac{4k(n + 1)}{n + 2},$$

and the proof is complete. □

Let $k \geq 1$ be an odd integer, and let H and \bar{H} be $(k - 1)$ -regular graphs of order $n = 2k - 1$. By Example 5, we have $\gamma_{sR}^k(H) = \gamma_{sR}^k(\bar{H}) = n$. Consequently,

$$\gamma_{sR}^k(H) + \gamma_{sR}^k(\bar{H}) = 2n = \frac{4kn}{n + 1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 13 is sharp for odd k .

4 The signed Roman k -Domination Number of $K_{p,p}$

Example 14 If $k \geq 1$ and $p \geq k + 2$ are integers, then $\gamma_{sR}^k(K_{p,p}) = 2k + 2$.

Proof Let X and Y be a bipartition of the complete bipartite graph $K_{p,p}$.

First we show that $\gamma_{sR}^k(K_{p,p}) \geq 2k + 2$. Let $f : V(K_{p,p}) \rightarrow \{-1, 1, 2\}$ be an SRkDF. If $f(u) \geq 1$ for each $u \in V(K_{p,p})$, then $\omega(f) \geq 2p \geq 2k + 2$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. Then $f(u) = 2$ for at least one vertex $u \in Y$. If $w \in Y$ with $w \neq u$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k + 2 + p - 2 \geq 2k + 2.$$

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. In addition, assume that f assigns a_1 vertices in X the value -1 , a_2 vertices in X the value 2, and to the remaining $p - a_1 - a_2$ vertices of X the value 1. Further assume that f assigns b_1 vertices in Y the value -1 , b_2 vertices in Y the value 2, and to the remaining $p - b_1 - b_2$ vertices of Y the value 1. Then $f[x] = -1 - b_1 + 2b_2 + p - b_1 - b_2 \geq k$ and thus $p + b_2 - 2b_1 \geq k + 1$. Analogously, $f[y] = -1 - a_1 + 2a_2 + p - a_1 - a_2 \geq k$ and thus $p + a_2 - 2a_1 \geq k + 1$. We conclude that

$$\begin{aligned} \omega(f) &= (-a_1 + 2a_2 + p - a_1 - a_2) + (-b_1 + 2b_2 + p - b_1 - b_2) \\ &= (p + a_2 - 2a_1) + (p + b_2 - 2b_1) \geq 2k + 2. \end{aligned}$$

Since we have discussed all possible case, we obtain $\gamma_{sR}^k(K_{p,p}) \geq 2k + 2$.

For the converse inequality $\gamma_{sR}^k(K_{p,p}) \leq 2k + 2$, we distinguish two cases.

Case 1. Assume that $p - k$ is even. Let the function $f : V(K_{p,p}) \rightarrow \{-1, 1, 2\}$ assign to one vertex of X and Y the value 2, to $(p + k - 2)/2$ vertices of X and Y the value 1 and to $(p - k)/2$ vertices of X and Y the value -1 . Then f is an SRkDF on $K_{p,p}$ of weight $\omega(f) = 2k + 2$ and so $\gamma_{sR}^k(K_{p,p}) \leq 2k + 2$.

Case 2. Assume that $p - k$ is odd. Let the function $f : V(K_{p,p}) \rightarrow \{-1, 1, 2\}$ assign to two vertices of X and Y the value 2, to $(p + k - 5)/2$ vertices of X and Y the value 1 and to $(p - k + 1)/2$ vertices of X and Y the value -1 . Then f is an SRkDF on $K_{p,p}$ of weight $\omega(f) = 2k + 2$ and so $\gamma_{sR}^k(K_{p,p}) \leq 2k + 2$.

These functions demonstrate that $\gamma_{sR}^k(K_{p,p}) \leq 2k + 2$ and consequently, $\gamma_{sR}^k(K_{p,p}) = 2k + 2$. □

For completeness we determine the signed Roman k -domination number of the complete bipartite graph $K_{p,p}$ for $k - 1 \leq p \leq k + 1$.

Example 15 Let $k \geq 1$ and $k - 1 \leq p \leq k + 1$ be integers.

- (a) If $k \geq 2$, then $\gamma_{sR}^k(K_{k-1,k-1}) = 2k - 2$.
- (b) $\gamma_{sR}^1(K_{1,1}) = 1$ and if $k \geq 2$, then $\gamma_{sR}^k(K_{k,k}) = 2k$.
- (c) $\gamma_{sR}^k(K_{k+1,k+1}) = 2k + 1$.

Proof (a) Example 5 leads to $\gamma_{sR}^k(K_{k-1,k-1}) = 2k - 2$ immediately.

(b) Clearly, $\gamma_{sR}^1(K_{1,1}) = 1$. Let now $k \geq 2$. By the first part of the proof of Example 14, we conclude that $\gamma_{sR}^k(K_{k,k}) \geq 2k$. Applying Theorem 3, we obtain $\gamma_{sR}^k(K_{k,k}) = 2k$.

(c) By the first part of the proof of Example 14, we conclude that $\gamma_{sR}^k(K_{k+1,k+1}) \geq 2k + 1$. If $K_{k+1,k+1}$ has the bipartition X and Y , then let the function $f : V(K_{k+1,k+1}) \rightarrow \{-1, 1, 2\}$ assign to one vertex of X the value -1 and to one vertex of Y the value 2 and to all other vertices the value 1. Then it is easy to verify that f is an SRkDF on $K_{k+1,k+1}$ of weight $2k + 1$ and so $\gamma_{sR}^k(K_{k+1,k+1}) \leq 2k + 1$. This implies $\gamma_{sR}^k(K_{k+1,k+1}) = 2k + 1$. □

5 Cycles and Cubic Graphs

As an immediate consequence of Theorem 3 and Corollary 4, we have that for $n \geq 3$, $\gamma_{sR}^3(C_n) = n$ where C_n denotes a cycle on n vertices. The following result establishes the signed Roman 2-domination number of a cycle.

Example 16 For $n \geq 3$, we have $\gamma_{sR}^2(C_n) = \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$.

Proof Case 1. Assume that $n = 3t$ with an integer $t \geq 1$. In view of Corollary 4, we have $\gamma_{sR}^2(C_{3t}) \geq 2t$.

Let $C_{3t} = v_0v_1 \dots v_{3t-1}v_0$. Define the function $f : V(C_{3t}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = -1$, $f(v_{3i+1}) = 1$ and $f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$. Then $f[v_j] = 2$ for each $0 \leq j \leq 3t - 1$ and therefore f is an SR2DF on C_{3t} of weight $\omega(f) = 2t$. Thus $\gamma_{sR}^2(C_{3t}) \leq 2t$. Consequently, $\gamma_{sR}^2(C_n) = 2t = \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$ in that case.

Case 2. Assume that $n = 3t + 1$ with an integer $t \geq 1$. Let $C_{3t+1} = v_1v_2 \dots v_{3t+1}v_1$. Define the function $f : V(C_{3t+1}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i-1}) = -1$, $f(v_{3i-2}) = 2$ for $1 \leq i \leq t$ and $f(v_{3t+1}) = 2$. Then $f[v_j] \geq 2$ for each $1 \leq j \leq 3t + 1$ and therefore f is an SR2DF on C_{3t+1} of weight $\omega(f) = 2t + 2$. Thus, $\gamma_{sR}^2(C_{3t+1}) \leq 2t + 2$.

Now we show that $\gamma_{sR}^2(C_{3t+1}) \geq 2t + 2$. Let f be a $\gamma_{sR}^2(C_{3t+1})$ -function. If $f(v) = 1$ for all vertices $v \in V(C_{3t+1})$, then $\omega(f) = n \geq 2t + 2$. Hence we may assume that $f(v) = 2$ for some vertex $v \in V(C_{3t+1})$. Renaming vertices if necessary, we may assume, without loss of generality, that $f(v_1) = 2$. Then,

$$\omega(f) = f(v_1) + \sum_{i=1}^t f[v_{3i}] \geq 2 + 2t.$$

Thus $\gamma_{sR}^2(C_{3t+1}) = \omega(f) \geq 2t + 2$. Consequently, $\gamma_{sR}^2(C_n) = 2t + 2 = \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$ also in that case.

Case 3. Assume that $n = 3t + 2$ with an integer $t \geq 1$. Let $C_{3t+2} = v_1v_2 \dots v_{3t+2}v_1$. Define the function $f : V(C_{3t+2}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i-1}) = -1$, $f(v_{3i-2}) = 2$ for $1 \leq i \leq t$, $f(v_{3t+1}) = 2$ and $f(v_{3t+2}) = 1$. Then $f[v_j] \geq 2$ for each $1 \leq j \leq 3t + 1$ and therefore f is an SR2DF on C_{3t+2} of weight $\omega(f) = 2t + 3$. Thus $\gamma_{sR}^2(C_{3t+2}) \leq 2t + 3$.

Next we show that $\gamma_{sR}^2(C_{3t+2}) \geq 2t + 3$. Let f be a $\gamma_{sR}^2(C_{3t+2})$ -function. If $f(v) = 1$ for all vertices $v \in V(C_{3t+2})$, then $\omega(f) = n \geq 2t + 3$. Hence we may assume that $f(v) = 2$ for some vertex $v \in V(C_{3t+2})$. Renaming vertices if necessary, we may assume, without loss of generality, that $f(v_1) = 2$. Since $f[v_1] \geq 2$, at least one of v_2 and v_{3t+2} has a positive weight under f . Renaming vertices if necessary, we may assume that $f(v_2) \geq 1$. Then,

$$\omega(f) = f(v_1) + f(v_2) + \sum_{i=1}^t f[v_{3i+1}] \geq 2 + 1 + 2t = 2t + 3.$$

Thus, $\gamma_{sR}^2(C_{3t+2}) = \omega(f) \geq 2t + 3$. Consequently, $\gamma_{sR}^2(C_n) = 2t + 3 = \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$ also in the last case. □

Next we establish lower and upper bounds on $\gamma_{sR}^k(G)$, when G is a cubic graph and $k \in \{2, 3\}$. We shall need the following result due to Favaron [3].

Theorem 17 ([3]) *If G is a connected cubic graph G of order n , then $\rho(G) \geq n/8$, unless G is the Petersen graph in which case $\rho(G) = (n - 2)/8 = 1$.*

Theorem 18 *Let G be a connected cubic graph of order n . Then the following holds.*

$$(a) \quad \frac{n}{2} \leq \gamma_{sR}^2(G) \leq \frac{7n}{8}.$$

$$(b) \quad \frac{3n}{4} \leq \gamma_{sR}^3(G) \leq n.$$

Proof The lower bounds follow from Corollary 4. The trivial upper bound in Part (b) follows from Theorem 3. To prove the upper bound in Part (a), let G be a connected cubic graph of order n . Let S be a maximum 2-packing in G , and so $|S| = \rho(G)$. If G is the Petersen graph, then $n = 10$ and $\rho(G) = 1$. In this case, let $S = \{v\}$ and let the function $f: V(G) \rightarrow \{-1, 1, 2\}$ assign the value 2 to v and one of its neighbors, the value -1 to the two other neighbors of v , and the value 1 to the remaining six vertices of G . Then, f is an SR2DF on G of weight $\omega(f) = 8 = 4n/5$, implying that $\gamma_{sR}^2(G) \leq 4n/5$. Suppose that G is not the Petersen graph. For each vertex $v \in S$, select one neighbor v' of v and let $S' = \bigcup_{v \in S} \{v'\}$. Since S is a 2-packing, the sets S and S' are vertex-disjoint and $|S| = |S'| = \rho(G)$. Let the function $f: V(G) \rightarrow \{-1, 1, 2\}$ assign to each vertex of S the value -1 , to each vertex of S' the value 2 and to all remaining vertices the value 1. For each vertex $v \in V(G)$, we have that $|N[v] \cap S| \leq 1$, implying that $f[v] \geq 3 - 1 = 2$. Thus, f is an SR2DF on G of weight $\omega(f) = 2|S'| + (n - |S| - |S'|) - |S| = n - 2|S| + |S'| = n - |S| = n - \rho(G)$. By Theorem 17, $\rho(G) \geq n/8$. Hence, $\omega(f) = n - \rho(G) \leq n - n/8 = 7n/8$.

That the lower bound of Theorem 18 (a) is tight may be seen by taking a cycle $v_1 v_2 \dots v_{3t}$, where $t \geq 1$, and adding t new vertices w_1, w_2, \dots, w_t and joining w_i to the three vertices $v_{3i-2}, v_{3i-1}, v_{3i}$ for $i = 1, 2, \dots, t$. Let G denote the resulting cubic graph of order $n = 4t$. Assigning the value -1 to each vertex w_i and to each vertex v_{3i-2} for all i , where $1 \leq i \leq t$, and assigning the value 2 to the remaining $2t$ vertices produces an SR2DF on G of weight $2t = n/2$ and so $\gamma_{sR}^2(G) \leq n/2$. By Corollary 4, we have that $\gamma_{sR}^2(G) \geq n/2$. Consequently, $\gamma_{sR}^2(G) = n/2$.

That the lower bound of Theorem 18 (b) is tight may be seen by taking a cycle $v_1 v_2 \dots v_{3t}$, where $t \geq 1$, and adding t new vertices w_1, w_2, \dots, w_t and joining w_i to the three vertices $v_{3i-2}, v_{3i-1}, v_{3i}$ for $i = 1, 2, \dots, t$. Let G denote the resulting cubic graph of order $n = 4t$. Assigning to each vertex w_i the value -1 , and to each vertex v_{3i-1} where $1 \leq i \leq t$ the value 2, and to the remaining $n/2$ vertices the value 1, we produce an SR3DF on G of weight $3t = 3n/4$ and so $\gamma_{sR}^3(G) \leq 3n/4$. By Corollary 4, we have that $\gamma_{sR}^3(G) \geq 3n/4$. Consequently, $\gamma_{sR}^3(G) = 3n/4$.

The upper bound of Corollary 18 (b) is achieved, for example, by the Petersen graph. However we believe the upper bound of Corollary 18 (a) is not best possible and pose the following question.

Question 1 *Is it true that if G is a cubic graph of order n , then $\gamma_{sR}^2(G) \leq 5n/6$.*

If Question 1 is true, then the bound is achieved, for example, by $K_{3,3}$ and the prism $K_3 \square K_2$.

In a graph G , a vertex dominates itself and its neighbors. Harary and Haynes [5] defined a subset $S \subseteq V(G)$ to be a *double dominating set*, abbreviated DD-set, of G if S dominates every vertex of G at least twice; that is, v is in S and has at least one neighbor

in S or v is in $V - S$ and has at least two neighbors in S . The *double domination number*, $\gamma_{\times 2}(G)$, of G is the minimum cardinality of a DD-set in G . A DD-set of cardinality $\gamma_{\times 2}(G)$ is called a $\gamma_{\times 2}(G)$ -set. Let G be a cubic graph of order n , and let S be a $\gamma_{\times 2}(G)$ -set. Assigning to each vertex in S the value 2, and to each vertex in $V - S$ the value -1 , we produce an SR2DF on G of weight $2|S| - (n - |S|) = 3|S| - n = 3\gamma_{\times 2}(G) - n$, implying that $\gamma_{sR}^2(G) \leq 3\gamma_{\times 2}(G) - n$. Hence we have the following observation.

Observation 19 *If G is a cubic graph of order n , then $\gamma_{sR}^2(G) \leq 3\gamma_{\times 2}(G) - n$.*

6 Trees

Our aim in the section is to determine $\gamma_{sR}^2(P_n)$ for the path P_n and to establish a lower bound on the signed Roman 2-domination number of a tree in terms of its order. As illustrated in Example 10, there are connected graphs whose signed Roman 2-domination number is negative. We show that this is not the case for the class of trees. We begin with the following observation.

Observation 20 *Let T be a tree of order n and let f be an SR2DF on T . Then the following holds.*

- (a) *If v is a leaf or a support vertex in T , then $f(v) \geq 1$.*
- (b) *If $2 \leq n \leq 5$, then $\gamma_{sR}^2(T) = n$.*
- (c) *If $6 \leq n \leq 7$ and $T = P_n$, then $\gamma_{sR}^2(P_n) = n$.*

By Observation 20, if P_n is a path of order $2 \leq n \leq 7$, then $\gamma_{sR}^2(P_n) = n$.

Example 21 For $n \geq 8$, we have $\gamma_{sR}^2(P_n) = \left\lceil \frac{2n+5}{3} \right\rceil$.

Proof Let $P_n = v_0v_1 \dots v_{n-1}$. By Observation 20, we note that $f(v_0), f(v_1), f(v_{n-2}), f(v_{n-1}) \geq 1$.

Case 1. Assume that $n = 3t + 2$ with an integer $t \geq 2$. Define the function $f : V(P_{3t+2}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1, f(v_{3i+1}) = 2$ for $0 \leq i \leq t$ and $f(v_{3i+2}) = -1$ for $0 \leq i \leq t - 1$. Then $f[v_j] \geq 2$ for each $0 \leq j \leq 3t + 1$ and therefore f is an SR2DF on P_{3t+2} of weight $\omega(f) = 2t + 3$. Thus $\gamma_{sR}^2(P_{3t+2}) \leq 2t + 3$.

Next we show that $\gamma_{sR}^2(P_{3t+2}) \geq 2t + 3$. Let f be a $\gamma_{sR}^2(P_{3t+2})$ -function. If $f(v_2) \geq 1$, then

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + f(v_2) + \left(\sum_{i=1}^{t-1} f[v_{3i+1}] \right) \\ &\quad + f(v_{3t}) + f(v_{3t+1}) \geq 3 + 2(t - 1) + 2 = 2t + 3. \end{aligned}$$

If $f(v_2) = -1$, then $f(v_0) + f(v_1) \geq 3$ and so

$$\omega(f) = f(v_0) + f(v_1) + \sum_{i=1}^t f[v_{3i}] \geq 3 + 2t.$$

Thus $\gamma_{sR}^2(P_{3t+2}) = \omega(f) \geq 2t + 3$. Consequently, $\gamma_{sR}^2(P_n) = 2t + 3 = \left\lceil \frac{2n+5}{3} \right\rceil$ in that case.

Case 2. Assume that $n = 3t$ with an integer $t \geq 3$. Define the function $f : V(P_{3t}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1, f(v_{3i+1}) = 2$ for $0 \leq i \leq t - 1, f(v_{3i+2}) = -1$ for $0 \leq i \leq t - 2$ and $f(v_{3t-1}) = 1$. Then $f[v_j] \geq 2$ for each $0 \leq j \leq 3t - 1$ and therefore f is an SR2DF on P_{3t} of weight $\omega(f) = 2t + 2$. Thus $\gamma_{sR}^2(P_{3t}) \leq 2t + 2$.

Next we show that $\gamma_{sR}^2(P_{3t}) \geq 2t + 2$. Let f be a $\gamma_{sR}^2(P_{3t})$ -function. If $f(v_2) = 2$, then

$$\omega(f) = f(v_0) + f(v_1) + f(v_2) + \sum_{i=1}^{t-1} f[v_{3i+1}] \geq 4 + 2(t - 1) = 2t + 2.$$

If $f(v_2) = 1$ and $f(v_3) \geq 1$, then

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + f(v_2) + f(v_3) + \left(\sum_{i=1}^{t-2} f[v_{3i+2}] \right) \\ &\quad + f(v_{3t-2}) + f(v_{3t-1}) \geq 6 + 2(t - 2) = 2t + 2. \end{aligned}$$

If $f(v_2) = 1$ and $f(v_3) = -1$, then $f(v_1) = 2$ and so

$$\omega(f) = f(v_0) + f(v_1) + f(v_2) + \sum_{i=1}^{t-1} f[v_{3i+1}] \geq 4 + 2(t - 1) = 2t + 2.$$

If $f(v_2) = -1$, then $f(v_0) + f(v_1) \geq 3$ and hence

$$\omega(f) = f(v_0) + f(v_1) + \left(\sum_{i=1}^{t-1} f[v_{3i}] \right) + f(v_{3t-1}) \geq 4 + 2(t - 1) = 2t + 2.$$

Thus $\gamma_{sR}^2(P_{3t}) = \omega(f) \geq 2t + 2$. Consequently, $\gamma_{sR}^2(P_n) = 2t + 2 = \left\lceil \frac{2n+5}{3} \right\rceil$ also in that case.

Case 3. Assume that $n = 3t + 1$ with an integer $t \geq 3$. Define the function $f : V(P_{3t+1}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$ for $0 \leq i \leq t, f(v_{3t-1}) = 1, f(v_{3i+1}) = 2$ for $0 \leq i \leq t - 1$ and $f(v_{3i+2}) = -1$ for $0 \leq i \leq t - 2$. Then $f[v_j] \geq 2$ for each $0 \leq j \leq 3t$ and therefore f is an SR2DF on P_{3t+1} of weight $\omega(f) = 2t + 3$. Thus $\gamma_{sR}^2(P_{3t+1}) \leq 2t + 3$.

Next we show that $\gamma_{sR}^2(P_{3t+1}) \geq 2t + 3$. Let f be a $\gamma_{sR}^2(P_{3t+1})$ -function. If $f(v_2) = 2$, then

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + f(v_2) + \left(\sum_{i=1}^{t-1} f[v_{3i+1}] \right) \\ &\quad + f(v_{3t}) \geq 5 + 2(t - 1) = 2t + 3. \end{aligned}$$

Let next $f(v_2) = 1$. If $f(v_3) = 2$, then

$$\omega(f) = f(v_0) + f(v_1) + f(v_2) + f(v_3) + \sum_{i=1}^{t-1} f[v_{3i+2}] \geq 5 + 2(t - 1) = 2t + 3.$$

If $f(v_3) = 1$, then $f(v_4) \geq 1$ and so

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + f(v_2) + f(v_3) + f(v_4) + \left(\sum_{i=2}^{t-1} f[v_{3i}]\right) \\ &\quad + f(v_{3t-1}) + f(v_{3t}) \geq 7 + 2(t - 2) = 2t + 3. \end{aligned}$$

If $f(v_3) = -1$, then $f(v_1) = f(v_4) = 2$ and hence

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + f(v_2) + f(v_3) + f(v_4) + \left(\sum_{i=2}^{t-1} f[v_{3i}]\right) \\ &\quad + f(v_{3t-1}) + f(v_{3t}) \geq 7 + 2(t - 2) = 2t + 3. \end{aligned}$$

Finally, let $f(v_2) = -1$. Then $f(v_0) + f(v_1) \geq 3$ and so

$$\begin{aligned} \omega(f) &= f(v_0) + f(v_1) + \left(\sum_{i=1}^{t-1} f[v_{3i}]\right) \\ &\quad + f(v_{3t-1}) + f(v_{3t}) \geq 5 + 2(t - 1) = 2t + 3. \end{aligned}$$

Thus $\gamma_{sR}^2(P_{3t+1}) = \omega(f) \geq 2t + 3$. Consequently, $\gamma_{sR}^2(P_n) = 2t + 3 = \lceil \frac{2n+5}{3} \rceil$ also in the last case. □

The next result provided a lower bound on the signed Roman 2-domination number of a tree in terms of its order.

Theorem 22 *If T is a tree of order $n \geq 4$, then*

$$\gamma_{sR}^2(T) \geq \frac{n + 4}{2}.$$

Proof We proceed by induction on the order $n \geq 4$. If $n = 4$, then by Observation 20 (b), $\gamma_{sR}^2(T) = n = (n + 4)/2$. This establishes the base case when $n = 4$. Let $n \geq 5$ and suppose that if T' is a tree of order n' where $4 \leq n' < n$, then $\gamma_{sR}^2(T) \geq (n' + 4)/2$. Let T be a tree of order n . Choose an optimal SR2DF f on T , and so $\gamma_{sR}^2(T) = \omega(f)$. If $f(x) \geq 1$ for each vertex $x \in V(T)$, then $\omega(f) \geq n > (n + 4)/2$. Now suppose that there is a vertex $v \in V(T)$ with $f(v) = -1$. Suppose that $T - v$ is the disjoint union of r trees T_1, T_2, \dots, T_r . Let f_i be the restriction of f on T_i for $1 \leq i \leq r$. Clearly, f_i is an SR2DF on T_i for $1 \leq i \leq r$. Since by Observation 20 (a) a leaf and its only neighbor has a positive label, $r \geq 2$ and each T_i has $n_i \geq 2$ vertices. If $n_i = 2$, then in fact $\omega(f_i) \geq 3 = (n_i + 4)/2$, and if $n_i = 3$, then $\omega(f_i) = 3$ or $\omega(f_i) \geq 4 > (n_i + 4)/2$. If $n_i \geq 4$, then by the induction hypothesis $\omega(f_i) \geq (n_i + 4)/2$. If $\omega(f_i) \geq (n_i + 4)/2$ for all i , then since $r \geq 2$,

$$\omega(f) = -1 + \sum_{i=1}^r \omega(f_i) \geq -1 + \sum_{i=1}^r \frac{n_i + 4}{2} = \frac{n + 4r - 3}{2} \geq \frac{n + 5}{2}.$$

Hence we may assume that for some i , $n_i = 3$ and $\omega(f_i) = 3$, for otherwise the desired result follows. Assume that T_1, T_2, \dots, T_q , $q \geq 1$, are exactly the trees with three vertices and with $\omega(f_i) = 3$, $1 \leq i \leq q$. We note that $f(w) = 1$ for each vertex w that belongs to such a tree T_i with $\omega(f_i) = 3$. Hence since v is adjacent to at least one vertex w with $f(w) = 2$, we have that $r > q$. Thus

$$\begin{aligned} \omega(f) &= -1 + \sum_{i=1}^q \omega(f_i) + \sum_{i=q+1}^r \omega(f_i) \\ &\geq -1 + 3q + \sum_{i=q+1}^r \frac{n_i + 4}{2} \\ &= \frac{n + 4(r - q) + 3(q - 1)}{2} \\ &\geq \frac{n + 4}{2}, \end{aligned}$$

as desired. □

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