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Small Matchings Extend to Hamiltonian Cycles in Hypercubes

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Abstract Ruskey and Savage asked the following question: does every matching of a hypercube Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Fink confirmed that the question is true for every perfect matching, thus solved Kreweras' conjecture. In this paper we prove that every matching of at most $3n - 10$ edges can be extended to a Hamiltonian cycle of Q_n for $n \geq 4$.

Keywords Hypercube · Hamiltonian cycle · Matching

1 Introduction

The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary strings of length n, with two vertices being adjacent whenever the corresponding strings differ in just one position.

It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement dates back to 1872 [9]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention.

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A set of edges in a graph G is called a *matching* if no two edges have an end-vertex in common. A matching is *perfect* if it covers all of $V(G)$. Ruskey and Savage [12] asked the following question: does every matching of Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Kreweras [11] conjectured that every perfect matching of Q_n for $n \geq 2$ can be extended to a Hamiltonian cycle of Q_n. Fink [4,6] solved the conjecture. Also, Fink in [4] pointed out that Ruskey and Savage's question is true for $n = 2, 3, 4$.

Gregor [7] strengthened Fink's result and obtained that given a partition of the hypercube into subcubes of nonzero dimensions, every perfect matching of the hypercube can be extended on these subcubes to a Hamiltonian cycle if and only if it interconnects them.

A complementary problem of Hamiltonian cycles in Q_n avoiding given matchings has been already settled for arbitrary matchings by Dimitrov et al. [2]. In particular, the authors in [2] proved that Q_n has a Hamiltonian cycle faulting a perfect matching M if and only if $Q_n - M$ is connected.

The *matching graph* $\mathcal{M}(G)$ of a graph G has a vertex set of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle of G. Fink [5,6] proved that the matching graph $\mathcal{M}(Q_n)$ of the n-dimensional hypercube is bipartite and connected for $n \geq 4$. This proved Kreweras' conjecture [11] that the graph M_n is connected, where M_n is obtained from $\mathcal{M}(Q_n)$ by contracting all vertices of $\mathcal{M}(Q_n)$ which correspond to isomorphic perfect matchings.

The following result obtained by Dvořák implied that Ruskey and Savage's question is true for every matching of at most $2n-3$ edges. A forest is *linear* if each component of it is a path.

Lemma 1.1 [3] *For $n \geq 2$, let $E \subseteq E(Q_n)$ with $|E| \leq 2n - 3$. Then there exists a Hamiltonian cycle of Q_n containing E if and only if the subgraph induced by E is a linear forest.*

In a bipartite graph G, a set $S \subseteq V(G)$ is *deficient* if $|N(S)| < |S|$. A matching M (with vertex set U) is k-*suitable* if $G - U$ has no deficient set of size less than k. Vandenbussche and West [14] proved that every k-suitable matching of at most $k(n-k) + \frac{(k-1)(k-2)}{2}$ edges for $k \leq n - 3$ and every induced matching can be extended to a perfect matching of Q_n, so can be extended to a Hamiltonian cycle of Q_n by Fink's result quoted above.

Ruskey and Savage's question has been resolved for perfect matchings only, while the general case is still widely open. In this paper, we consider this question and obtain the following result.

Theorem 1.2 *For $n \geq 4$, let M be a matching of Q_n with $|M| \leq 3n - 10$. Then there exists a Hamiltonian cycle containing M in Q_n.*

The rest of this paper is organized as follows. In Sect. 2 we introduce some conclusions about path partitions of Q_n. The main result is proved in Sect. 3.

2 Path Partitions of Q_n

This section is devoted to auxiliary results about path partitions which are applied in the constructions of Hamiltonian cycles in the main result. First, we introduce some necessary definitions.

The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For a vertex $v \in V(G)$ and a set $E \subseteq E(G)$, let $G - v$ denote the subgraph of G induced by $V(G)\backslash\{v\}$, let $G - E$ denote the graph with vertices $V(G)$ and edges $E(G)\backslash E$, let $\langle E \rangle$ denote the subgraph of G induced by E and let $N(v)$ denote the set $\{u \in V(G) \mid uv \in E(G)\}$. The distance of vertices u and v is denoted by $d(u, v)$. The distance $d(u, xy)$ of a vertex u and an edge xy is defined by $d(u, xy) := \min\{d(u, x), d(u, y)\}$ and the distance $d(uv, xy)$ of edges uv and xy is defined by $d(uv, xy) := \min\{d(u, xy), d(v, xy)\}$. For a vertex $u = (\delta_1, \delta_2, \ldots, \delta_n) \in V(Q_n)$, we define the *parity* of u by $p(u) = (-1)^{|\{i \in [n] | \delta_i = 1\}|}$. Note that vertices of each parity form bipartite sets of Q_n. Consequently, $p(u) = p(v)$ if and only if $d(u, v)$ is even.

An edge in Q_n is called an *i-dimensional edge* if its endvertices differ in the ith position. The set of all i-dimensional edges of Q_n is denoted by E_i. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For any given $j \in [n]$, let $Q_{n-1}^{0,j}$ and $Q_{n-1}^{1,j}$, with the superscript j being omitted when the context is clear, be two $(n-1)$-dimensional subcubes of Q_n induced by all the vertices with the jth positions being 0 or 1, respectively. Clearly, $Q_n - E_j = Q_{n-1}^0 \cup Q_{n-1}^1$, we say that Q_n is decomposed into two $(n-1)$-dimensional subcubes Q_{n-1}^0 and Q_{n-1}^1 by E_j. For $\delta \in \{0, 1\}$, any vertex $u \in V(Q_{n-1}^\delta)$ has in $Q_{n-1}^{1-\delta}$ a unique neighbor, denoted by $u_{1-\delta}$ and for any edge $e = uv \in E(Q_{n-1}^\delta)$, $e_{1-\delta}$ denotes the edge $u_{1-\delta}v_{1-\delta} \in E(Q_{n-1}^{1-\delta})$. Given $M \subseteq E(Q_n)$, let $M_\delta := M \cap E(Q_{n-1}^\delta)$.

Lemma 2.1 [13] *For $n \geq 2$, let e and f be two disjoint edges in Q_n. Then Q_n can be decomposed into two $(n-1)$-dimensional subcubes such that one contains e and the other contains f.*

A *path partition* of a graph G is a set of vertex-disjoint paths that cover all vertices of G. Given a set E of edges, a path P *passes through* E if $E \subseteq E(P)$. Similarly, a set $\{P_i\}_{i=1}^k$ of paths *passes through* E if $E \subseteq \bigcup_{i=1}^k E(P_i)$. We use P_{uv} to denote a path between vertices u and v.

First, let us recall the following classical result, originally obtained by Havel in [10].

Lemma 2.2 [10] *Let $x, y \in V(Q_n)$ such that $p(x) \neq p(y)$. Then there exists a Hamiltonian path between x and y in Q_n.*

Lemma 2.3 [3] *For $n \geq 2$, let $x, y \in V(Q_n)$ and $e \in E(Q_n)$ such that $p(x) \neq p(y)$ and $e \neq xy$. Then there is a Hamiltonian path of Q_n between x and y passing through edge e.*

Lemma 2.4 *For $n \geq 4$, let e and f be two disjoint edges in Q_n and $x, y \in V(Q_n)$ such that $p(x) \neq p(y)$ and $xy \notin \{e, f\}$. Then there is a Hamiltonian path of Q_n between x and y passing through $\{e, f\}$.*

Proof By Lemma 2.1, Q_n can be decomposed into subcubes Q^0_{n-1} and Q^1_{n-1} such that $e \in E(Q^0_{n-1})$ and $f \in E(Q^1_{n-1})$. Without loss of generality, assume that $x \in V(Q^0_{n-1})$.

If $y \in V(Q^0_{n-1})$, since $p(x) \neq p(y)$ and $xy \neq e$, by Lemma 2.3 there is a Hamiltonian path P_{xy} of Q^0_{n-1} passing through edge e. Let $uv \in E(P_{xy})\backslash\{e\}$ such that $u_1v_1 \neq f$ and using Lemma 2.3 again, there is a Hamiltonian path $P_{u_1v_1}$ of Q^1_{n-1} passing through f. Then the desired Hamiltonian path of Q_n is induced by edges of $(E(P_{xy}) \cup E(P_{u_1v_1}) \cup \{uu_1, vv_1\})\backslash\{uv\}$.

If $y \in V(Q^1_{n-1})$, choose a vertex $w \in V(Q^0_{n-1})$ such that $p(x) \neq p(w)$ and $xw \neq e$, $yw_1 \neq f$. Note that as $|\{w \in V(Q^0_{n-1}) \mid p(x) \neq p(w)\}| = 2^{n-2} > 2$ for $n \geq 4$, this is always possible. Since $p(w_1) = p(x) \neq p(y) = p(w)$, by Lemma 2.3 there exist Hamiltonian paths P_{xw} of Q^0_{n-1} and P_{w_1y} of Q^1_{n-1} passing through e and f, respectively. Hence the desired Hamiltonian path of Q_n is induced by edges of $E(P_{xw}) \cup \{ww_1\} \cup E(P_{w_1y})$. □

Lemma 2.5 [3] *For $n \geq 2$, let x, y, u, v be pairwise distinct vertices of Q_n such that $p(x) \neq p(y)$ and $p(u) \neq p(v)$. Then (i) there exists a path partition $\{P_{xy}, P_{uv}\}$ of Q_n; (ii) moreover, in the case when $d(x, y) = 1$, path P_{xy} can be chosen such that $P_{xy} = xy$, unless $n = 3$, $d(u, v) = 1$ and $d(xy, uv) = 2$.*

Lemma 2.6 *For $n \geq 5$, let uv and f be two disjoint edges in Q_n and $x, y \in V(Q_n)\backslash\{u, v\}$ such that $p(x) \neq p(y)$ and $xy \neq f$. Then there exists a Hamiltonian path between x and y passing through edge f in $Q_n - \{u, v\}$.*

Proof By Lemma 2.1, Q_n can be decomposed into subcubes Q^0_{n-1} and Q^1_{n-1} such that $uv \in E(Q^0_{n-1})$ and $f \in E(Q^1_{n-1})$.

If $x, y \in V(Q^0_{n-1})$, since $n-1 \geq 4$, by Lemma 2.5 there is a Hamiltonian path P_{xy} in $Q^0_{n-1} - \{u, v\}$. Let $st \in E(P_{xy})$ such that $s_1t_1 \neq f$ and apply Lemma 2.3 to find a Hamiltonian path $P_{s_1t_1}$ passing through f in Q^1_{n-1}. Then the desired Hamiltonian path in $Q_n - \{u, v\}$ is induced by edges of $(E(P_{xy}) \cup E(P_{s_1t_1}) \cup \{ss_1, tt_1\})\backslash\{st\}$.

If $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$, choose a vertex $w \in V(Q^0_{n-1})$ such that $p(x) \neq p(w)$ and $w \notin \{u, v\}$, $w_1y \neq f$. Note that as $|\{w \in V(Q^0_{n-1}) \mid p(x) \neq p(w)\}| \geq 2^{n-2} \geq 8$ for $n \geq 5$, this is always possible. Since $p(w_1) = p(x) \neq p(y) = p(w)$, by Lemmas 2.5 and 2.3 there exist a Hamiltonian path P_{xw} in $Q^0_{n-1} - \{u, v\}$ and a Hamiltonian path P_{w_1y} passing through f in Q^1_{n-1}. Then the desired Hamiltonian path in $Q_n - \{u, v\}$ is induced by edges of $E(P_{xw}) \cup E(P_{w_1y}) \cup \{ww_1\}$.

If $x, y \in V(Q^1_{n-1})$, apply Lemma 2.3 to find a Hamiltonian path P_{xy} passing through f in Q^1_{n-1}. Since $|E(P_{xy})\backslash\{f\}| \geq 2^{n-1} - 2 > 4$, there exists an edge $wt \in E(P_{xy})\backslash\{f\}$ such that $\{w_0, t_0\} \cap \{u, v\} = \emptyset$. Since $n - 1 \geq 4$, by Lemma 2.5 there is a Hamiltonian path $P_{w_0t_0}$ in $Q^0_{n-1} - \{u, v\}$. Then the desired Hamiltonian path in $Q_n - \{u, v\}$ is induced by edges of $(E(P_{w_0t_0}) \cup E(P_{xy}) \cup \{w_0w, t_0t\})\backslash\{wt\}$. □

A set $\{\{a_i, b_i\}\}^k_{i=1}$ of pairs of distinct vertices of Q_n is called a *balanced pair set* if $\sum^k_{i=1}(p(a_i) + p(b_i)) = 0$.

Lemma 2.7 [1] *For $n \geq 4$, let x, y, u, v be pairwise distinct vertices of Q_n such that $p(x) = p(y) \neq p(u) = p(v)$. Then there exists a path partition $\{P_{xy}, P_{uv}\}$ of Q_n.*

Lemma 2.8 [8] *For $n \geq 1$, let $\{\{a_i, b_i\}\}_{i=1}^k$ be a balanced pair set in Q_n such that $2k - |\{a_i b_i\}_{i=1}^k \cap E(Q_n)| < n$. Then there exists a path partition $\{P_{a_i b_i}\}_{i=1}^k$ of Q_n.*

Let $P = v_1 v_2 \ldots v_i \ldots v_j \ldots v_k$ be a path, we use $P[v_i, v_j]$ to denote the subpath $v_i \ldots v_j$ of P from v_i to v_j. For an edge e, we use $V(e)$ to denote the set of two endvertices of e and for a matching M, let $V(M) := \bigcup_{e \in M} V(e)$.

Lemma 2.9 *For $n \geq 5$, let $e \in E(Q_n)$ and $x, y, u, v \in V(Q_n)$ be pairwise distinct vertices such that $p(x) \neq p(y)$, $p(u) \neq p(v)$ and $\{u, v\} \cap V(e) = \emptyset$. Then there exists a path partition $\{P_{xy}, P_{uv}\}$ of Q_n passing through edge e.*

Proof If $xy = e$, by Lemma 2.5 there exists a path partition $\{P_{xy} = xy, P_{uv}\}$ of Q_n, and hence the conclusion holds. So in what follows we shall assume that $xy \neq e$. Without loss of generality we may assume $y \notin V(e)$ and $p(y) = p(v) \neq p(x) = p(u)$. Since $d(x, u) \geq 2$, there exists $j \in [n]$ such that $x \in V(Q_{n-1}^0)$, $u \in V(Q_{n-1}^1)$ and $e \notin E_j$. If $x \in V(e)$, then $e \in E(Q_{n-1}^0)$. If $x \notin V(e)$, then without loss of generality we may assume $e \in E(Q_{n-1}^0)$.

First, we suppose $v \in V(Q_{n-1}^0)$. Since $p(x) \neq p(v)$ and $xv \neq e$, by Lemma 2.3 there is a Hamiltonian path P_{xv} passing through edge e in Q_{n-1}^0. If $y \in V(Q_{n-1}^0)$, let w be the neighbor of y such that $w \in V(P_{xv}[y, v])$. Since $y \notin V(e)$, we have $yw \neq e$. Since $p(w_1) = p(y) \neq p(u)$, by Lemma 2.2 there is a Hamiltonian path $P_{w_1 u}$ in Q_{n-1}^1. Then $\{P_{xv}[x, y], \langle E(P_{xv}[v, w]) \cup \{ww_1\} \cup E(P_{w_1 u})\rangle\}$ is the desired path partition of Q_n. If $y \in V(Q_{n-1}^1)$, since $|E(P_{xv}) \backslash \{e\}| = 2^{n-1} - 2 > 4$ for $n \geq 5$, there exists an edge $st \in E(P_{xv}) \backslash \{e\}$ such that $\{s_1, t_1\} \cap \{u, y\} = \emptyset$. Assume that s lies on $P_{xv}[x, t]$, which means that s is closer to x on P_{xv} than t. Since $p(s_1) \neq p(t_1)$, $p(y) \neq p(u)$ and $n - 1 \geq 4$, using Lemma 2.5 in case $p(s_1) \neq p(y)$ or Lemma 2.7 in case $p(s_1) = p(y)$, there exists a path partition $\{P_{s_1 y}, P_{t_1 u}\}$ of Q_{n-1}^1. Then $\{\langle E(P_{xv}[x, s]) \cup \{ss_1\} \cup E(P_{s_1 y})\rangle, \langle E(P_{xv}[v, t]) \cup \{tt_1\} \cup E(P_{t_1 u})\rangle\}$ is the desired path partition of Q_n.

Next, we consider $v \in V(Q_{n-1}^1)$. If $y \in V(Q_{n-1}^0)$, by Lemmas 2.3 and 2.2 there exist Hamiltonian paths P_{xy} of Q_{n-1}^0 passing through e and P_{uv} of Q_{n-1}^1, then $\{P_{xy}, P_{uv}\}$ is the desired path partition of Q_n. If $y \in V(Q_{n-1}^1)$, since $|\{w \in V(Q_{n-1}^1) \backslash \{u\} \mid p(w) = p(u)\}| \geq 2^{n-2} - 1 > 1$, there exists a vertex $w \in V(Q_{n-1}^1) \backslash \{u\}$ such that $p(w) = p(u)$ and $xw_0 \neq e$. Note that then $p(x) = p(u) = p(w) \neq p(y) = p(v) = p(w_0)$. By Lemma 2.3 there is a Hamiltonian path P_{xw_0} passing through e in Q_{n-1}^0 and by Lemma 2.5 there exists a path partition $\{P_{wy}, P_{uv}\}$ of Q_{n-1}^1. Then $\{\langle E(P_{xw_0}) \cup \{w_0 w\} \cup E(P_{wy})\rangle, P_{uv}\}$ is the desired path partition of Q_n. □

Lemma 2.10 *Let $ux, vy \in E(Q_4)$ be two disjoint edges and $e \in E(Q_4)$ such that $\{u, v\} \cap V(e) = \emptyset$ and $xy \neq e$. Then there exists a path partition $\{P_{ux}, P_{vy}\}$ of Q_4 passing through edge e.*

Proof By Lemma 2.1, Q_4 can be decomposed into Q_3^0 and Q_3^1 by some E_j such that $ux \in E(Q_3^0)$ and $vy \in E(Q_3^1)$. If $e \in E_j$, let $e = ww_1$. Since $xy \neq e = ww_1$, by symmetry, we may assume that $x \neq w$. Since $|\{s \in V(Q_3^0) \mid p(s) \neq p(w)\}| = 4$, there exists a vertex $s \in V(Q_3^0)$ such that $p(s) \neq p(w)$ and $s \notin \{u, x\}$, $s_1 \neq v$. By Lemma 2.5 there exists a path partition $\{P_{ux}, P_{ws}\}$ of Q_3^0. Since $v \neq w_1$ and $v \neq s_1$, we have $w_1 s_1 \neq vy$. Since $p(w_1) \neq p(s_1)$, by Lemma 2.3 there exists a Hamiltonian path $P_{w_1 s_1}$ of Q_3^1 passing through vy. Then $\{P_{ux}, \langle (E(P_{w_1 s_1}) \cup E(P_{ws}) \cup \{ww_1, ss_1\}) \setminus \{vy\} \rangle\}$ is the desired path partition of Q_4. If $e \notin E_j$, by symmetry, we may assume that $e \in E(Q_3^0)$. Since $p(u) \neq p(x)$, $p(v) \neq p(y)$ and $ux \neq e$, by Lemmas 2.3 and 2.2 there exist Hamiltonian paths P_{ux} of Q_3^0 passing through e and P_{vy} of Q_3^1. Then $\{P_{ux}, P_{vy}\}$ is the desired path partition of Q_4. □

Lemma 2.11 *For $n \geq 4$, let M be a matching of Q_n with $|M| \leq 2n - 8$ and $ux, vy \in E(Q_n)$ be two disjoint edges such that $\{u, v\} \cap V(M) = \emptyset$ and $xy \notin M$. Then there exists a path partition $\{P'_{ux}, P'_{vy}\}$ of Q_n passing through M.*

Proof We prove the lemma by induction on n. The lemma holds for $n = 4$ by Lemma 2.5. Assume that the lemma holds for $n - 1 (\geq 4)$, we are to show it holds for $n (\geq 5)$.

Select $j \in [n]$ such that $|(M \cup \{ux, vy\}) \cap E_j|$ is as small as possible. Since $|M \cup \{ux, vy\}| \leq 2n - 6$, we have $|(M \cup \{ux, vy\}) \cap E_j| \leq 1$. If $|(M \cup \{ux, vy\}) \cap E_j| = 1$, since $|(M \cup \{ux, vy\}) \cap E_i| \geq 1$ for any $i \in [n]$, there are at least six possibilities of such j and therefore, moreover, we can choose j such that $(M \cup \{ux, vy\}) \cap E_j = M \cap E_j := \{ww_1\}$ and $\{x, y\} \cap \{w, w_1\} = \emptyset$. Decompose Q_n into Q_{n-1}^0 and Q_{n-1}^1 by E_j. Since $\{ux, vy\} \subseteq E(Q_{n-1}^0) \cup E(Q_{n-1}^1)$, by symmetry, we may assume that $ux \in E(Q_{n-1}^0)$.

Case 1. Suppose $vy \in E(Q_{n-1}^1)$.

If $M \cap E_j = \emptyset$, since $M_0 \cup \{ux\}$ and $M_1 \cup \{vy\}$ are both linear forests of at most $2(n-1) - 5$ edges, by Lemma 1.1 there exist Hamiltonian cycles C_0 of Q_{n-1}^0 and C_1 of Q_{n-1}^1 containing $M_0 \cup \{ux\}$ and $M_1 \cup \{vy\}$, respectively. Then $\{C_0 - ux, C_1 - vy\}$ is the desired path partition of Q_n.

If $M \cap E_j = \{ww_1\}$, since $\{u, v\} \cap V(M) = \emptyset$ and $\{x, y\} \cap \{w, w_1\} = \emptyset$, we have $\{u, v, x, y\} \cap \{w, w_1\} = \emptyset$. By symmetry, we may assume that $|M_0| \geq |M_1|$. Since $|N(w_1) \cap V(Q_{n-1}^1)| = n - 1 > 3$ for $n \geq 5$ and $p(v) \neq p(y)$, $p(u) \neq p(x)$, there exists a vertex $s \in N(w_1) \cap V(Q_{n-1}^1)$ such that $s \notin \{v, y\}$, $s_0 \notin \{u, x\}$ and $sy \notin M_1$. Then w_1, s, v, y are pairwise distinct vertices satisfying $\{w_1, v\} \cap V(M_1) = \emptyset$, $w_1 s, vy \in E(Q_{n-1}^1)$ and $sy \notin M_1$. Since $|M_0| + |M_1| \leq 2n - 9$ and $|M_0| \geq |M_1|$, we have $|M_1| \leq n - 5 \leq 2(n-1) - 8$ and therefore by the induction hypothesis, there exists a path partition $\{P_{w_1 s}, P_{vy}\}$ of Q_{n-1}^1 passing through M_1. Since M_0 is a matching, $\{u, w\} \cap V(M_0) = \emptyset$ and u, x, w, s_0 are pairwise distinct vertices, we obtain that $M_0 \cup \{ux, ws_0\}$ is a linear forest of at most $2(n-1) - 5$ edges and therefore by Lemma 1.1, there is a Hamiltonian cycle C_0 containing $M_0 \cup \{ux, ws_0\}$ in Q_{n-1}^0. Then $\{P'_{ux} := \langle (E(C_0) \cup E(P_{w_1 s}) \cup \{ww_1, s_0 s\}) \setminus \{ux, ws_0\} \rangle, P'_{vy} := P_{vy}\}$ is the desired path partition of Q_n.

Case 2. Suppose $vy \in E(Q_{n-1}^0)$.

Claim 1 *If there is a path partition $\{P_{ux}, P_{vy}\}$ of Q_{n-1}^0 passing through M_0, then we can construct a path partition $\{P'_{ux}, P'_{vy}\}$ of Q_n passing through M.*

If $M \cap E_j = \{ww_1\}$, without loss of generality, assume that $w \in V(P_{ux})$. Select a neighbor t of w on P_{ux}. Since M is a matching and $ww_1 \in M$, we have $\{wt, w_1 t_1\} \cap$

$M = \emptyset$. If $M \cap E_j = \emptyset$, without loss of generality, assume that $|E(P_{ux})| \geq |E(P_{vy})|$, then $|E(P_{ux})| \geq 2^{n-2} - 1$. Since $|E(P_{ux})\backslash M_0| - |M_1| \geq 2^{n-2} - 1 - (2n - 8) \geq 5$ for $n \geq 5$, we can choose an edge $wt \in E(P_{ux})\backslash M_0$ with $w_1t_1 \notin M_1$. In both cases, $M_1 \cup \{w_1t_1\}$ is a linear forest of at most $2(n - 1) - 5$ edges. By Lemma 1.1 there is a Hamiltonian cycle C_1 containing $M_1 \cup \{w_1t_1\}$ in Q^1_{n-1}. Then $\{P'_{ux} := \langle(E(P_{ux}) \cup E(C_1) \cup \{ww_1, tt_1\})\backslash\{wt, w_1t_1\}\rangle, P'_{vy} := P_{vy}\}$ is the desired path partition of Q_n. Claim 1 is proved.

If $|M_0| \leq 2(n - 1) - 8$, by the induction hypothesis, there exists a path partition $\{P_{ux}, P_{vy}\}$ of Q^0_{n-1} passing through M_0. Then the lemma holds by Claim 1. So in what follows it suffices to consider the case that $|M_0| \geq 2n - 9$. Since $|M| \leq 2n - 8$, we have $|M_1| \leq 1$. We distinguish two subcases to consider.

Subcase 2.1. $M_1 = \emptyset$. Since $M_0 \cup \{ux, vy\}$ is a linear forest of at most $2(n - 1) - 4$ edges, by Lemma 1.1 there is a Hamiltonian cycle C_0 containing $M_0 \cup \{ux, vy\}$ in Q^0_{n-1}. Observe that $C_0 - \{ux, vy\}$ is a disjoint union of two paths P_u and P_x with endvertices u and x, respectively. If $M \cap E_j = \{ww_1\}$, since $\{u, v, x, y\} \cap \{w, w_1\} = \emptyset$, without loss of generality, we may assume that $w \in V(P_u)$. Let t be a neighbor of w on P_u. Since M is a matching and $ww_1 \in M$, we have $wt \notin M$. Since $xy \notin M$ and $v \notin V(M)$, there exists an edge $sr \in E(P_x)\backslash M$. If $M \cap E_j = \emptyset$, since $\{u, v\} \cap V(M) = \emptyset$ and $xy \notin M$, there exist edges $wt \in E(P_u)\backslash M$ and $sr \in E(P_x)\backslash M$.

Without loss of generality, assume that $w \in V(P_u[u, t])$ and $s \in V(P_x[x, r])$. Since $p(w_1) \neq p(t_1)$ and $p(s_1) \neq p(r_1)$, using Lemma 2.5 in case $p(w_1) \neq p(s_1)$ or Lemma 2.7 in case $p(w_1) = p(s_1)$, there exists a path partition $\{P_{w_1s_1}, P_{t_1r_1}\}$ of Q^1_{n-1}. Then $\{P'_{ux} := \langle E(P_u[u, w]) \cup E(P_x[x, s]) \cup \{ww_1, ss_1\} \cup E(P_{w_1s_1})\rangle, P'_{vy} := \langle(E(P_u)\backslash E(P_u[u, t])) \cup (E(P_x)\backslash E(P_x[x, r])) \cup \{tt_1, rr_1\} \cup E(P_{t_1r_1})\rangle\}$ is the desired path partition of Q_n.

Subcase 2.2. $|M_1| = 1$. Since $|M_0| \geq 2n - 9$ and $|M| \leq 2n - 8$, we have $|M_0| = 2n - 9$ and $M \cap E_j = \emptyset$. If $n = 5$, since $|M_0| = 1$, by Lemma 2.10 there is a path partition $\{P_{ux}, P_{vy}\}$ of Q^0_{n-1} passing through M_0 and therefore, the lemma holds by Claim 1. If $n \geq 6$, since $|M_0| = 2n - 9 \geq 3$, there exists an edge $wt \in M_0$ such that $\{x, y\} \cap \{w, t\} = \emptyset$. Apply the induction to find a path partition $\{P_{ux}, P_{vy}\}$ of Q^0_{n-1} passing through $M_0\backslash\{wt\}$. If $wt \in E(P_{ux}) \cup E(P_{vy})$, then $\{P_{ux}, P_{vy}\}$ is a path partition of Q^0_{n-1} passing through M_0 and therefore, the lemma holds by Claim 1. If $wt \notin E(P_{ux}) \cup E(P_{vy})$, without loss of generality, we may assume that $w \in V(P_{ux})$.

If $t \in V(P_{ux})$, let s, r be neighbors of w, t on P_{ux}, respectively, such that exactly one of s and r lies on $P_{ux}[w, t]$ and $s_1r_1 \notin M_1$. Since $\{u, x\} \cap \{w, t\} = \emptyset$, this is always possible. Since $p(s_1) \neq p(r_1)$, by Lemma 2.3 there is a Hamiltonian path $P_{s_1r_1}$ passing through M_1 in Q^1_{n-1}. Then $\{P'_{ux} := \langle(E(P_{ux}) \cup E(P_{s_1r_1}) \cup \{wt, ss_1, rr_1\})\backslash\{sw, rt\}\rangle, P'_{vy} := P_{vy}\}$ is the desired path partition of Q_n.

If $t \in V(P_{vy})$, without loss of generality, assume that $|E(P_{ux})| \geq |E(P_{vy})|$, then $|E(P_{ux})| \geq 2^{n-2} - 1$. Choose a vertex $b \in V(P_{ux})\backslash\{w\}$ such that $p(b) = p(w)$ and $b \notin V(M_0)$. Since $|\{b \in V(P_{ux})\backslash\{w\} \mid p(b) = p(w)\}| \geq 2^{n-3} - 1 \geq (2n - 9) + 4$ for $n \geq 6$, there are at least four ways to choose such b. Let a be the neighbor of b such that $a \in V(P_{ux}[b, w])$. Then, moreover, we can choose b such that $a_1 \notin V(M_1)$. Since $b \notin V(M_0)$, we have $ba \notin M_0$. Let r be the neighbor of w on P_{ux} such that w lies on $P_{ux}[r, a]$ and let s be a neighbor of t on P_{vy} such that $s_1 \notin V(M_1)$. Since $\{u, v, x, y\} \cap$

$\{w, t\} = \emptyset$, this is always possible. Then in Q^1_{n-1}, $p(r_1) \neq p(b_1)$, $p(s_1) \neq p(a_1)$ and $\{s_1, a_1\} \cap V(M_1) = \emptyset$. Since $n - 1 \geq 5$, by Lemma 2.9 there exists a path partition $\{P_{r_1b_1}, P_{s_1a_1}\}$ of Q^1_{n-1} passing through M_1. Since M is a matching and $wt \in M$, we have $\{wr, ts\} \cap M = \emptyset$. Hence $\{P'_{ux} := \langle(E(P_{ux})\backslash E(P_{ux}[r, b])) \cup \{rr_1, bb_1\} \cup E(P_{r_1b_1})\rangle$, $P'_{vy} := \langle(E(P_{vy}) \cup E(P_{ux}[w, a]) \cup E(P_{s_1a_1}) \cup \{wt, aa_1, ss_1\})\backslash\{st\}\rangle\}$ is the desired path partition of Q_n. □

3 Proof of Theorem 1.2

We prove the theorem by induction on n. The theorem holds for $4 \leq n \leq 7$ by Lemma 1.1. Suppose that the theorem holds for $n - 1(\geq 7)$. We are to show that it holds for $n(\geq 8)$. Since $|M| \leq 3n-10$, there exists $j \in [n]$ such that $|M \cap E_j| \leq 2$. Decompose Q_n into subcubes Q^0_{n-1} and Q^1_{n-1} by E_j such that $|M_0| \geq |M_1|$. When $|M \cap E_j| = 1$, let $M \cap E_j = \{uu_1\}$; when $|M \cap E_j| = 2$, let $M \cap E_j = \{uu_1, vv_1\}$.

Claim 1 *If there exists a Hamiltonian cycle C_0 containing M_0 in Q^0_{n-1}, then we can construct a Hamiltonian cycle containing M in Q_n.*

If $M \cap E_j = \{uu_1\}$, select a neighbor v of u on C_0. Since M is a matching and $uu_1 \in M$, we have $\{uv, u_1v_1\} \cap M = \emptyset$. If $M \cap E_j = \emptyset$, since $|E(C_0)| - (|M_0| + |M_1|) \geq 2^{n-1} - (3n - 10) > 1$ for $n \geq 8$, there exists an edge $uv \in E(C_0)\backslash M_0$ such that $u_1v_1 \notin M_1$. If $M \cap E_j = \{uu_1, vv_1\}$ and $d_{C_0}(u, v) = 1$, then $uv \in E(C_0)\backslash M_0$ and $u_1v_1 \notin M_1$. In all cases, $M_1 \cup \{u_1v_1\}$ is a linear forest with $|M_1 \cup \{u_1v_1\}| \leq \frac{3n-10}{2} + 1 < 2(n - 1) - 3$ for $n \geq 8$. By Lemma 1.1 there exists a Hamiltonian cycle C_1 containing $M_1 \cup \{u_1v_1\}$ in Q^1_{n-1}. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(C_1) \cup \{uu_1, vv_1\})\backslash\{uv, u_1v_1\}$.

If $M \cap E_j = \{uu_1, vv_1\}$ and $d_{C_0}(u, v) > 1$, let x, y be neighbors of u, v on C_0, respectively, such that $x \neq y$ and $x_1y_1 \notin M_1$. Since $\{u_1, v_1\} \cap V(M_1) = \emptyset$ and $|M_1| \leq \frac{3n-12}{2} \leq 2(n - 1) - 8$ for $n \geq 8$, by Lemma 2.11 there exists a path partition $\{P_{u_1x_1}, P_{v_1y_1}\}$ of Q^1_{n-1} passing through M_1. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{u_1x_1}) \cup E(P_{v_1y_1}) \cup \{uu_1, vv_1, xx_1, yy_1\})\backslash\{ux, vy\}$. Claim 1 is proved.

Claim 2 *Let $xy \in M_0$ and moreover, when $|M \cap E_j| \geq 1$, let xy be such that $d(u, xy) \neq 1$. Let C_0 be a Hamiltonian cycle of Q^0_{n-1} containing $M_0\backslash\{xy\}$. If $xy \notin E(C_0)$ and $|M \cap E_j| + |M_1| \leq 2$, then we can construct a Hamiltonian cycle containing M in Q_n.*

If $M \cap E_j = \emptyset$, then $|M_1| \leq 2$. Let s and r be neighbors of x and y on C_0, respectively, such that one of the paths between x and y on C_0 contains s and the other contains r. If $s_1r_1 \notin M_1$, since $p(s_1) \neq p(r_1)$, by Lemma 2.4 there is a Hamiltonian path $P_{s_1r_1}$ passing through M_1 in Q^1_{n-1}, then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{s_1r_1}) \cup \{xy, ss_1, rr_1\})\backslash\{xs, yr\}$. If, however, $s_1r_1 \in M_1$, since $|E(C_0)\backslash M_0| - |M_1| \geq 2^{n-1} - (3n - 10) > 4$, there exists an edge $wt \in E(C_0)\backslash M_0$ such that $\{w, t\} \cap \{s, r\} = \emptyset$ and $w_1t_1 \notin M_1$. Using Lemma 2.6 in case $|M_1| = 2$ or Lemma 2.5 in case $|M_1| = 1$, there is a Hamiltonian path $P_{w_1t_1}$ passing

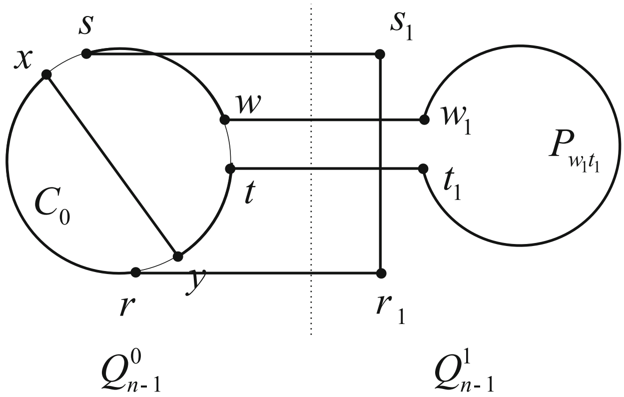

Fig. 1 A Hamiltonian cycle containing M in Q_n when $M \cap E_j = \emptyset$ and $s_1r_1 \in M_1$ in Claim 2

through $M_1 \backslash \{s_1r_1\}$ in $Q^1_{n-1} - \{s_1, r_1\}$. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{w_1t_1}) \cup \{xy, ss_1, rr_1, s_1r_1, ww_1, tt_1\}) \backslash \{xs, yr, wt\}$, see Fig. 1.

If $M \cap E_j = \{uu_1\}$, then $|M_1| \leq 1$. Select a neighbor v of u on C_0 such that $v_1 \notin V(M_1)$, then $\{u_1, v_1\} \cap V(M_1) = \emptyset$. If $M \cap E_j = \{uu_1, vv_1\}$ and $d_{C_0}(u, v) = 1$, note that then $M_1 = \emptyset$. In the above two cases, since u is not adjacent to x or y on C_0, we can choose neighbors s, r of x, y on C_0, respectively, such that one of the paths between x and y on C_0 contains s and the other contains r and $\{s, r\} \cap \{u, v\} = \emptyset$. Since $p(s_1) \neq p(r_1)$ and $p(u_1) \neq p(v_1)$, using Lemma 2.5 in case $M_1 = \emptyset$ or Lemma 2.9 in case $|M_1| = 1$, there exists a path partition $\{P_{s_1r_1}, P_{u_1v_1}\}$ of Q^1_{n-1} passing through M_1. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{s_1r_1}) \cup E(P_{u_1v_1}) \cup \{xy, ss_1, rr_1, uu_1, vv_1\}) \backslash \{xs, yr, uv\}$.

If $M \cap E_j = \{uu_1, vv_1\}$ and $d_{C_0}(u, v) > 1$, then $M_1 = \emptyset$. Select neighbors w, t of x, y on C_0, respectively, such that one of the paths between x and y contains w and the other contains t. Note that there are two ways to choose such w and t. Since Q_n is bipartite and $p(x) \neq p(y)$, the neighbors of x on C_0 and neighbors of y on C_0 are disjoint, so, moreover, we can choose w, t such that $v \notin \{w, t\}$. Since u is not adjacent to x or y on C_0, we have $\{u, v\} \cap \{w, t\} = \emptyset$. Choose neighbors s, r of u, v on C_0, respectively, such that $s \neq r$ and $\{s, r\} \cap \{w, t\} = \emptyset$. Since $d_{C_0}(u, v) > 1$, this is always possible. Note that then s, r, u, v, w, t are pairwise distinct vertices and $p(w_1) \neq p(t_1)$, $p(u_1) \neq p(s_1)$, $p(v_1) \neq p(r_1)$. Since $\{\{w_1, t_1\}, \{u_1, s_1\}, \{v_1, r_1\}\}$ is a balanced pair set and $n - 1 > 6$, by Lemma 2.8 there exists a path partition $\{P_{w_1t_1}, P_{u_1s_1}, P_{v_1r_1}\}$ of Q^1_{n-1}. Hence the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{w_1t_1}) \cup E(P_{u_1s_1}) \cup E(P_{v_1r_1}) \cup \{xy, uu_1, vv_1, rr_1, ss_1, ww_1, tt_1\}) \backslash \{xw, yt, us, vr\}$. Claim 2 is proved.

Claim 3 *Let xy and wt be two edges of M_0. Moreover, when $|M \cap E_j| \geq 1$, let xy, wt be such that $d(u, xy) \neq 1$ and $d(u, wt) \neq 1$. Let C_0 be a Hamiltonian cycle of Q^0_{n-1} containing $M_0 \backslash \{xy, wt\}$. If $\{xy, wt\} \cap E(C_0) = \emptyset$ and $|M \cap E_j| + |M_1| \leq 1$, then we can construct a Hamiltonian cycle containing M in Q_n.*

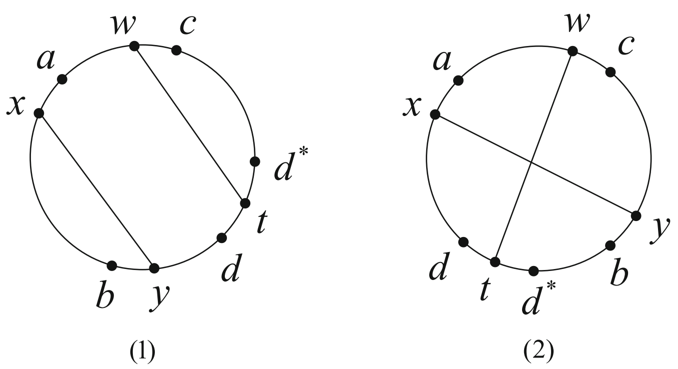

Fig. 2 Two possibilities of $\{xy, wt\}$ and neighbors of x, y, w, t in Claim 3

Since $\{xy, wt\} \cap E(C_0) = \emptyset$, there are two possibilities up to isomorphism, see Fig. 2. Let a, b, c, d be neighbors of x, y, w, t on C_0, respectively, see Fig. 2. Note that a, b, c, d are not arbitrarily chosen neighbors of x, y, w, t on C_0. Their order on the cycle is specified as depicted on Fig. 2. Also, it may happen that $a = w$ or $d = y$ on Fig. 2(1), or even some of $a = w, b = t, c = y, d = x$ on Fig. 2(2), while the constructions described below are still valid.

If $M \cap E_j = \{uu_1\}$, then $M_1 = \emptyset$. Observe that a, b, c and d are pairwise distinct. Since u is adjacent to none of $\{x, y, w, t\}$ on C_0, we have $u \notin \{a, b, c, d\}$. Let v be a neighbor of u on C_0 such that $v \notin \{a, b, c, d\}$. If $\{xy, wt\}$ is as on Fig. 2(1), let z_1, z_2, z_3, z_4 denote the vertices a_1, b_1, c_1, d_1, respectively. If $\{xy, wt\}$ is as on Fig. 2(2), let z_1, z_2, z_3, z_4 denote the vertices a_1, c_1, b_1, d_1, respectively. Since $p(a_1) \neq p(b_1)$ and $p(c_1) \neq p(d_1)$, we have $\{\{z_1, z_2\}, \{z_3, z_4\}, \{u_1, v_1\}\}$ is a balanced pair set. Since $n - 1 > 6$, by Lemma 2.8 there exists a path partition $\{P_{z_1z_2}, P_{z_3z_4}, P_{u_1v_1}\}$ of Q^1_{n-1}. Hence the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{z_1z_2}) \cup E(P_{z_3z_4}) \cup E(P_{u_1v_1}) \cup \{xy, wt, aa_1, bb_1, cc_1, dd_1, uu_1, vv_1\}) \setminus \{xa, yb, wc, td, uv\}$.

It remains to settle the case $M \cap E_j = \emptyset$. We distinguish two cases to consider. Note that $|M_1| \leq 1$.

Case 1. Suppose $\{xy, wt\}$ *is as on Fig.* 2(1). If $\{a_1, b_1\} \cap V(M_1) = \emptyset$ or $\{c_1, d_1\} \cap V(M_1) = \emptyset$, using Lemma 2.5 in case $M_1 = \emptyset$ or Lemma 2.9 in case $|M_1| = 1$, there exists a path partition $\{P_{a_1b_1}, P_{c_1d_1}\}$ of Q^1_{n-1} passing through M_1. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1b_1}) \cup E(P_{c_1d_1}) \cup \{xy, wt, aa_1, bb_1, cc_1, dd_1\}) \setminus \{xa, yb, wc, td\}$, see Fig. 3. Otherwise, $|\{a_1, b_1\} \cap V(M_1)| = |\{c_1, d_1\} \cap V(M_1)| = 1$. Note that in this case, $|M_1| = 1$. Let $M_1 = \{e\}$.

If $a_1d_1 = e$ or $b_1c_1 = e$, since $p(a_1) \neq p(b_1)$ and $p(c_1) \neq p(d_1)$, we have $p(a_1) \neq p(d_1)$ and $p(b_1) \neq p(c_1)$. Since $\{b_1, c_1\} \cap V(e) = \emptyset$ or $\{a_1, d_1\} \cap V(e) = \emptyset$, by Lemma 2.9, there is a path partition $\{P_{a_1d_1}, P_{b_1c_1}\}$ of Q^1_{n-1} passing through e. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1d_1}) \cup E(P_{b_1c_1}) \cup \{xy, wt, aa_1, bb_1, cc_1, dd_1\}) \setminus \{xa, yb, wc, td\}$. If $a_1c_1 = e$, let d^* be the neighbor of t on C_0 distinct with d. Since Q_n is bipartite and $p(w) \neq p(t)$, we have $p(d^*) \neq p(c)$, and therefore, $d^* \neq c$. Since $p(c_1) = p(b_1) \neq p(a_1) = p(d^*_1)$, by Lemma 2.5 there is a path partition $\{P_{a_1c_1} = a_1c_1 = e, P_{b_1d^*_1}\}$ of Q^1_{n-1}. Then the

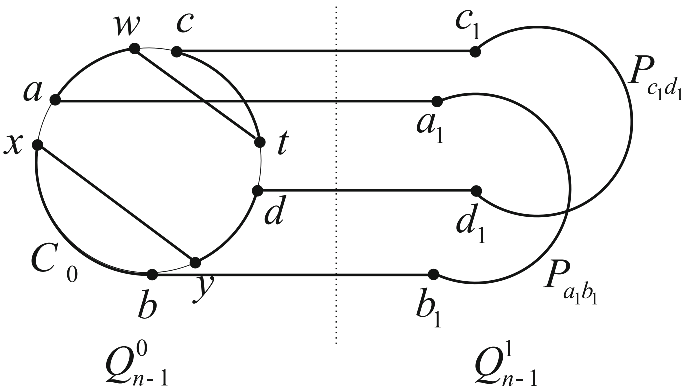

Fig. 3 A Hamiltonian cycle containing M in Q_n when $\{xy, wt\}$ is as on Fig. 2(1) and $\{a_1, b_1\} \cap V(M_1) = \emptyset$ or $\{c_1, d_1\} \cap V(M_1) = \emptyset$

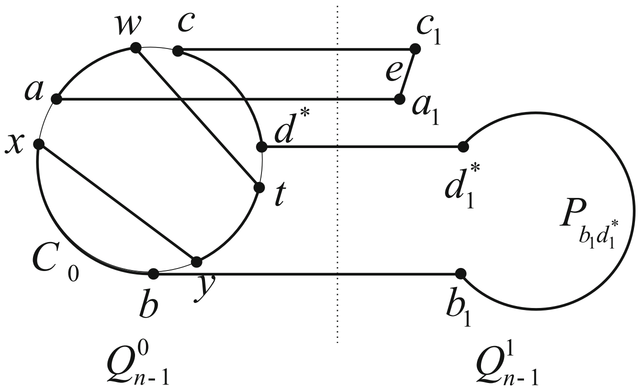

Fig. 4 A Hamiltonian cycle containing M in Q_n when $\{xy, wt\}$ is as on Fig. 2(1) and $a_1c_1 = e$

desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1c_1}) \cup E(P_{b_1d_1^*}) \cup \{xy, wt, aa_1, bb_1, cc_1, d^*d_1^*\}) \setminus \{xa, yb, wc, td^*\}$, see Fig. 4. If $b_1d_1 = e$, this case is isomorphic to the case $a_1c_1 = e$.

Case 2. Suppose $\{xy, wt\}$ *is as on Fig.* 2(2). Since $p(w) \neq p(t)$, we have $p(x) = p(w)$ or $p(x) = p(t)$, without loss of generality, we may assume that $p(x) = p(t)$, then $p(x) = p(t) \neq p(w) = p(y)$ and $p(a_1) = p(d_1) \neq p(c_1) = p(b_1)$.

If $\{a_1, c_1\} \cap V(M_1) = \emptyset$ or $\{b_1, d_1\} \cap V(M_1) = \emptyset$, using Lemma 2.5 in case $M_1 = \emptyset$ or Lemma 2.9 in case $|M_1| = 1$, there exists a path partition $\{P_{a_1c_1}, P_{b_1d_1}\}$ of Q^1_{n-1} passing through M_1. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1c_1}) \cup E(P_{b_1d_1}) \cup \{xy, wt, aa_1, bb_1, cc_1, dd_1\}) \setminus \{xa, yb, wc, td\}$, see Fig. 5.

Otherwise, $|\{a_1, c_1\} \cap V(M_1)| = |\{b_1, d_1\} \cap V(M_1)| = 1$. In this case, $|M_1| = 1$. Let $M_1 = \{e\}$. Since $p(a_1) = p(d_1) \neq p(c_1) = p(b_1)$, we have $a_1b_1 = e$ or $c_1d_1 = e$.

If $d_{C_0}(t, y) = 1$, since $p(a_1) \neq p(c_1)$ and $a_1c_1 \neq e$, by Lemma 2.3 there exists a Hamiltonian path $P_{a_1c_1}$ of Q^1_{n-1} passing through e. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1c_1}) \cup \{xy, wt, aa_1, cc_1\}) \setminus \{xa, wc, yt\}$. If

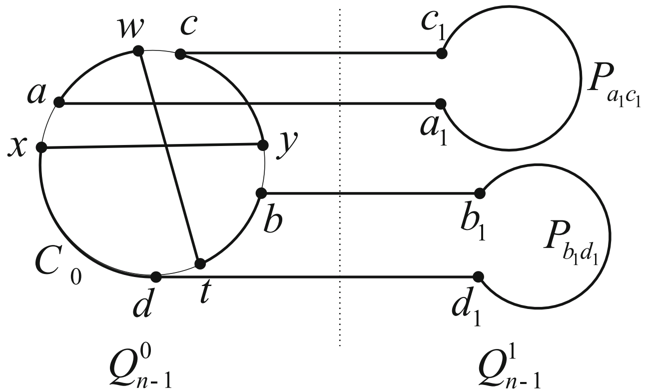

Fig. 5 A Hamiltonian cycle containing M in Q_n when $\{xy, wt\}$ is as on Fig. 2(2) and $\{a_1, c_1\} \cap V(M_1) = \emptyset$ or $\{b_1, d_1\} \cap V(M_1) = \emptyset$

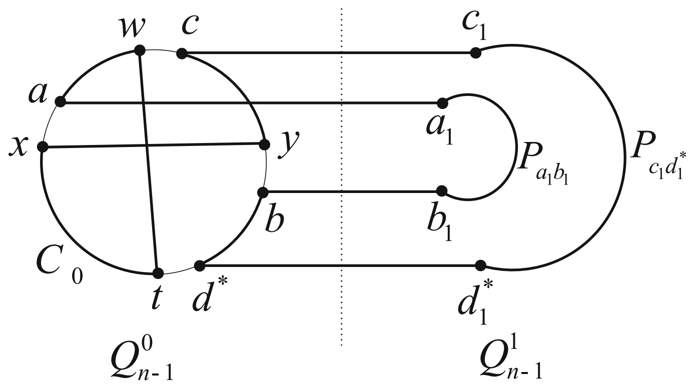

Fig. 6 A Hamiltonian cycle containing M in Q_n when $\{xy, wt\}$ is as on Fig. 2(2) and $|\{a_1, c_1\} \cap V(M_1)| = |\{b_1, d_1\} \cap V(M_1)| = 1$, $d_{C_0}(t, y) \neq 1$

$d_{C_0}(t, y) \neq 1$, then let d^* be the neighbor of t on C_0 distinct with d. Since $p(t) \neq p(y)$ by the above assumption, we have $p(d^*) \neq p(b)$, and therefore, $d^* \neq b$. Observe that a, b, c, d and d^* are distinct. When $a_1b_1 = e$, we have $\{c_1, d_1^*\} \cap V(e) = \emptyset$. When $c_1d_1 = e$, we have $\{a_1, b_1\} \cap V(e) = \emptyset$. Since $p(d_1^*) = p(t) \neq p(w) = p(c_1)$, by Lemma 2.9 there exists a path partition $\{P_{a_1b_1}, P_{c_1d_1^*}\}$ of Q^1_{n-1}. Then the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{a_1b_1}) \cup E(P_{c_1d_1^*}) \cup \{xy, wt, aa_1, bb_1, cc_1, d^*d_1^*\}) \setminus \{xa, yb, wc, td^*\}$, see Fig. 6. Claim 3 is proved.

If $|M_0| \leq 3n - 13 = 3(n - 1) - 10$, by the induction hypothesis there exists a Hamiltonian cycle C_0 containing M_0 in Q^0_{n-1} and therefore, the theorem holds by Claim 1. If $|M_0| = 3n - 12$, choose an edge $xy \in M_0$. Since $|M_0| = 3n - 12 > (n - 1) + 1$ for $n \geq 8$, moreover, in the case when $|M \cap E_j| \geq 1$, we may choose xy such that $d(u, xy) \neq 1$. Apply the induction to find a Hamiltonian cycle C_0 containing $M_0 \setminus \{xy\}$ in Q^0_{n-1}. If $xy \in E(C_0)$, then C_0 is a Hamiltonian cycle of Q^0_{n-1} containing M_0 and therefore, the theorem holds by Claim 1. If $xy \notin E(C_0)$, since $|M \cap E_j| + |M_1| \leq 2$, the theorem holds by Claim 2. If $|M_0| = 3n - 11$, then $|M \cap E_j| + |M_1| \leq 1$. Let

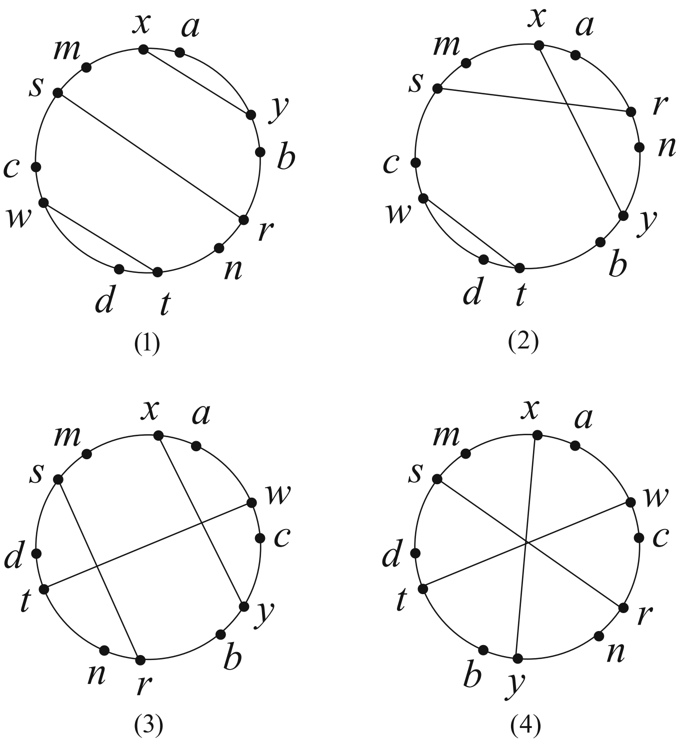

Fig. 7 Four possibilities when $\{xy, wt, sr\} \cap E(C_0) = \emptyset$ up to isomorphism

xy, wt be two edges of M_0. Since $|M_0| = 3n - 11 > (n-1)+2$, moreover, in the case when $|M \cap E_j| \geq 1$, we can choose xy, wt such that $d(u, xy) \neq 1$ and $d(u, wt) \neq 1$. Apply the induction to find a Hamiltonian cycle C_0 containing $M_0 \backslash \{xy, wt\}$ in Q^0_{n-1}. Using Claim 1 in case $\{xy, wt\} \subseteq E(C_0)$, or Claim 2 in case $|\{xy, wt\} \cap E(C_0)| = 1$, or Claim 3 in case $\{xy, wt\} \cap E(C_0) = \emptyset$, the theorem holds.

If $|M_0| = 3n - 10$, then $M \cap E_j = \emptyset = M_1$. Let $xy, wt, sr \in M_0$. Apply the induction to find a Hamiltonian cycle C_0 containing $M_0 \backslash \{xy, wt, sr\}$ in Q^0_{n-1}. If $\{xy, wt, sr\} \cap E(C_0) \neq \emptyset$, the conclusion holds by Claim 1–3. If, however, $\{xy, wt, sr\} \cap E(C_0) = \emptyset$, there are four possibilities up to isomorphism, see Fig. 7. Let a, b, c, d, m, n be neighbors of x, y, w, t, s, r on C_0, respectively, see Fig. 7. Since M is a matching and $xy, wt, sr \in M_0$, we have a, b, c, d, m, n are pairwise distinct vertices and $\{xa, yb, wc, td, sm, rn\} \cap M = \emptyset$. If $\{xy, wt, sr\}$ is as on Fig. 7(1) or (3), let $z_1, z_2, z_3, z_4, z_5, z_6$ denote the vertices $a_1, b_1, c_1, d_1, m_1, n_1$, respectively. If $\{xy, wt, sr\}$ is as on Fig. 7(2), let $z_1, z_2, z_3, z_4, z_5, z_6$ denote the vertices $a_1, m_1, b_1, n_1, c_1, d_1$, respectively. If $\{xy, wt, sr\}$ is as on Fig. 7(4), let $z_1, z_2, z_3, z_4, z_5, z_6$ denote the vertices $a_1, c_1, b_1, n_1, d_1, m_1$, respectively.

Since $p(a_1) \neq p(b_1)$, $p(c_1) \neq p(d_1)$ and $p(m_1) \neq p(n_1)$, we have $\{\{z_1, z_2\}, \{z_3, z_4\}, \{z_5, z_6\}\}$ is a balanced pair set. Since $n - 1 > 6$, by Lemma 2.8 there exists a path partition $\{P_{z_1z_2}, P_{z_3z_4}, P_{z_5z_6}\}$ of Q^1_{n-1}. Hence the desired Hamiltonian cycle of Q_n is induced by edges of $(E(C_0) \cup E(P_{z_1z_2}) \cup E(P_{z_3z_4}) \cup E(P_{z_5z_6}) \cup \{xy, wt, sr, aa_1, bb_1, cc_1, dd_1, mm_1, nn_1\}) \setminus \{xa, yb, wc, td, sm, rn\}$.

The proof of Theorem 1.2 is complete.

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