Graphs and Combinatorics (2015) 31:2103–2111
DOI 10.1007/s00373-014-1516-z

ORIGINAL PAPER

Graphs with a 3-Cycle-2-Cover

Zhi-Hong Chen · Miaomiao Han · Hong-Jian Lai · Mingquan Zhan

Received: 26 May 2014 / Revised: 3 November 2014 / Published online: 13 January 2015

Abstract If a graph G has three even subgraphs C_1, C_2 and C_3 such that every edge of G lies in exactly two members of $\{C_1, C_2, C_3\}$, then we say that G has a 3-cycle-2-cover. Let S_3 denote the family of graphs that admit a 3-cycle-2-cover, and let $\mathcal{S}(h, k) = \{G : G$ is at most h edges short of being k-edge-connected$\}$. Catlin (J Gr Theory 13:465–483, 1989) introduced a reduction method such that a graph $G \in S_3$ if its reduction is in S_3; and proved that a graph in the graph family $\mathcal{S}(5, 4)$ is either in S_3 or its reduction is in a forbidden collection consisting of only one graph. In this paper, we introduce a weak reduction for S_3 such that a graph $G \in S_3$ if its weak reduction is in S_3, and identify several graph families, including $\mathcal{S}(h, 4)$ for an integer $h \geq 0$, with the property that any graph in these families is either in S_3, or its weak reduction falls into a finite collection of forbidden graphs.

Keywords 3-cycle-2-cover · Nowhere zero flows · Collapsible graphs · Reduction

1 Introduction

We study finite and loopless graphs with undefined terms and notations following Bondy and Murty [1]. For graphs G and H, $H \subseteq G$ means that H is a subgraph of

Z.-H. Chen (✉)
Department of Computer Science and Software Engineering, Butler University, Indianapolis, IN 46208, USA
e-mail: chen@butler.edu

M. Han · H-J. Lai
Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

M. Zhan
Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, USA

G. If X is an edge subset not in G but every edge in X has its end vertices in G, then $G + X$ is the graph with vertex set $V(G)$ and edge set $E(G) \bigcup X$. For a graph G, let $\kappa'(G)$ denote the edge-connectivity of G. A **circuit** is defined to be a nontrivial 2-regular connected graph, and a **cycle** to be an edge-disjoint union of circuits. A circuit of length n will be denoted as C^n. Often a cycle is also called an **even** graph. A 3-**cycle**-2-**cover** of G is a collection of 3 cycles of G such that each edge of G is in exactly two cycles of the collection.

The study of graphs with a 3-cycle-2-cover is motivated by the theory of nowhere zero flows, initiated by Tulle [23] more than half a century ago. Let $D = D(G)$ be an orientation of a graph G. For a vertex $v \in V(D)$, let $E_D^+(v)$ ($E_D^-(v)$, respectively) denote the set of all edges oriented outgoing from v (oriented incoming into v, respectively). Let $k > 1$ be an integer. A function f from $E(D)$ to the set of integers is a nowhere zero k-flow if for any $e \in E(D)$, $f(e) \neq 0$ and $|f(e)| < k$ and for any $v \in V(D)$, $\sum_{e \in E_D^+(v)} f(e) = \sum_{e \in E_D^-(v)} f(e)$. It is well known (for example, see [5,15,22]) that a connected graph G admitting a nowhere zero 4-flow if and only if G has a 3-cycle-2-cover.

For a graph G, let $O(G)$ be the set of odd-degree vertices of G. Thus G is a cycle if and only if $O(G) = \emptyset$. A graph G is *collapsible* ([4], see also Proposition 1 of [17]) if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - E(\Gamma_R)$ is connected. Following Catlin [5], we use $\mathcal{CL}$ to denote the family of collapsible graphs. An edge subset $X \subseteq E(G)$ is an $O(G)$-**join** if $O(G[X]) = O(G)$. We have the following observations.

Observation 1.1 *Let G be a graph.*

(i) *An edge subset $X \subseteq E(G)$ is an $O(G)$-join of G if and only if $G - X$ is a cycle.*
(ii) *If $E(G) = E_1 \bigcup E_2 \bigcup E_3$ is a disjoint union of 3 $O(G)$-joins, then G has 3 cycles $C_i = G - E_i$, $i = 1, 2, 3$, such that every edge $e \in E(G)$ is in exactly two members of the set (possibly a multiset) $\{C_1, C_2, C_3\}$. (In this case, $\{C_1, C_2, C_3\}$ is a 3-cycle-2-cover of G).*

Following Catlin [5], we define S_3 to be the family of connected graphs admitting a 3-cycle-2-cover. A graph G in S_3 will be called an S_3-**graph**. As mentioned above, S_3 is the family of connected graphs that admit nowhere zero 4-flows.

Jaeger [14] proved that every 4-edge-connected graph is in S_3. It is known (see [5,15, 22]) that 3-edge-connectedness does not warrant a membership in S_3, as evidenced by the Petersen graph. Hence, characterizing S_3-graphs among 3-edge connected graphs has been a problem for investigation. Such problem is not just interesting by itself, it is also closely related to the study on Chinese Postman problem and Traveling Salesman problem [2].

Catlin in [5] defined a graph reduction and identified a family $\mathcal{F}$ of 3-edge-connected graphs that are closed to be 4-edge-connected, with the property that a graph $G \in \mathcal{F}$ is either in S_3 or its reduction is in $\{P(10)\}$, where $P(10)$ is the Petersen graph.

Graph contraction is needed to describe Catlin's reduction. For $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If $H \subseteq G$, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H, and is denoted by $PI_G(v_H)$. Given a family $\mathcal{F}$ of connected graphs, for any graph G, an $\mathcal{F}$-**reduction** of G is obtained from G by successively contracting nontrivial subgraphs in $\mathcal{F}$ until none left.

Catlin in [4] showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$, and the $\mathcal{CL}$-**reduction** of G is exactly $G' = G/(\cup_{i=1}^{c} E(H_i))$, which is unique. For a family $\mathcal{F}$ of graphs, Catlin in [7] defined

$$\mathcal{F}^o = \{H | H \text{is connected, and for graph} G \text{ with } H \subseteq G, G/H \in \mathcal{F} \\ \text{if and only if } G \in \mathcal{F}\}. \quad (1.1)$$

Let C^4 denote a circuit of length 4. For the family S_3, Catlin [5] showed $\mathcal{CL} \bigcup \{C^4\} \subseteq S_3^o$. In [5], Catlin defined, for integers $k, t > 0$,

$$\mathcal{S}(h, k) = \{G : \text{ for some edge set} X \cap E(G) = \emptyset \text{ with } |X| \le h, \\ \text{and } \kappa'(G + X) \ge k\}. \quad (1.2)$$

Theorem 1.2 (Catlin, Theorem 14 of [5]) *Let G be a graph in $\mathcal{S}(5, 4)$. Then exactly one of the following holds:*

(i) $G \in S_3$.
(ii) *G has at least one cut-edge.*
(iii) *The $\mathcal{CL} \bigcup \{C^4\}$-reduction of G is the Petersen graph.*

Theorem 1.2 indicates that within certain graph families, one can characterize S_3-graphs in term of excluding a finite list of reductions. The purpose of this paper is to continue such investigation by studying more general families of graphs and to give a characterization of S_3-graphs within these families by excluding a finite list of certain reductions. To this aim, we define, for integers $h, k > 0$,

$$N_h(k) = \{G : G \text{ is simple}, |V(G)| \le k, \kappa'(G) \ge h, \text{ and } G \notin S_3\}.$$

In Theorem 3.10 of [9], it is shown that under certain general and necessary condition of $\mathcal{F}$, the $\mathcal{F}^o$-reduction is unique. In particular, the S_3^o-reduction of any graph G is uniquely determined by G. We in the next section will define a weak reduction for the family S_3 (called **weak** S_3-**reduction**) in which we might not have the uniqueness.

Suppose that a, b are real numbers with $0 < a < 1$, and $f_{a,b}(n) = an + b$ is a function of n. Let $C(h, a, b)$ denote the family of simple graphs G of order n with $\kappa'(G) \ge h$ such that for any edge cut X of G with $|X| \le 3$, each component of $G - X$ has at least $f_{a,b}(n)$ vertices.

If a graph G has a spanning eulerian subgraph, then G is **supereulerian**. It is well known that all supereulerian graphs are in S_3 (see, for example, Section 7 of [6]). The prior results of graph families $C(h, a, b)$ are summarized in the theorem below.

Theorem 1.3 *Let $G \in C(h, a, b)$ be a graph. Then each of the following holds.*

(i) *(Catlin and Li [11]) If $h = 2$, $a = \frac{1}{5}$ and $b = 0$, then G is supereulerian or the reduction of G is in $\{K_{2,3}, K_{2,5}\}$. Hence in any case, $G \in S_3$.*

(ii) *(Broersma and Xiong [3]) If* $h = 2, a = \frac{1}{5}$ *and* $b = -\frac{2}{5}$*, then G is supereulerian or the reduction of G is in a family of 3 exceptional cases, all of which are in* S_3.
(iii) *(Li et al, [18]) If* $h = 2, a = \frac{1}{6}$ *and* $b = -\frac{2}{5}$*, then G is supereulerian or the reduction of G is in a finite family of exceptional cases. Thus any such G is in* S_3 *if and only if the* $\mathcal{CL}$*-reduction of G is not in a finite forbidden family of graphs.*
(iv) *(Lai and Liang [16]) If* $h = 2, a = \frac{1}{6}$ *and b is any fixed number, then G is supereulerian or the reduction of G is in a finite family of exceptional cases. Thus any such G is in* S_3 *if and only if the* $\mathcal{CL}$*-reduction of G is not in a finite forbidden family of graphs.*
(v) *(Li et al [19]) If* $h = 2, a = \frac{1}{7}$ *and* $b = 0$*, then G is supereulerian or the reduction of G is in a finite family of exceptional cases. Thus any such G is in* S_3 *if and only if the* $\mathcal{CL}$*-reduction of G is not in a finite forbidden family of graphs.*
(vi) *(Niu and Xiong [21]) If* $h = 3, a = \frac{1}{10}$ *and b is any fixed number, then G is supereulerian or the reduction of G is in a finite family of exceptional cases. Thus any such G is in* S_3 *if and only if the* $\mathcal{CL}$*-reduction of G is not in a finite forbidden family of graphs.*

Theorems 1.2 and 1.3 motivate our research. The main results of this paper are the following.

Theorem 1.4 *Let G be a graph of order n. For any real numbers a and b with* $0 < a < 1$*, if* $G \in C(2, a, b)$*, then one of the following holds.*

(i) $G \in S_3$.
(ii) *Every weak* S_3*-reduction of G is in* $N_2(\lceil \frac{3}{a} \rceil)$.

For a graph G, let $t_3(G)$ be the number of 3-edge-cuts of G. For a given integer k, define

$$\mathcal{W}(k) = \{G \mid G \text{ is simple and } t_3(G) \le k\}.$$

Theorem 1.5 *Let G be a graph of order n with* $\kappa'(G) \ge 3$*. For a given integer* $k \ge 0$*, if* $G \in \mathcal{W}(k)$*, then one of the following holds.*

(i) $G \in S_3$.
(ii) $k \ge 10$ *and every weak* S_3*-reduction of G is in* $N_3(2k - 10)$.

Theorem 1.6 *Let G be a graph of order n. For an integer* $h \ge 0$*, if* $G \in \mathcal{S}(h, 4)$ *satisfies* $\kappa'(G) \ge 3$*, then one of the following holds.*

(i) $G \in S_3$.
(ii) $h \ge 5$*, and every weak* S_3*-reduction of G is in* $N_3(4h - 10)$.

It is well known that the Petersen graph is the only 3-edge-connected graph with at most 10 vertices that is not in S_3. Hence when $h = 5$, Theorem 1.6 implies that a graph $G \in \mathcal{S}(5, 4)$ is not in S_3 if and only if the only weak S_3-reduction of G is the Petersen graph. This fact relates our result to Catlin's Theorem 14 of [5]. Furthermore, for given a, b, k and h, each graph in $N_2(\lceil \frac{3}{a} \rceil) \cup N_3(2k-10) \cup N_3(4h-10)$ has order independent on n. Thus, the number of graphs in $N_2(\lceil \frac{3}{a} \rceil) \cup N_3(2k-10) \cup N_3(4h-10)$

is fixed and finite. From a computational point of view, for given a, b, k and h, each of these families: $N_2(\lceil \frac{3}{a} \rceil)$ or $N_3(2k-10)$ or $N_3(4h-10)$, can be determined in a constant time. Like the characterization of planar graphs, people view that K_5 and $K_{3,3}$ are the only two nonplanar graphs. By Theorems 1.4, 1.5 and 1.6, in some sense, we can see that only a finite number of graphs in $C(2, a, b)$ or 3-edge-connected graphs in $\mathcal{W}(k) \cup \mathcal{S}(h, 4)$ are not in S_3.

In Sect. 2, weak S_3-reduction of graphs will be introduced and certain reduction results will be reviewed and developed. The proofs of the main theorems are given in the last section.

2 Reductions

We will introduce weak S_3-reduction of graphs in this section. Let G be a graph and $i \geq 0$ be an integer. Define

$$V_i(G) = \{v \in V(G) | d_G(v) = i\}; \quad \text{and} \quad d_i(G) = |V_i(G)|.$$

For a vertex $v \in V(G)$, $N_G(v)$, the neighborhood of v, is the set of vertices adjacent to v in G. For a vertex $u \in V(G)$ with $N_G(u) = \{v_1, v_2, v_3, v_4\}$, let $\pi = \langle \{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\} \rangle$ be a 2-partition of $N_G(u)$ into a pair of 2-subsets. Define G_π to be the graph obtained from $G - u$ by adding new edges $v_{i_1}v_{i_2}$, $v_{i_3}v_{i_4}$. We say that G_π is obtained from G by **dissolving** u (via a 2-partition π).

Theorem 2.1 (Fleischer [12], Mader [20]) *If $u \in V_4(G)$ with $|N_G(u)| = 4$, then for some 2-partition π of $N_G(u)$, $\kappa'(G_\pi) = \kappa'(G)$.*

Theorem 2.2 (Catlin) *Let G be a graph, H be a collapsible subgraph of G, G_π be the graph obtained from G by dissolving a vertex $u \in V_4(G)$, and G' be the $\mathcal{CL}$-reduction of G. Then each of the following holds.*

(i) *(Corollary 13A of [5]) $\mathcal{CL} \cup \{C^4\} \subset S_3^o$. In particular, $G' \in S_3$ if and only if $G \in S_3$.*
(ii) *(Lemma 3 of [5]) If $G_\pi \in S_3$, then $G \in S_3$.*
(iii) *(Theorem 8 of [4]) G' is simple.*

For a graph G, let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with 2-edge-disjoint spanning trees. The following has been proved.

Theorem 2.3 *Let G be a connected graph. Each of the following holds.*

(i) *(Catlin, Theorem 7 of [4]) If $F(G) \leq 1$, then either G is collapsible or the reduction of G is K_2.*
(ii) *(Catlin et al, Theorem 1.3 of [8]) If $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.*

It follows from Theorems 2.2 and 2.3 that

$$\text{if } \kappa'(G) \geq 2 \quad \text{and} \quad F(G) \leq 2, \text{ then } G \in S_3. \tag{2.1}$$

Let G' be the $\mathcal{CL}$-reduction of G. By Lemma 2.3 of [8], we have

$$F(G') = 2|V(G')| - |E(G')| - 2. \tag{2.2}$$

As $|V(G')| = \sum_{i\geq 1} d_i(G')$ and $2|E(G')| = \sum_{i\geq 1} id_i(G')$, it follows from (2.2) that

$$2F(G') = 4\sum_{i\geq 1} d_i(G') - \sum_{i\geq 1} id_i(G') - 4 = \sum_{i\geq 1}(4-i)d_i(G') - 4,$$

and so

$$3d_1(G') + 2d_2(G') + d_3(G') = 2F(G') + 4 + \sum_{i\geq 5}(i-4)d_i(G'). \tag{2.3}$$

Let G be a graph and G' be the $\mathcal{CL}$-reduction of G. A **weak S_3-reduction** of G is obtained from G' by repeatedly dissolving vertices of degree 4 in G' while preserving the edge-connectivity of G', until no vertices of degree 4 are left. Parts (i) and (ii) of the following lemma are immediate consequences of the definition of weak S_3-reduction and Theorem 2.2. Part (iii) is a consequence of (2.3) and Part (i).

Lemma 2.4 *Let G' be the $\mathcal{CL}$-reduction of G and G'' be a weak S_3-reduction of G.*

(i) $V_4(G'') = \emptyset$, *and for any* $i \neq 4$, $d_i(G'') = d_i(G')$.
(ii) *If* $G'' \in S_3$, *then* $G \in S_3$.
(iii) $3d_1(G'')+2d_2(G'')+d_3(G'') = 2F(G')+4+\sum_{i\geq 5}(i-4)d_i(G'')$. *In particular, if* $\kappa'(G) \geq 3$, *then* $d_3(G'') = 2F(G') + 4 + \sum_{i\geq 5}(i-4)d_i(G'')$.

To prove our main results, we need to show that

$$\text{a graph } G \text{ is in } S_3 \text{ if and only if } G \text{ has one weak } S_3\text{-reduction in } S_3. \tag{2.4}$$

Theorem 2.2 indicates that if a weak S_3-reduction of G is in S_3, then $G \in S_3$. To show the necessity of (2.4), we will prove the following lemma to justify (2.4).

Lemma 2.5 *Let G be a connected graph. If $G \in S_3$, then G has one weak S_3-reduction in S_3.*

Proof Let $G \in S_3$, and let G' be the $\mathcal{CL}$-reduction of G. By Theorem 2.2, $G' \in S_3$. We shall show that a weak reduction G'' of G is in S_3. If $V_4(G') = \emptyset$, then $G'' = G'$ is the weak S_3-reduction of G. As $G' \in S_3$, we are done. Hence we argue by induction on $|V_4(G')|$ and assume that $V_4(G') \neq \emptyset$.

Pick a vertex $u \in V_4(G')$. By Theorem 2.2, G' is simple and so we may assume that $N_{G'}(u) = \{v_1, v_2, v_3, v_4\}$ and $E_{G'}(u) = \{uv_1, uv_2, uv_3, uv_4\}$. To complete inductive argument, we shall find a 2-partition π of $N_{G'}(u)$ such that $G'_\pi \in S_3$. Note that by the definition of G'_π, we can view $V(G') - \{u\} = V(G'_\pi)$. As $u \in V_4(G')$, $O(G') = O(G'_\pi)$.

Since $G' \in S_3$, there exist edge-disjoint $O(G')$-joins $E'_1, E'_2, E'_3 \subseteq E(G')$ such that $E'_1 \bigcup E'_2 \bigcup E'_3 = E(G')$. For $i = 1, 2, 3$, since $u \notin O(G')$ and since E'_i is an

$O(G')$-join, $|E_{G'}(u) \bigcap E'_i| \equiv 0 \pmod 2$. Since $\{E'_1, E'_2, E'_3\}$ is a partition of $E(G')$, we may assume that either $E_{G'}(u) \subseteq E'_1$ and $|E_{G'}(u) \bigcap E'_i| = 0$ for $i \in \{2, 3\}$, or $|E_{G'}(u) \bigcap E'_1| = |E_{G'}(u) \bigcap E'_2| = 2$ and $|E_{G'}(u) \bigcap E'_3| = 0$.

Case 1. $E_{G'}(u) \subseteq E'_1$ and $|E_{G'}(u) \bigcap E'_i| = 0$ for $i \in \{2, 3\}$.
Define $\pi = \langle\{v_1, v_2\}, \{v_3, v_4\}\rangle$, and let $E''_1 = (E'_1 - E_{G'}(u)) \bigcup\{v_1v_2, v_3v_4\}$, $E''_2 = E'_2$ and $E''_3 = E'_3$. As $O(G') = O(G'_\pi)$, each E''_i is an $O(G'_\pi)$-join. Since E'_1, E'_2, E'_3 are edge-disjoint in $E(G')$ with $E'_1 \bigcup E'_2 \bigcup E'_3 = E(G')$, we conclude that E''_1, E''_2, E''_3 are edge-disjoint in $E(G'_\pi)$ with $E''_1 \bigcup E''_2 \bigcup E''_3 = E(G'_\pi)$. By definition, $G'_\pi \in S_3$.

Case 2. $|E_{G'}(u) \bigcap E'_1| = |E_{G'}(u) \bigcap E'_2| = 2$ and $|E_{G'}(u) \bigcap E'_3| = 0$.
Without loss of generality, we assume that $uv_1, uv_2 \in E'_1$ and $uv_3, uv_4 \in E'_2$. Define $\pi = \langle\{v_1, v_2\}, \{v_3, v_4\}\rangle$, and let $E''_1 = (E'_1 - E_{G'}(u)) \bigcup\{v_1v_2\}$, $E''_2 = (E'_2 - E_{G'}(u)) \bigcup\{v_3v_4\}$ and $E''_3 = E'_3$. As $O(G') = O(G'_\pi)$, each E''_i is an $O(G'_\pi)$-join. Since E'_1, E'_2, E'_3 are edge-disjoint in $E(G')$ with $E'_1 \bigcup E'_2 \bigcup E'_3 = E(G')$, we conclude that E''_1, E''_2, E''_3 are edge-disjoint in $E(G'_\pi)$ with $E''_1 \bigcup E''_2 \bigcup E''_3 = E(G'_\pi)$. By definition, $G'_\pi \in S_3$.

As in either case, we can always find a 2-partition π of $N_{G'}(u)$ such that $G'_\pi \in S_3$, the lemma is proved by induction. □

3 Proof of The Main Results

We shall prove the main results in this section. Throughout this section, a, b denote two real numbers with $0 < a < 1$, and $h, k > 0$ denote two integers. Let G be a graph in $C(2, a, b) \cup S(h, 4) \cup \{G : \kappa'(G) \geq 2, t_3(G) \leq k\}$. Assume that G is not in $\mathcal{S}_3$, by (2.1), we have $F(G') \geq 3$. Let G'' be a weak S_3-reduction of G. We shall show that $|V(G'')|$ must be bounded by the quantities given in Theorems 1.4, 1.5 and 1.6, respectively. To simplify notations, for each i, let $d_i = d_i(G'')$.

Proof of Theorem 1.4 Assume first that $G \in C(2, a, b)$. By Lemma 2.4 (i) and (iii) and by $\kappa'(G) \geq 2$, we have

$$2(d_2 + d_3) \geq 2d_2 + d_3 = 2F(G') + 4 + \sum_{i \geq 4}(i - 4)d_i. \tag{3.1}$$

By (2.1) and (3.1), we have

$$2(d_2 + d_3) \geq 10 + \sum_{i \geq 4}(i - 4)d_i \geq 10 + \sum_{i \geq 5} d_i. \tag{3.2}$$

By the definition of $C(2, a, b)$, then the edges incident to a vertex of degree two (or three) in G' correspond to a 2-edge-cut (or 3-edge-cut) in G. We have $(d_2 + d_3)(an + b) \leq n$, and so $d_2 + d_3 \leq \dfrac{n}{an + b} \leq \left\lceil \dfrac{1}{a} \right\rceil$ (if $b < 0, n > -\frac{b}{a}(1 + \frac{1}{a})$). It follows by

(3.2) that

$$|V(G'')| = (d_2 + d_3) + \sum_{i \geq 5} d_i \leq 3(d_2 + d_3) \leq \left\lceil \frac{3}{a} \right\rceil,$$

which implies Theorem 1.4. □

Proof of Theorem 1.5 Next we assume that $\kappa'(G) \geq 3$ and $t_3(G) \leq k$. By the definition of contraction, every 3-edge-cut of G' is a 3-edge-cut of G, and so $k \geq t_3(G) \geq t_3(G') \geq d_3$. By Lemma 2.4 (i) and (iii) and $\kappa'(G) \geq 3$, we have

$$k \geq d_3 = 2F(G') + 4 + \sum_{i \geq 5} (i - 4)d_i.$$

By (2.1) and $\kappa'(G) \geq 3$, we have $F(G') \geq 3$, and

$$k - 10 \geq d_3 - 10 \geq \sum_{i \geq 5} (i - 4)d_i.$$

It follows that

$$|V(G'')| = d_3 + \sum_{i \geq 5} d_i \leq d_3 + (d_3 - 10) \leq 2k - 10,$$

which implies Theorem 1.5. □

Proof of Theorem 1.6 Assume that $G \in S(h, 4)$ with $\kappa'(G) \geq 3$. By the definition of $S(h, 4)$, for any $G \in S(h, 4)$, there exists an edge subset X not in G such that $\kappa'(G + X) \geq 4$ with $|X| \leq h$. Since $\delta(G + X) \geq \kappa'(G + X) \geq 4$, we have $d_3 \leq 2h$. By Lemma 2.4 (i) and (iii), we have

$$d_3 = 2F(G') + 4 + \sum_{i \geq 5} (i - 4)d_i. \tag{3.3}$$

By (2.1), $F(G') \geq 3$. This, together with (3.3), implies

$$d_3 \geq 10 + \sum_{i \geq 5} (i - 4)d_i \geq 10 + \sum_{i \geq 5} d_i. \tag{3.4}$$

By (3.4),

$$|V(G'')| = d_3 + \sum_{i \geq 5} d_i \leq 2h + 2h - 10 = 4h - 10,$$

which implies Theorem 1.6. □

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, New York (2008)
2. Boyd, S., Sitters, R., Ster, S., Stougie, L.: The traveling salesman problem on cubic and subcubic graphs. Math. Progr. Ser. A **144**, 227–247 (2014)
3. Broersma, H.J., Xiong, L.: A note on minimum degree conditions for supereulerian graphs. Discret. Appl. Math. **120**, 35–43 (2002)
4. Catlin, P.A.: A reduction method to find spanning Eulerian subgraphs. J. Gr. Theory **12**, 29–44 (1988)
5. Catlin, P.A.: Double cycle covers and the Petersen graph. J. Gr. Theory **13**, 465–483 (1989)
6. Catlin, P.A.: Super-Eulerian graphs: a survey. J. Gr. Theory **16**, 177–196 (1992)
7. Catlin, P.A.: The reduction of graph families under contraction. Discret. Math. **160**, 67–80 (1996)
8. Catlin, P.A., Han, Z., Lai, H.-J.: Graphs without spanning closed trails. Discret. Math. **160**, 81–91 (1996)
9. Catlin, P.A., Hobbs, A.M., Lai, H.-J.: Graph families operations. Discret. Math. **230**, 71–97 (2001)
10. Catlin, P.A., Lai, H.-J., Shao, Y.: Edge-connectivity and edge-disjoint spanning trees. Discret. Math. **309**, 1033–1040 (2009)
11. Catlin, P.A., Li, X.W.: Supereulerian graphs of minimum degree at least 4. Adv. Math. (China) **28**, 65–69 (1999)
12. Fleischer, H.: Eine gemeinsame Basis für die Theorie der Eulerschen Graphen und den Satz von Petersen. Monats. Matt. **81**, 267–278 (1976)
13. Hungerford, T.W.: Algebra. Springer, New York (1974)
14. Jaeger, F.: Flows and generalized coloring theorems in graphs. J. Comb. Theory (B) **26**, 205–216 (1979)
15. Jaeger, F.: Nowhere-Zero Flow Problems, Topics in Graph Theory, vol. 3. Academic Pess, London (1988)
16. Lai, H.-J., Liang, Y.: Supereulerian graphs in the graph family $C_2(6, k)$. Discret. Appl. Math. **159**, 467–477 (2011)
17. Lai, H.-J., Shao, Y., Yan, H.: An update on super Eulerian graphs. WSEAS Trans. Math. **12**, 926–940 (2013)
18. Li, D.X., Lai, H.-J., Zhan, M.Q.: Eulerian subgraphs and Hamilton-connected line graphs. Discret. Appl. Math. **145**, 422–428 (2005)
19. Li, X., Wang, C., Fan, Q., Niu, Z., Xiong, L.: Spanning Eulerian subgraphs of 2-edge-connected graphs. Gr. Combin. **29**, 275–280 (2013)
20. Mader, W.: Edge-connectivity preserving reductions. In: Proceedings of Advances in Graph Theory, North-Holland (1978)
21. Niu, Z., Xiong, L.: Supereulerianity of k-edge-connected graphs with a restriction on small bonds. Discret. Appl. Math. **158**, 37–43 (2010)
22. Zhang, C.Q.: Integer Flows and Cycle Covers of Graphs. Marcel Dekker Inc, New York (1997)
23. Tutte, W.T.: A contribution to the theory of chromatic polynomials. Can. J. Math. **6**, 80–91 (1954)