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Total Dominator Colorings and Total Domination in Graphs

Michael A. Henning

Abstract A total dominator coloring of a graph G is a proper coloring of the vertices of G in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic number $\chi_d^t(G)$ of G is the minimum number of colors among all total dominator coloring of G. A total dominating set of G is a set S of vertices such that every vertex in G is adjacent to at least one vertex in S. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. We establish lower and upper bounds on the total dominator chromatic number of a graph in terms of its total domination number. In particular, we show that every graph G with no isolated vertex satisfies $\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + \chi(G)$, where $\chi(G)$ denotes the chromatic number of G. We establish properties of total dominator colorings in trees. We characterize the trees T for which $\gamma_t(T) = \chi_d^t(T)$. We prove that if T is a tree of $n \geq 2$ vertices, then $\chi_d^t(T) \leq 2(n+1)/3$ and we characterize the trees achieving equality in this bound.

Keywords Graph colorings · Total domination · Total dominator coloring

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1 Introduction

A proper vertex coloring of a graph G is an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned

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distinct colors. If k colors are used, then the coloring is referred to as a k-coloring. In a given coloring of G, a color class of the coloring is a set consisting of all those vertices assigned the same color. The vertex chromatic number $\chi(G)$ of G is the minimum integer k such that G is k-colorable. In this paper, we simply call a proper vertex coloring a proper coloring and we refer to the vertex chromatic number as the chromatic number. The concept of a graph coloring is very well-studied in graph theory. Indeed, the chromatic number is perhaps the most studied of all graph theoretic parameters.

A dominator coloring of a graph G is a proper coloring of G in which every vertex dominates every vertex of at least one color class; that is, every vertex in $V(G)$ is adjacent to all other vertices in its own color class or is adjacent to all vertices from at least one (other) color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. Since every dominator coloring of G is a coloring of G, we observe that $\chi(G) \leq \chi_d(G)$. The concept of a dominator coloring in a graph was introduced and studied by Gera et al. [5] and studied further, for example, by Gera [3,4] and Chellali and Maffray [1].

The dominator chromatic number of a graph is related to its domination number. A dominating set of a graph G is a set $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [6,7]. Gera [3,4] established the following upper and lower bounds on the dominator chromatic number of an arbitrary graph in terms of its domination number and chromatic number.

Theorem 1 ([3,4]) *Every graph G satisfies*

$$\max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G).$$

Recently, Kazemi [10] studied the new concept of a total dominator coloring in a graph. A total dominator coloring, abbreviated TD-coloring, of a graph G with no isolated vertex is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number $\chi_d^t(G)$ of G is the minimum number of color classes in a TD-coloring of G. A $\chi_d^t(G)$-coloring of G is any total dominator coloring with $\chi_d^t(G)$ colors. A color class C in a given TD-coloring $\mathcal{C}$ of G is free if each vertex of G is adjacent to every vertex of some color class different from C.

The total dominator chromatic number of a graph is related to its total domination number. A total dominating set, abbreviated TD-set, of G is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination is now well studied in graph theory. The literature on the subject on total domination in graphs has been surveyed and detailed in the recent book [9]. A survey of total domination in graphs can also be found in [8].

Consider an arbitrary $\chi_d^t(G)$-coloring of G, and let S be a set consisting of one vertex from each of the $\chi_d^t(G)$ color classes. Since every vertex in G is adjacent to

every vertex of some color class (different from its own color class), the set S is a TD-set in G, implying that $\gamma_t(G) \leq |S| = \chi_d^t(G)$. Hence we have the following result, first observed by Kazemi [10].

Observation 2 ([10]) For every graph G with no isolated vertex, $\gamma_t(G) \leq \chi_d^t(G)$.

In this paper, we continue the study of total dominator colorings in graphs. We show that the total dominator chromatic number of a graph is bounded above by the sum of its chromatic number and total domination number. In particular, if G is a bipartite graph, we show that $\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + 2$. For each $t \in \{0, 1, 2\}$, we construct an infinite family $\mathcal{G}_t$ of (bipartite) graphs such that each graph $G \in \mathcal{G}_t$ satisfies $\chi_d^t(G) = \gamma_t(G) + t$. We study properties of TD-colorings in a tree. We characterize trees T satisfying $\gamma_t(T) = \chi_d^t(T)$. We also establish a tight upper bound on the total dominator chromatic number of a tree in terms of its order and we characterize the trees with maximum possible total dominator chromatic number.

1.1 Notation and Terminology

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [9]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let v be a vertex in V. The graph G is nontrivial if $n \geq 2$. The open neighborhood of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its open neighborhood is the set $N_G(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. The degree of a vertex v in G is $d_G(v) = |N_G(v)|$. The neighborhood of an edge $e = uv$ is the set $N_G[e] = N_G[u] \cup N_G[v]$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N[e]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G[e]$, $N_G(S)$ and $N_G[S]$, respectively.

A cycle on n vertices is denoted by C_n and a path on n vertices by P_n. A complete graph on n vertices is denoted by K_n. A star is the tree $K_{1,n-1}$ of order $n \geq 2$. We call a vertex of degree one a leaf, its adjacent vertex a support vertex and its incident edge a pendant edge. The set of support vertices in a graph G we denote by $S(G)$. A strong support vertex is a support vertex with at least two leaf-neighbors. A double star is a tree with exactly two non-leaves. A graph is isolate-free if it contains no isolated vertex (of degree 0).

For any subset $S \subset V(G)$, we let $G - S$ denote the graph obtained from G by removing S and all edges incident with vertices in S. We denote the subgraph induced by S in G by $G[S]$.

A rooted tree distinguishes one vertex r called the root. For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique (r, v)-path, while a child of v is any other neighbor of v. A descendant of v is a vertex u such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. The maximal subtree at v is the subtree of T induced by v and its descendants, and is denoted by T_v.

Let H be a graph. The corona $H \circ K_1$ of H, also denoted $\mathrm{cor}(H)$ in the literature, is the graph obtained from H by adding a pendant edge to each vertex of H. The

2-corona $H \circ P_2$ of H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint.

A subset S of vertices in a graph G is a packing (respectively, an open packing) if the closed (respectively, open) neighborhoods of vertices in S are pairwise disjoint. Further the set S is a perfect packing (respectively, a perfect open packing) if every vertex belongs to at exactly one of the closed (respectively, open) neighborhoods of vertices in S. The open packing number $\rho^o(G)$ is the maximum cardinality of an open packing in G.

1.2 Known Results and Observations

The total domination number of a cycle or path is easy to compute.

Observation 3 For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

The following bound on the total domination number of a connected graph in terms of the order of the graph is due to Cockayne et al. [2].

Theorem 4 ([2]) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

Since any TD-set in a graph G must have a nonempty intersection with every open neighborhood, we observe that if G is a graph with no isolated vertex, then $\rho^o(G) \leq \gamma_t(G)$. We state this formally.

Observation 5 Every isolate-free graph G satisfies $\rho^o(G) \leq \gamma_t(G)$.

2 General Bounds

In this section, we establish bounds on the total dominator chromatic number of a graph. As an immediate consequence of Observation 2 and Observation 5, we see that every graph G with no isolated vertex satisfies $\rho^o(G) \leq \chi_d^t(G)$. One can say slightly more. First we present the following observation.

Observation 6 If v is an arbitrary vertex in an isolate-free graph G, then in every TD-coloring of G the neighborhood $N(v)$ of v contains a color class.

Theorem 7 *Let G be a graph with no isolated vertex. Then, $\rho^o(G) \leq \chi_d^t(G)$, with strict inequality if there is no perfect open packing in G.*

Proof As an immediate consequence of Observation 6, we see that if S is an open packing in G, then the open neighborhoods of vertices in S contain at least $|S|$ color classes, and so $\chi_d^t(G) \geq |S|$. Choosing S to be a maximum open packing, we see that $\chi_d^t(G) \geq \rho^o(G)$. Further, if G does not have a perfect open packing, then at least one additional color class is needed to contain the vertices that do not belong to the open neighborhood of any vertex of S, and so $\chi_d^t(G) \geq \rho^o(G) + 1$.   □
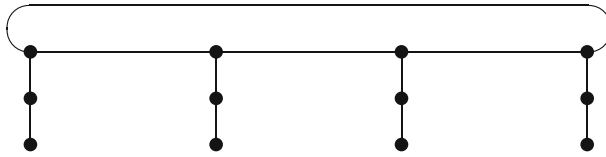
Fig. 1 The graph $C_4 \circ P_2$

If H is any connected graph of order $k \geq 1$, then the 2-corona $G = H \circ P_2$ satisfies $\rho^o(G) = 2k = \chi_d^t(G)$, illustrating that there do exist graphs that contain a perfect open packing and satisfy $\rho^o(G) = \chi_d^t(G)$. The graph $C_4 \circ P_2$, for example, is shown in Fig. 1 (here, $H = C_4$).

However we remark that if a graph G contains a perfect open packing, then it is not necessary true that $\rho^o(G) = \chi_d^t(G)$. The simplest example illustrating this is a path $G = P_4$, with $\rho^o(G) = 2$ and $\chi_d^t(G) = 3$. More generally, if $G = P_n$ where $n \equiv 0 \pmod 4$ and $n \geq 4$, then G has a perfect open packing and $\rho^o(G) = \gamma_t(G)$. However, by Observation 19 and Proposition 20, we have $\gamma_t(G) < \chi_d^t(G)$.

We next present an upper bound on the total dominator chromatic number. For a given graph G, let $\mathcal{A}_t(G)$ denote the set of all $\gamma_t(G)$-sets in G.

Theorem 8 *Let G be a connected graph of order at least* 3. *Then,*

$$\chi_d^t(G) \leq \gamma_t(G) + \min_{S \in \mathcal{A}_t(G)} \{\chi(G - S)\}.$$

Further this bound is tight.

Proof Let S be an arbitrary $\gamma_t(G)$-set and let $\mathcal{C}$ be a proper coloring of the graph $G - S$ using $\chi(G - S)$ colors. We now extend the coloring $\mathcal{C}$ to a coloring of the vertices of G by assigning to each vertex in S a new and distinct color. Let $\mathcal{C}'$ denote the resulting coloring of G and note that $\mathcal{C}'$ uses $\gamma_t(G) + \chi(G - S)$ colors. Since S is a TD-set of G, every vertex in G is adjacent to at least one vertex of S. Since the color class of $\mathcal{C}'$ containing a given vertex of S consists only of that vertex, each vertex in G is therefore adjacent to every vertex of some (other) color class in the coloring $\mathcal{C}'$. Hence, $\mathcal{C}'$ is a TD-coloring of G using $\gamma_t(G) + \chi(G - S)$ colors. This is true for every $\gamma_t(G)$-set S. The desired result now follows by choosing S to be a $\gamma_t(G)$-set that minimizes $\chi(G - S)$.

That the bound is tight may be seen as follows. Let G be obtained from a connected graph H of order $k \geq 1$ as follows: For each vertex $v \in V(H)$ add a path P_7 and identify the vertex v with a vertex on this path at distance 2 from a leaf. When $H = P_4$ (and so, $k = |V(H)| = 4$), for example, the graph G is illustrated in Fig. 2, where the darkened vertices form a $\gamma_t(G)$-set. Then, $\gamma_t(G) = 4k$. Let T be the set of all support vertices in G together with a non-leaf neighbor of each support vertex. Then, T is a $\gamma_t(G)$-set. Further, $G - T$ consists of isolated vertices, and so $\chi(G - T) = 1$. Hence,

$$\gamma_t(G) + \min_{S \in \mathcal{A}_t(G)} \{\chi(G - S)\} = 4k + 1.$$

Fig. 2 A graph G with $\chi_d^t(G) = \gamma_t(G) + 1$
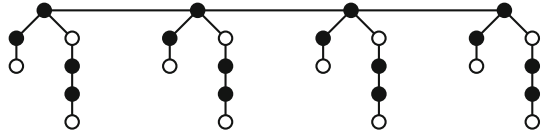
As a consequence of our upper bound on $\chi_d^t(G)$, we therefore have that $\chi_d^t(G) \leq 4k + 1$. It remains for us to show that $\chi_d^t(G) \geq 4k + 1$. Let $\mathcal{C}$ be a $\chi_d^t(G)$-coloring. By Observation 6, each support vertex in G forms a color class that consists only of that vertex. Further, for each support vertex v in G, the two neighbors of v form a color class or there is a color class consisting of exactly one neighbor of v and no other vertex. Since the support vertices of G are at distance at least 3 apart, and since there are $2k$ support vertices in G, this implies that there are at least $4k$ color classes in $\mathcal{C}$. Further the set of k vertices of G that do not belong to the closed neighborhood of any support vertex must be assigned at least one new color, implying that $\mathcal{C}$ contains at least $4k + 1$ color classes. Hence, $\chi_d^t(G) \geq 4k + 1$, as desired. Consequently,

$$\chi_d^t(G) = 4k + 1 = \gamma_t(G) + \min_{S \in \mathcal{A}_t(G)} \{\chi(G - S)\}.$$

This completes the proof of Theorem 8.   □

We observe that $\chi(G - S) \leq \chi(G)$ for every proper subset $S \subset V(G)$. We also observe that every total dominator coloring of G is a dominator coloring of G, and so $\chi(G) \leq \chi_d(G) \leq \chi_d^t(G)$. Hence as a consequence of Observation 2, we have the following analogous result to Theorem 1, thereby establishing upper and lower bounds on the total dominator chromatic number of an arbitrary graph in terms of its total domination number and chromatic number.

Theorem 9 *Every isolate-free graph G satisfies*

$$\max\{\gamma_t(G), \chi(G)\} \leq \chi_d^t(G) \leq \gamma_t(G) + \chi(G).$$

As an immediate consequence of Theorem 9, we have the following result.

Corollary 10 *If G is a bipartite isolate-free graph, then $\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + 2$.*

For each $t \in \{0, 1, 2\}$, we construct an infinite family $\mathcal{G}_t$ of (bipartite) graphs such that each graph $G \in \mathcal{G}_t$ satisfies $\chi_d^t(G) = \gamma_t(G) + t$.

Let $\mathcal{G}_0$ be the family of isolate free-graphs G that contain a TD-set S that is a perfect open packing in G and such that the neighborhood of each edge e in $G[S]$ induces a complete bipartite graph in G. Let $G \in \mathcal{G}_0$. We note that every edge in $G[S]$ is an isolated edge in $G[S]$. Further if $e = uv$ is an edge in $G[S]$, then the subgraph of G induced by the neighborhood, $N[e]$, of e is a complete bipartite graph K_{n_1, n_2} where $d(u) = n_1$ and $d(v) = n_2$. For example, if H is an arbitrary graph, then the graph $G = H \circ P_2$ belongs to the family $\mathcal{G}_0$ since the set $S = V(G) \backslash V(H)$ is a TD-set that is a perfect open packing in G and the neighborhood of each edge e in $G[S]$ induces a complete bipartite graph $K_{1,2}$ in G.

Proposition 11 *If $G \in \mathcal{G}_0$, then $\chi_d^t(G) = \gamma_t(G)$.*

Proof Let $G \in \mathcal{G}_0$. Let S be a TD-set in G that is a perfect open packing and such that the neighborhood of each edge e in $G[S]$ induces a complete bipartite graph. Then, $\gamma_t(G) \le |S| = \rho^o(G)$, implying by Observation 5 that $\gamma_t(G) = \rho^o(G)$. We now color the vertices of G with $\gamma_t(G)$ colors, one color to the open neighborhood of each vertex in S, and let $\mathcal{C}$ denote the resulting coloring. Since $N(v)$ is an independent set for each vertex $v \in S$, we note that the coloring $\mathcal{C}$ is a proper coloring of G. Let $w \in V(G)$ and let v be the (unique) vertex in S adjacent to w. Further, let u be the (unique) neighbor of v in S (possibly $u = w$). Since the neighborhood of the edge uv induces a complete bipartite graph with partite sets $N(u)$ and $N(v)$, the vertex w is adjacent to every vertex in the color class $N(u)$. Since w is an arbitrary vertex in G, every vertex in G is therefore adjacent to every vertex of some color class. The coloring $\mathcal{C}$ is therefore a TD-coloring of G, and so $\chi_d^t(G) \le \gamma_t(G)$. By Observation 2, $\chi_d^t(G) \ge \gamma_t(G)$. Consequently, $\chi_d^t(G) = \gamma_t(G)$. □

Let $\mathcal{G}_1$ be the family of graphs that can be obtained from an isolate-free graph H by attaching any number of pendant edges, but at least one, to each vertex of H. For example, if H is an arbitrary isolate-free graph, then the corona $G = H \circ P_1$ of H belongs to the family $\mathcal{G}_1$.

Proposition 12 *If $G \in \mathcal{G}_1$, then $\chi_d^t(G) = \gamma_t(G) + 1$.*

Proof Let $G \in \mathcal{G}_1$. Then, G can be obtained from an isolate-free graph H by attaching any number of pendant edges, but at least one, to each vertex of H. Let $\mathcal{C}$ be a TD-coloring in G. By construction, the set $V(H)$ is the set of support vertices in G, while each vertex not in H is a leaf in G. By Observation 6, each vertex in $V(H)$ therefore forms a color class of $\mathcal{C}$ that consists only of that vertex. At least one additional color class is needed to color the leaves in G, and so $\chi_d^t(G) \ge |V(H)| + 1$. However, assigning a distinct color to each vertex of H and assigning an additional color to the leaves of G produces a TD-coloring of G, implying that $\chi_d^t(G) \le |V(H)| + 1$. Consequently, $\chi_d^t(G) = |V(H)| + 1$. Since $\gamma_t(G) = |V(H)|$ (the set $V(H)$ forms the unique $\gamma_t(G)$-set), we therefore have that $\chi_d^t(G) = \gamma_t(G) + 1$. □

Letting $\mathcal{G}_2$ be the family of all paths P_n and cycles C_n, where $n \equiv 0 \pmod 4$ and $n \ge 8$, we have the following result (see Observation 19, Proposition 20 and Observation 21).

Proposition 13 *If $G \in \mathcal{G}_2$, then $\chi_d^t(G) = \gamma_t(G) + 2$.*

3 Trees

In this section, we consider total dominator colorings in trees. By Corollary 10, if T is a tree of order $n \ge 2$, then $\gamma_t(T) \le \chi_d^t(T) \le \gamma_t(T) + 2$. We have four aims in this section. First to characterize trees T satisfying $\gamma_t(T) = \chi_d^t(T)$. Secondly, to establish properties of $\chi_d^t(T)$-colorings in a tree T. Thirdly, to establish a tight upper bound on the total dominator chromatic number of a tree in terms of its order and to characterize the trees with maximum possible total dominator chromatic number. Fourthly, to determine the total dominator chromatic number of a path.

3.1 Trees T Satisfying $\gamma_t(T) = \chi_d^t(T)$

The first aim in this section is to characterize trees T satisfying $\gamma_t(T) = \chi_d^t(T)$. For this purpose, let $\mathcal{T}$ be the family of trees constructed as follows. Let $\mathcal{T}$ consist of the tree P_2 and all trees that can be obtained from a disjoint union of $k \geq 1$ stars each of order at least 3 by adding $k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original k stars remains a support vertex. If $T \in \mathcal{T}$ and $T \neq P_2$, then we call each of the original k stars used to construct the tree T an underlying star of T. We remark that $\mathcal{T}$ is precisely the subfamily of $\mathcal{G}_0$ consisting of all trees; that is, $\mathcal{T} = \{T : T \in \mathcal{G}_0 \text{ and } T \text{ is a tree }\}$. We proceed further with the following property of trees in the family $\mathcal{T}$.

Lemma 14 *If $T \in \mathcal{T}$, then $\gamma_t(T) = \chi_d^t(T)$. Further, the color classes of a $\chi_d^t(T)$-coloring are unique, and consist of the support vertices of T and the open neighborhoods of support vertices of T.*

Proof Let $T \in \mathcal{T}$. If $T = P_2$, then the result is immediate. Hence we may assume that T has order at least 3. Thus, T can be obtained from a disjoint union of $k \geq 1$ stars each of order at least 3 by adding $k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original k stars remains a support vertex. We note that the set of support vertices, $S(T)$, of T is precisely the set of central vertices of the underlying stars of T. In particular, $|S(T)| = k$. Further, the set $S(T)$ forms a perfect packing in T. Moreover, the set $S(T)$, together with one leaf-neighbor of each support vertex, forms a perfect open packing in T, implying that $\gamma_t(T) = 2k$. By Proposition 11, $\chi_d^t(T) = \gamma_t(T) = 2k$.

Let $\mathcal{C}$ be a $\chi_d^t(T)$-coloring of the tree T. By Observation 6, each support vertex in T forms a color class that consists only of that vertex. Further, for each support vertex v in T, the neighborhood $N(v)$ of v forms a color class or contains a color class as a proper subset (in order for the vertex v to be adjacent to every vertex in some color class). Hence, $\mathcal{C}$ contains at least $2k$ color class. Further if a neighborhood $N(v)$ for some vertex $v \in S(T)$ does not form a color class in $\mathcal{C}$, then an additional color class is needed, contradicting the fact that $\chi_d^t(T) = 2k$ and $\mathcal{C}$ is a $\chi_d^t(T)$-coloring. Hence the color classes in $\mathcal{C}$ are uniquely determined, and consist of the support vertices of T and the open neighborhoods of support vertices of T. □

Theorem 15 *Let T be a nontrivial tree. Then, $\gamma_t(T) = \chi_d^t(T)$ if and only if $T \in \mathcal{T}$.*

Proof The sufficiency follows from Lemma 14. To prove the necessity, we proceed by induction on the order $n \geq 2$ of a tree T that satisfies $\gamma_t(T) = \chi_d^t(T)$. If $n \in \{2, 3\}$, then T is a star and $T \in \mathcal{T}$, as desired. This establishes the base case. Suppose, then, that $n \geq 4$ and that if T' is a tree of order n', where $2 \leq n' < n$, satisfying $\gamma_t(T') = \chi_d^t(T')$, then $T' \in \mathcal{T}$. Let T be a tree of order n that satisfies $\gamma_t(T) = \chi_d^t(T)$. If T is a star, then $T \in \mathcal{T}$, as desired. Hence we may assume that $\mathrm{diam}(T) \geq 3$. If $\mathrm{diam}(T) = 3$, then T is a double star. But then $\gamma_t(T) = 2$ and $\chi_d^t(T) = 3$, a contradiction. Hence, $\mathrm{diam}(T) \geq 4$. Let $\mathcal{C}$ be a $\chi_d^t(T)$-coloring in T.

We proceed further with the following series of claims that we may assume are satisfied by the tree T. More precisely, we show in the proof of each claim that if T

does not satisfy the statement of the claim, then $T \in \mathcal{T}$ and the desired result follows. Hence we may assume that T does indeed satisfy the statement of each claim, for otherwise there is nothing left to prove.

Claim A *We may assume that the tree T has no strong support vertex.*

Proof Suppose that T has a strong support vertex v that is adjacent to at least two leaves, u and w. Let $T' = T - u$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices in T'. Suppose that $\mathcal{C}'$ is not a TD-coloring of T'. Then since $\mathcal{C}$ is a TD-coloring of T, the only vertex in T' that is not adjacent to every vertex of some color class in $\mathcal{C}'$ is the vertex v, implying that in the coloring $\mathcal{C}$ the vertex u has a unique color. But then considering the tree $T' = T - w$, we see that the restriction of the coloring $\mathcal{C}$ to the vertices in T' is a TD-coloring of T'. Hence by renaming the vertices u and w, if necessary, we may assume that $\mathcal{C}'$ is a TD-coloring of T'. Since the number of color classes in $\mathcal{C}'$ is at most the number of color classes in $\mathcal{C}$, and since $\mathcal{C}$ has $\chi_d^t(T)$ color classes and is a TD-coloring of T', this implies that $\chi_d^t(T') \leq \chi_d^t(T)$. Further, since every TD-set of a graph contains every support vertex in the graph, every TD-set of T' contains the vertex v and is therefore a TD-set of T, implying that $\gamma_t(T) \leq \gamma_t(T')$. Therefore, by Corollary 10, we see that

$$\chi_d^t(T) = \gamma_t(T) \leq \gamma_t(T') \leq \chi_d^t(T') \leq \chi_d^t(T).$$

Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T') = \chi_d^t(T')$. Applying the inductive hypothesis to the tree T', we see that $T' \in \mathcal{T}$. Thus, T' can be obtained from a disjoint union of $k \geq 2$ stars each of order at least 3 by adding $k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original k stars remains a support vertex. Since the vertex v is a support vertex in T' with w as a leaf-neighbor, the vertex v is the center vertex of one of the underlying stars of T'. Thus adding the deleted vertex u back to the tree, we see that $T \in \mathcal{T}$ (where the underlying star containing v is extended to contain the leaf u). Hence we may assume that T has no strong support vertex, for otherwise $T \in \mathcal{T}$ and the desired result follows. $\qquad \square$

Let r and u be two vertices at maximum distance apart in T and root the tree at the vertex r. Recall that $\mathrm{diam}(T) \geq 4$. Let v be the parent of u, let w be the parent of v and let x be the parent of w. As an immediate consequence of Claim A, we see that $d_T(v) = 2$.

Claim B $d_T(w) = 2$.

Proof Suppose that $d_T(w) \geq 3$. Suppose that w has a child, v', distinct from v, that is not a leaf. Analogously as with the vertex v, we see that $d_T(v') = 2$. Let u' be the leaf-neighbor of v' and consider the tree $T' = T - \{u', v'\}$. We note that in the coloring $\mathcal{C}$, both (support vertices) v and v' are assigned unique colors. The restriction, $\mathcal{C}'$, of the coloring $\mathcal{C}$ to the vertices of T' is therefore a TD-coloring, implying that $\chi_d^t(T') \leq \chi_d^t(T) - 1$. Let D' be a $\gamma_t(T')$-set. Then, $v \in D'$. If $u' \in D'$, then we can simply replace the vertex u' in D' with the vertex w. Hence we may choose D' so that $w \in D'$. But

then $D' \cup \{v'\}$ is a TD-set of T, implying that $\gamma_t(T) \le |D'| + 1 = \gamma_t(T') + 1$. Therefore, by Corollary 10, we see that $\chi_d^t(T) = \gamma_t(T) \le \gamma_t(T') + 1 \le \chi_d^t(T') + 1 \le \chi_d^t(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T') = \chi_d^t(T')$. Further, $\chi_d^t(T') = \chi_d^t(T) - 1$, implying that $\mathcal{C}'$ is a $\chi_d^t(T')$-coloring. Applying the inductive hypothesis to the tree T', we see that $T' \in \mathcal{T}$. Since the vertex v is a support vertex in T' with u as a leaf-neighbor, the vertex v is the center vertex of one of the underlying stars of T'. By Lemma 14, the color classes of a $\chi_d^t(T)$-coloring are unique and consist of the support vertices of T and the open neighborhoods of support vertices of T. In particular, $N(v) = \{u, w\}$ is a color class in $\mathcal{C}'$. In order for the vertex v' to be adjacent to every vertex in some color class in the coloring $\mathcal{C}$, this implies that u' is assigned a unique color in $\mathcal{C}$. But then the TD-coloring $\mathcal{C}'$ of T' contains two fewer color classes than does $\mathcal{C}$, implying that $\chi_d^t(T') \le \chi_d^t(T) - 2$, a contradiction. Therefore, since T has no strong support vertex, we see that $d_T(w) = 3$ and the child, v', of w distinct from v is a leaf.

In the coloring $\mathcal{C}$, the two support vertices v and w are assigned unique colors. We now consider the tree $T' = T - v'$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices of T'. We note that $\mathcal{C}'$ is a TD-coloring, implying that $\chi_d^t(T') \le \chi_d^t(T)$. Further, $\gamma_t(T) \le \gamma_t(T')$ since as before we can choose a $\gamma_t(T')$-set to contain the vertex w. Therefore, by Corollary 10, we see that $\chi_d^t(T) = \gamma_t(T) \le \gamma_t(T') \le \chi_d^t(T') \le \chi_d^t(T)$, implying that $\gamma_t(T') = \chi_d^t(T')$, $\chi_d^t(T') = \chi_d^t(T)$, and that $\mathcal{C}'$ is a $\chi_d^t(T')$-coloring. As before, we see that $T' \in \mathcal{T}$ and that $N(v) = \{u, w\}$ is a color class in $\mathcal{C}'$. But then in the coloring $\mathcal{C}$ the vertex v' is not adjacent to every vertex in some color class, a contradiction. Therefore, $d_T(w) = 2$.   □

By Claims A and B, we see that $d_T(v) = d_T(w) = 2$. We consider the tree $T' = T - \{u, v, w\}$. Recall that $\mathcal{C}$ is defined earlier to be a $\chi_d^t(T)$-coloring in T.

Claim C *Both sets $\{v\}$ and $N(v) = \{u, w\}$ form a color class in $\mathcal{C}$.*

Proof Since $\mathcal{C}$ is a TD-coloring in T, every set consisting of exactly one, but an arbitrary, vertex from each color class of $\mathcal{C}$ is a TD-set of T. Since $\gamma_t(T) = \chi_d^t(T)$, such a TD-set is in fact a $\gamma_t(T)$-set. Suppose that all three vertices u, v and w are assigned different colors. In this case, we choose a set D to consist of u, v and w, and one vertex from every color class that does not contain u, v or w. The resulting set D contains one vertex from each color class of $\mathcal{C}$ and is therefore a $\gamma_t(T)$-set. However, $D \setminus \{u\}$ is a TD-set of T, contradicting the minimality of the set D. Hence, at most two colors are used to color the vertices u, v and w. Since $\mathcal{C}$ is a TD-coloring of T, the support vertex v is assigned a unique color. Therefore, the vertices u and w are assigned the same color. In order for the vertex v to be adjacent to every vertex of some color, the neighborhood $N(v) = \{u, w\}$ of v forms a color class. Thus the set $\{v\}$ and the set $\{u, w\}$ both form a color class in $\mathcal{C}$.   □

Claim D $\gamma_t(T) = \gamma_t(T') + 2$ *and* $\gamma_t(T') = \chi_d^t(T')$.

Proof Let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices in T'. By Claim C, the set $\{v\}$ and the set $\{u, w\}$ both form a color class in the coloring $\mathcal{C}$. Thus the coloring $\mathcal{C}'$ has two fewer color classes than does the coloring $\mathcal{C}$. The only possible vertex in T' that

is not adjacent to every vertex of some color class in $\mathcal{C}'$ is the vertex x. However this would imply that in the coloring $\mathcal{C}$ the vertex w has a unique color, contradicting our earlier observation that $\{u, w\}$ is a color class in $\mathcal{C}$. Therefore, $\mathcal{C}'$ is a TD-coloring of T', implying that $\chi_d^t(T') \leq \chi_d^t(T) - 2$. Every TD-set in T' can be extended to a TD-set in T by adding to it the vertices v and w, implying that $\gamma_t(T) \leq \gamma_t(T') + 2$. Therefore, by Corollary 10, we see that $\chi_d^t(T) = \gamma_t(T) \leq \gamma_t(T') + 2 \leq \chi_d^t(T') + 2 \leq \chi_d^t(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_t(T') = \chi_d^t(T')$.   □

We now return to the proof of Theorem 15. By Claim D, $\gamma_t(T') = \chi_d^t(T')$. Since $\gamma_t(T) = \chi_d^t(T)$, we note that $T \neq P_5$, and so T' has order at least 3. Applying the inductive hypothesis to the tree T', we see that $T' \in \mathcal{T}$. Thus, T' can be obtained from a disjoint union of $k \geq 1$ stars each of order at least 3 by adding $k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original k stars remains a support vertex. We note that $\gamma_t(T') = 2k$. Further, every $\gamma_t(T')$-set contains the k support vertices of T' and an arbitrary leaf-neighbor of every support vertex. Let S_x be the underlying star of the tree T' that contains the vertex x.

Claim E *The following holds.*

(a) *The vertex x is a leaf in the star S_x.*
(b) *The star S_x contains a vertex that is a leaf in T.*

Proof (a) Suppose to the contrary that x is the center vertex of S_x. Let R be a $\gamma_t(T')$-set. As observed earlier, the set R contains the k support vertices of T' and one leaf-neighbor of every support vertex. In particular, $x \in R$. Let x' be a leaf-neighbor of x that belongs to the set R. Then the set $(R \setminus \{x'\}) \cup \{v, w\}$ is a TD-set of T, implying that $\gamma_t(T) \leq |R| + 1 = 2k + 1 = \gamma_t(T') + 1$, contradicting Claim D.

(b) Suppose to the contrary that the star S_x contains no leaf of T. Let R be a $\gamma_t(T')$-set. Let y be the center of S_x, and let $y_1, \ldots, y_b$ be the leaves of S_x different from x. For each $i = 1, \ldots, b$, let z_i be a neighbor of y_i different from y and note that each vertex z_i is a leaf in some underlying star of the tree T'. Further we note that in the underlying star of T' containing the vertex z_i, we can clearly choose the set R so that $z_i \in R$ (for otherwise, we simply replace the leaf in the underlying star that belongs to R with the vertex z_i). Further we can choose the set R so that x is the leaf in the star S_x that belongs to R. But then the set $(R \setminus \{y\}) \cup \{v, w\}$ is a TD-set of T, implying that $\gamma_t(T) \leq |R| + 1 = 2k + 1 = \gamma_t(T') + 1$, once again contradicting Claim D.   □

By Claim E, the vertex x is a leaf of the star S_x. Further, the star S_x contains a vertex that is leaf in T. This implies that $T \in \mathcal{T}$, where the underlying stars in the tree T consist of the star induced by $\{u, v, w\}$ and the k stars of the tree T'. This completes the proof of Theorem 15.   □

3.2 Properties of $\chi_d^t(T)$-Colorings in a Tree T

Our second aim in this section is to establish properties of $\chi_d^t(T)$-colorings in a tree T. Recall that a color class C in a given TD-coloring of a graph G is free if each vertex

of G is adjacent to every vertex of some color class different from C. Recall further by Corollary 10, that if T is a nontrivial tree, then $\gamma_t(T) \le \chi_d^t(T) \le \gamma_t(T) + 2$. We establish next the following property of $\chi_d^t(T)$-colorings in a tree T.

Lemma 16 *Let T be a nontrivial tree. Then the following holds.*

(a) *If $\gamma_t(T) = \chi_d^t(T)$, then no $\chi_d^t(T)$-coloring contains a free color class.*
(b) *If $\chi_d^t(T) = \gamma_t(T) + 1$, then there exists a $\chi_d^t(T)$-coloring that contains a free color class.*
(c) *If $\chi_d^t(T) = \gamma_t(T) + 2$, then there exists a $\chi_d^t(T)$-coloring that contains two free color class.*

Proof Part (a) is an immediate consequence of Lemmma 14 and Theorem 15.

To prove Part (c), suppose that $\chi_d^t(T) = \gamma_t(T) + 2$ and let S be a $\gamma_t(T)$-set. If $V \setminus S$ is an independent set, then assigning to each vertex of S a unique color and assigning to the vertices in $V \setminus S$ an additional color, produces a TD-coloring in T with $\gamma_t(T) + 1$ color classes, implying that $\chi_d^t(T) \le \gamma_t(T) + 1$, a contradiction. Hence, $V \setminus S$ is not an independent set. We now 2-color the vertices in the forest $T[V \setminus S]$ and we color each vertex of S with a unique color. The resulting coloring is a TD-coloring of T with $\chi_d^t(T) = \gamma_t(T) + 2$ color classes. However both color classes used to color the vertices in $V \setminus S$ are free color classes. This proves Part (c).

It remains for us to prove Part (b). We proceed by induction on the order n of a nontrivial tree satisfying $\chi_d^t(T) = \gamma_t(T) + 1$. The smallest such tree is a path $T = P_4$, for which the desired result holds. This establishes the base case. Suppose, then, that $n \ge 5$ and that if T' is a nontrivial tree of order less than n satisfying $\chi_d^t(T') = \gamma_t(T') + 1$, then there exists a $\chi_d^t(T')$-coloring that contains a free color class. Let T be a tree of order n satisfying $\chi_d^t(T) = \gamma_t(T) + 1$. Let $\mathcal{C}$ be a $\chi_d^t(T)$-coloring.

If T is a star, then $\chi_d^t(T) = \gamma_t(T)$, a contradiction. Hence, $\text{diam}(T) \ge 3$. Suppose $\text{diam}(T) = 3$. Then, T is a double star and $\chi_d^t(T) = \gamma_t(T) + 1 = 3$. In this case, assigning a unique color to both central vertices of T and a third color to the leaves of T produces a $\chi_d^t(T)$-coloring that contains a free color class, as desired. Hence we may assume that $\text{diam}(T) \ge 4$.

We proceed further with the following series of claims that we may assume are satisfied by the tree T.

Claim I *We may assume that the tree T has no strong support vertex.*

Proof Suppose that T has a strong support vertex w and let u and v be two leaf-neighbors of w. Then, $\chi_d^t(T - v) = \chi_d^t(T)$ and $\gamma_t(T - v) = \gamma_t(T)$, implying that $\chi_d^t(T - v) = \gamma_t(T - v) + 1$. Applying the inductive hypothesis to the tree $T - v$, there exists a $\chi_d^t(T - v)$-coloring $\mathcal{C}'$ that contains a free color class, C say. Assigning to the leaf v the same color as used to color the leaf u in $\mathcal{C}'$ extends this coloring to a $\chi_d^t(T)$-coloring $\mathcal{C}^*$. On the one hand, if the vertex v was not added to the color class C, then the color class C remains a free color class in $\mathcal{C}^*$. On the other hand, if v was added to the color class C, then this new color class is a free color class in $\mathcal{C}^*$. In both cases, we produce a $\chi_d^t(T)$-coloring that contains a free color class. Hence

we may assume that T has no strong support vertex, for otherwise the desired result follows.   □

Let r and u be two vertices at maximum distance apart in T and root the tree at the vertex r. Let v be the parent of u, let w be the parent of v and let x be the parent of w. By Claim I, the tree T has no strong support vertex. Hence, $d_T(v) = 2$. We note that v is assigned a unique coloring in $\mathcal{C}$.

Claim II *We may assume that $d_T(w) = 2$.*

Proof Suppose that $d_T(w) \geq 3$. Let v' be a child of w different from v. If v' is a leaf, then w is a support vertex and is assigned a unique color in the coloring $\mathcal{C}$. But then the color class containing the vertex u is free. Hence we may assume that v' is not a leaf, implying that $d_T(v') = 2$. Let u' be the leaf-neighbor of v'. We note that v' is assigned a unique coloring in $\mathcal{C}$.

Suppose that u and w are colored with the same color. Then, $\{u, w\}$ is the unique color class contained in $N(v)$. Further, the color class containing the vertex u' is unique; that is, $\{u'\}$ is the unique color class which is contained in $N(v')$. Renaming the colors, if necessary, we may assume that u and w are colored with color 1 and the vertex u' is colored with color 2. But then recoloring the vertex u with color 2, produces a new $\chi_d^t(T)$-coloring in which the color class $\{u, u'\}$ is free. Hence we may assume that u and w belong to different color classes. Analogously, we may assume that u' and w belong to different color classes.

If w is assigned a unique color, then the color class containing the vertex u is free. Hence we may assume that w is not assigned a unique color. This implies that both vertices u and u' are assigned a unique color. Renaming the colors, if necessary, we may assume that u, u' and w are colored with colors 1, 2 and 3, respectively. But then recoloring the vertex u' with color 1 and recoloring the vertex w with color 2, produces a new $\chi_d^t(T)$-coloring in which the color class $\{u, u'\}$ is free. Therefore, we may assume that $d_T(w) = 2$, for otherwise the desired result follows.   □

We now consider the tree $T' = T - \{u, v, w\}$. Let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the tree T'.

Claim III *We may assume that $\mathcal{C}'$ is a TD-coloring of T'.*

Proof Suppose that $\mathcal{C}'$ is not a TD-coloring of T'. This implies that the vertex w has a unique color. This in turn implies that the color class of $\mathcal{C}$ that contains the vertex u is free. Hence we may assume that $\mathcal{C}'$ is a TD-coloring of T', for otherwise the desired result follows.   □

Claim IV $\chi_d^t(T') \leq \chi_d^t(T) - 2$ *and* $\gamma_t(T) \leq \gamma_t(T') + 2$.

Proof By Claim III, the restriction, $\mathcal{C}'$, of the coloring $\mathcal{C}$ to the tree T' is a TD-coloring of T'. Since $\mathcal{C}'$ contains at least two fewer color class than does $\mathcal{C}$, we see that $\chi_d^t(T') \leq \chi_d^t(T) - 2$. Every $\gamma_t(T')$-set can be extended to a TD-set of T' by adding to it the vertices v and w, and so $\gamma_t(T) \leq \gamma_t(T') + 2$.   □

By assumption, $\chi_d^t(T) = \gamma_t(T) + 1$. By Corollary 10, we see that $\chi_d^t(T') = \gamma_t(T') + \ell$ for some $\ell \in \{0, 1, 2\}$. If $\chi_d^t(T') = \gamma_t(T') + 2$, then by Claim IV, $\gamma_t(T) \leq$

$\gamma_t(T') + 2 = \chi_d^t(T') \le \chi_d^t(T) - 2 = \gamma_t(T) - 1$, which is impossible. Hence, $\chi_d^t(T') = \gamma_t(T')$ or $\chi_d^t(T') = \gamma_t(T') + 1$.

Claim V *We may assume that* $\chi_d^t(T') = \gamma_t(T')$.

Proof Suppose that $\chi_d^t(T') = \gamma_t(T') + 1$. Then, by Claim IV, we see that $\gamma_t(T) \le \gamma_t(T') + 2 = \chi_d^t(T') + 1 \le \chi_d^t(T) - 1 = \gamma_t(T)$. Hence we must have equality throughout this inequality chain. In particular, $\chi_d^t(T') = \chi_d^t(T) - 2$. Applying the inductive hypothesis to the tree T', there exists a $\chi_d^t(T')$-coloring, $\mathcal{D}'$, that contains a free color class, C say. The coloring $\mathcal{D}'$ can be extended to a $\chi_d^t(T)$-coloring $\mathcal{D}^*$ by adding to it two new color classes, namely $\{v\}$ and $\{u, w\}$. The free color class, C, of $\mathcal{D}'$ is a free color class in $\mathcal{D}^*$. Hence we may assume that $\chi_d^t(T') = \gamma_t(T')$, for otherwise the desired result follows.   □

By Claim V, $\chi_d^t(T') = \gamma_t(T')$. By Theorem 15, $T' \in \mathcal{T}$. Recall that x is the parent of w in T and r is the root of T.

Claim VI *We may assume that* $T' \ne P_2$.

Proof Suppose $T' = P_2$. Then, $T = P_5$ and T is the path $uvwxr$. In this case, $\gamma_t(T) = 3$ and $\chi_d^t(T) = 4$. Coloring u and r with the color 1, and coloring v, w and x with the colors 2, 3 and 4, respectively, produces a $\chi_d^t(T)$-coloring that contains a free color class, namely the color class $\{u, r\}$. Hence we may assume that $T' \ne P_2$, for otherwise the desired result follows.   □

By Claim VI, $T' \ne P_2$. Hence by definition of the family $\mathcal{T}$, the tree T' can be obtained from a disjoint union of $k \ge 1$ stars each of order at least 3 by adding $k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original k stars remains a support vertex.

Let $\mathcal{D}'$ be a $\chi_d^t(T')$-coloring. By Lemma 14, the color classes of $\mathcal{D}'$ are unique, and consist of the k support vertices of T' (namely, the centers of the k original stars used to construct T') and the open neighborhoods of these support vertices in T'. In particular, we note that $\chi_d^t(T') = 2k = \gamma_t(T')$. Let S_x be the original star used to construct T' that contains x. Let x' be the center of the star S_x. (Possibly, $x = x'$.) If $T \in \mathcal{T}$, then $\chi_d^t(T) = \gamma_t(T)$, a contradiction. Hence, $T \notin \mathcal{T}$, implying that either $x = x'$ or x is a leaf of S_x and every leaf of S_x different from x has degree at least 2 in T'.

Claim VI *We may assume that* $x = x'$.

Proof Suppose that x is not the central vertex in S_x; that is, $x \ne x'$. Then, x is a leaf of S_x and every leaf of S_x different from x has degree at least 2 in T'. Let R be the set of all vertices in $V(T') \setminus V(S_x)$ that are adjacent in T' to a vertex in $V(S_x)$. We note that each vertex in R is a leaf from one of the underlying stars of T'. Further, each underlying star of T' contains at most one vertex from R. Hence we can choose a $\gamma_t(T')$-set, D', to contain the k support vertices of T' and one neighbor of each of these support vertices in such a way that $R \subset D'$. Further, we may choose D' so that $x \in D'$. With this choice of D' we see that $(D' \setminus \{x'\}) \cup \{v, w\}$ is a TD-set of T, and so $\gamma_t(T) \le |D'| + 1 = 2k + 1$. However, $\gamma_t(T) \ge \rho^o(T) \ge 2k + 1$. Consequently, $\gamma_t(T) = 2k + 1$ and $\chi_d^t(T) = 2k + 2$. We may assume that the center, x', of the star

Fig. 3 A tree in the family $\mathcal{F}$
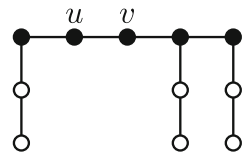
S_x is colored with the color 1 in $\mathcal{D}'$. We now modify the coloring $\mathcal{D}'$ as follows. Let L be the set of vertices of T' that belong to an underlying star of T' that contains a vertex of R and are leaves in that underlying star not in R. Since T' is a tree, the set L is an independent set in T'. We now recolor all the vertices that belong to L with the color 1 (and note that the color of vertices in R remains unchanged). Let $\mathcal{D}^*$ denote the resulting modified coloring of T'. We now extend the coloring $\mathcal{D}^*$ to a $\chi_d^t(T)$-coloring by coloring u with the color 1 and coloring v and w with two new colors. In the resulting $\chi_d^t(T)$-coloring the color class consisting of vertices colored 1 is free. Hence we may assume that x is the central vertex in S_x, for otherwise the desired result follows.   □

By Claim VII, x is the central vertex in S_x. Then, $\gamma_t(T) = 2k + 1$ and $\chi_d^t(T) = 2k + 2$. We now extend the coloring $\mathcal{D}'$ to a $\chi_d^t(T)$-coloring by adding u to the color class, $N_{T'}(x)$, that consists of the neighbors of x in T', and coloring v and w with two new colors. In the resulting $\chi_d^t(T)$-coloring the color class $N_{T'}(x) \cup \{u\}$ is free. This completes the proof of Part (b) and therefore of Lemma 16.   □

3.3 Trees with Large Total Dominator Chromatic Number

Our third aim in this section is to establish a tight upper bound on the total dominator chromatic number of a tree in terms of its order and to characterize the trees with maximum possible total dominator chromatic number. By Theorem 4 and Corollary 10, we see that if T is a tree of order $n \geq 3$, then $\chi_d^t(T) \leq 2n/3 + 2$. We show next that this bound can be improved slightly. For this purpose, let $\mathcal{F}$ be the family of all trees T that can be obtained from a tree H of order at least 2 by selecting an arbitrary edge $e = uv$ in H and attaching a path of length 2 to each vertex of $V(H)\backslash\{u, v\}$ so that the resulting paths are vertex-disjoint. We call H the underlying tree of T. A tree in the family $\mathcal{F}$ with underlying tree $H = P_5$, for example, is illustrated in Fig. 3 (here the darkened vertices form the vertices of H).

We remark that the smallest trees in the family $\mathcal{F}$ are the trees P_2 and P_5. We observe that if the underlying tree H used to construct a tree $T \in \mathcal{F}$ has order $k \geq 3$, then T has order $n = 3k - 4$. Further, if $x \in V(H)\backslash\{u, v\}$ and xyz is the path added to x when constructing T, then in every TD-coloring of T the set $\{x, y, z\}$ contains at least two color classes. Further the adjacent vertices u and v are assigned two distinct colors, implying that there are at least $2k - 2$ color classes, and so $\chi_d^t(T) \geq 2k - 2$. Renaming the vertices u and v, if necessary, we may assume that $d_H(v) \geq d_H(u)$. Assigning the same color to the vertex u and to each of the added $k - 2$ leaves, and assigning a unique color to each remaining vertex of T, produces a TD-coloring of T with $2k - 2$ color classes, and so $\chi_d^t(T) \leq 2k - 2$. Consequently, $\chi_d^t(T) = 2k - 2 = 2(n + 1)/3$. We state this formally as follows.

Observation 17 If $T \in \mathcal{F}$ has order n, then $\chi_d^t(T) = 2(n+1)/3$.

We are now in a position to present a tight upper bound on the total dominator chromatic number of a tree in terms of its order and to characterize the extremal trees.

Theorem 18 Let T be a tree of order $n \geq 2$. Then, $\chi_d^t(T) \leq 2(n+1)/3$ with equality if and only if $T \in \mathcal{F}$.

Proof We proceed by induction on the order $n \geq 2$ of a tree T. If $n = 2$, then $T = P_2 \in \mathcal{F}$ and $\chi_d^t(T) = 2 = 2(n+1)/3$. This establishes the base case. Suppose that $n \geq 3$ and that for every tree T' of order n', where $2 \leq n' < n$, we have $\chi_d^t(T') \leq 2(n'+1)/3$, with equality if and only if $T' \in \mathcal{F}$. Let T be a tree of order $n \geq 3$. If T is a star, then $\chi_d^t(T) = 2 < 2(n+1)/3$. If T is a double-star, then $n \geq 4$ and $\chi_d^t(T) = 3 < 2(n+1)/3$. Hence we may assume that $\operatorname{diam}(T) \geq 4$, for otherwise the desired result follows.

If T has a strong support vertex v and if u is a leaf-neighbor of v, then $\chi_d^t(T) = \chi_d^t(T - u)$. Applying the inductive hypothesis to the tree $T - u$ of order $n - 1$, we see that $\chi_d^t(T - u) \leq 2n/3$, and so $\chi_d^t(T) < 2(n+1)/3$. Hence we may assume that T has no strong support vertex, for otherwise the desired result holds.

Let r and u be two vertices at maximum distance apart in T and root the tree at the vertex r. Let v be the parent of u, let w be the parent of v and let x be the parent of w. Since T has no strong support vertex, we see that $d_T(v) = 2$. Let T' be the tree obtained from T by deleting the vertex w and all its descendants; that is, $T' = T - V(T_w)$, where recall that T_w denotes the maximal subtree of T at w. Let T' have order n'. Since $\operatorname{diam}(T) \geq 4$, we note that $n' \geq 2$. We proceed further with the following series of claims.

Claim 1 If $n' = 2$ and $T \neq P_5$, then $\chi_d^t(T) < 2(n+1)/3$.

Proof Suppose that $n' = 2$ and $T \neq P_5$. Then, T can be obtained from a star $K_{1,k}$, where $k \geq 3$, by subdividing at least $k - 1$ edges exactly once. If exactly $k - 1$ edges are subdivided, then $n = 2k$ and $\chi_d^t(T) = k + 1 < 2(n+1)/3$. If exactly k edges are subdivided, then $n = 2k + 1$ and $\chi_d^t(T) = k + 2 < 2(n+1)/3$. □

Suppose that $n' = 2$. By Claim 1, if $T \neq P_5$, then $\chi_d^t(T) < 2(n+1)/3$. If $T = P_5$, then $T \in \mathcal{F}$ and $\chi_d^t(T) = 4 = 2(n+1)/3$. Hence we may assume that $n' \geq 3$.

Claim 2 If no $\chi_d^t(T')$-coloring contains a free color class, then $\chi_d^t(T) \leq 2(n+1)/3$. Further if $\chi_d^t(T) = 2(n+1)/3$, then $T \in \mathcal{F}$.

Proof Suppose that no $\chi_d^t(T')$-coloring contains a free color class. Then, since $n' \geq 3$, we have by Lemma 16 that $\gamma_t(T') = \chi_d^t(T')$. Thus, by Theorem 15, we see that $T' \in \mathcal{T}$, implying that $\chi_d^t(T') \leq 2n'/3$.

If $d_T(w) = 2$, then $n' = n - 3$ and every $\chi_d^t(T')$-coloring can be extended to a TD-coloring of T by adding two new color classes, namely the color classes $\{v\}$ and $\{u, w\}$, implying that $\chi_d^t(T) \leq \chi_d^t(T') + 2 \leq 2(n-3)/3 + 2 = 2n/3$.

Suppose that $d_T(w) \geq 3$. Then, T_w can be obtained from a star $K_{1,k}$, where $k \geq 2$, by subdividing at least $k - 1$ edges exactly once. Suppose that exactly $k - 1$ edges are

subdivided in order to construct T_w. Then, $n' = n - 2k$ and every $\chi_d^t(T')$-coloring can be extended to a TD-coloring of T by coloring each of the k support vertices in T_w with a unique color and adding one additional color to color the k leaves of T. Therefore, in this case, $\chi_d^t(T) \leq \chi_d^t(T') + k + 1 \leq 2(n-2k)/3 + k + 1 = (2n - k + 3)/3 \leq (2n+1)/3$.

Suppose that exactly k edges are subdivided in order to construct T_w. Then, $n' = n - 2k - 1$ and every $\chi_d^t(T')$-coloring can be extended to a TD-coloring of T by coloring each of the k support vertices in T_w with a unique color, coloring the vertex w with a unique color, and adding one additional color to color the k leaves of T. Therefore, in this case,

$$\begin{aligned}
\chi_d^t(T) &\leq \chi_d^t(T') + k + 2 \\
&\leq 2n'/3 + k + 2 \\
&= 2(n - 2k - 1)/3 + k + 2 \\
&= (2n - k + 4)/3 \\
&\leq 2(n + 1)/3.
\end{aligned}$$

Suppose, further, that $\chi_d^t(T) = 2(n+1)/3$. Then we must have equality throughout the previous inequality chain. In particular, $\chi_d^t(T') = 2n'/3$ and $k = 2$. Thus, $T_w = P_5$ and T' can be obtained from a disjoint union of $\ell \geq 1$ copies of P_3 by adding $\ell - 1$ edges joining leaf vertices in such a way that the resulting graph is connected and the center of each of the original ℓ copies of P_3 remains a support vertex. Thus, T' is the 2-corona $T' = H' \circ P_2$ of a tree H' of order ℓ. Further, the support vertices of T' are the central vertices of the copies of P_3 used to construct the tree T'. Let $uvwv'u'$ be the path T_w, and so v' is the child of w different from v, and u' is the child of v'. There are two cases to consider.

Suppose that x is a support vertex of T'. Then, $d_T(x) = 3$. We now color the ℓ support vertices of T' with a unique color, and we color each of v, v' and w with a unique color. For each support vertex x' of T' different from x we color both neighbors of x' with the same, but unique, color. Finally we color the two neighbors of x in T' and the two leaves u and u' all with the same, but unique, color. The resulting coloring is a TD-coloring of T with $2\ell + 3$ color classes. Since $n = 3\ell + 5$, we therefore have that $\chi_d^t(T) \leq 2\ell + 3 < 2(n+1)/3$, a contradiction. Hence, x is not a support vertex of T'. Let y be the support vertex of T' that is adjacent to x and let z denote the remaining neighbor of y in T'. We note that either x is a leaf in T' or $x \in V(H')$.

Suppose that $\ell = 1$. Then, $T' = P_3$ and T is obtained from a star $K_{1,3}$ by subdividing two edges once and the remaining edge twice. In this case, $T \in \mathcal{F}$ (for example, take the underlying tree in $T \in \mathcal{F}$ to be the path $uvwx$).

Suppose that $\ell \geq 2$. Suppose that x is a leaf in T'. Then, $z \in V(H')$. In this case, we color each support vertex of T', different from y, with a unique color. Further we color each vertex of H', different from z, with a unique color and we color both x and z with the same, but unique, color. Finally, we color each of v, v' and w with a unique color, and we color all remaining vertices (namely, the vertices u, u', y and all leaves of T' different from x) with the same, but unique, color. The resulting coloring is a TD-coloring of T with $2\ell + 3$ color classes. Since $n = 3\ell + 5$, we therefore have that $\chi_d^t(T) \leq 2\ell + 3 < 2(n + 1)/3$, a contradiction. Hence, z must be a leaf in T',

implying that $x \in V(H')$. But then $T \in \mathcal{F}$ (take, for example, the underlying tree in $T \in \mathcal{F}$ to be the tree induced by the vertices $V(H') \cup \{u, v, w\}$). □

By Claim 2, we may assume that there exists a $\chi_d^t(T')$-coloring that contains a free color class, for otherwise the desired result follows. Applying the inductive hypothesis to the tree T', we see that $\chi_d^t(T') \leq 2(n'+1)/3$, with equality if and only of $T' \in \mathcal{F}$.

Claim 3 *If $d_T(w) \geq 3$, then $\chi_d^t(T) < 2(n+1)/3$.*

Proof Suppose that $d_T(w) \geq 3$. Then, T_w can be obtained from a star $K_{1,k}$, where $k \geq 2$, by subdividing at least $k-1$ edges exactly once. Let $\mathcal{C}'$ be a $\chi_d^t(T')$-coloring that contains a free color class, C. Renaming colors, if necessary, we may assume that the vertices in C are colored with the color 1.

If exactly $k-1$ edges are subdivided when constructing T_w, then $n' = n - 2k$ and the coloring $\mathcal{C}'$ can be extended to a TD-coloring of T by coloring each of the k support vertices in T_w with a unique color and coloring each of the k leaves in T_w with the color 1 used to color the vertices in the color class C of $\mathcal{C}'$. Therefore, in this case, $\chi_d^t(T) \leq \chi_d^t(T') + k \leq 2(n'+1)/3 + k = 2(n-2k+1)/3 + k = (2n-k+2)/3 \leq 2n/3$.

If exactly k edges are subdivided, then $n' = n - 2k - 1$ and the coloring $\mathcal{C}'$ can be extended to a TD-coloring of T by coloring each of the k support vertices in T_w with a unique color, coloring the vertex w with a unique color, and coloring each of the k leaves in T_w with the color 1 (and adding them to the color class C). Therefore, in this case, $\chi_d^t(T) \leq \chi_d^t(T') + k + 1 \leq 2(n'+1)/3 + k + 1 = 2(n-2k)/3 + k + 1 = (2n-k+3)/3 \leq (2n+1)/3$.

In both cases, we see that $\chi_d^t(T) < 2(n+1)/3$. □

By Claim 3, we may assume that $d_T(w) = 2$, for otherwise $\chi_d^t(T) < 2(n+1)/3$. Thus, T_w consists of the path wvu. Hence, $n' = n - 3$. Every TD-coloring in T' can be extended to a TD-coloring of T by coloring v with a unique color and coloring both u and w with the same, but unique, color, implying that

$$
\begin{aligned}
\chi_d^t(T) &\leq \chi_d^t(T') + 2 \\
&\leq 2(n'+1)/3 + 2 \\
&= 2(n-2)/3 + 2 \\
&= 2(n+1)/3.
\end{aligned}
$$

This establishes the desired upper bound. Suppose, further, that $\chi_d^t(T) = 2(n+1)/3$. Then we must have equality throughout the previous inequality chain. In particular, $\chi_d^t(T) = \chi_d^t(T') + 2$ and $\chi_d^t(T') = 2(n'+1)/3$. Applying the inductive hypothesis to the tree T', we see that $T' \in \mathcal{F}$. Let H' be the underlying tree of T'. Further, let u' and v' be the two adjacent vertices in H' whose degrees are unchanged when constructing the tree T'. Renaming the vertices u' and v', if necessary, we may assume that $d_{H'}(v') \geq d_{H'}(u')$. Since $n' \geq 3$, we note that $d_{H'}(v') \geq 2$. Let $\mathcal{C}'$ be the $\chi_d^t(T')$-coloring of T' that colors the leaves of T' and the vertex u' with the color 1, and colors all remaining vertices of T' with a unique color.

Claim 4 *If x is a support vertex of T' and $x \notin V(H')$, then $T \in \mathcal{F}$.*

Proof Suppose that $x \notin V(H')$ and that x is a support vertex of T'. Let y_1 and y_2 be the two neighbors of x in T', where $y_1 \in V(H')$.

Suppose that v' is adjacent to a vertex in H' different from u' and y_1. In this case, let $\mathcal{C}$ be the coloring obtained from $\mathcal{C}'$ by recoloring the vertex y_1 with the color 1, coloring the vertex u with the color 1, and coloring both v and w with a unique color. Then, $\mathcal{C}$ is a TD-coloring of T with one more color class than the coloring $\mathcal{C}'$, implying that $\chi_d^t(T) \leq \chi_d^t(T') + 1$, contradicting our earlier observation that $\chi_d^t(T) = \chi_d^t(T') + 2$. Therefore, $d_{H'}(v') = 2$, and so $N_T(v') = \{u', y_1\}$.

Suppose that u' has degree at least 2 in H'. In this case, let $\mathcal{C}$ be the coloring obtained from $\mathcal{C}'$ by recoloring the vertex u' with the same color used to color y_1, recoloring the vertex v' with the color 1, coloring the vertex u with the color 1, and coloring both v and w with a unique color. Then, $\mathcal{C}$ is a TD-coloring of T with one more color class than the coloring $\mathcal{C}'$, a contradiction. Therefore, u' is a leaf in H' (and hence in T).

Since u' is a leaf in T and $N_T(v') = \{u', y_1\}$, we see that $T \in \mathcal{F}$, where the underlying tree, H^*, of T has vertex set $(V(H') \setminus \{u', v'\}) \cup \{w, x, y_2\}$ and where x and y_2 are the two adjacent vertices in H^* whose degrees are unchanged when constructing the tree T. This completes the proof of Claim 4. □

Claim 5 *If x is a leaf of T' and $x \notin V(H')$, then $T \in \mathcal{F}$.*

Proof Suppose that $x \notin V(H')$ and that x is a leaf of T'. Let y be the neighbor of x in T' and let z be the neighbor of y different from x in T'. (We note that in the rooted tree T, the vertex y is the parent of x and the vertex z is the parent of y).

Suppose that v' is adjacent to a vertex in H' different from u' and z. In this case, let $\mathcal{C}$ be the coloring obtained from $\mathcal{C}'$ by recoloring both vertices x and z with the color assigned to y in $\mathcal{C}'$, recoloring the vertex y with the color 1, coloring the vertex u with the color 1, and coloring both v and w with a unique color. Then, $\mathcal{C}$ is a TD-coloring of T with one more color class than the coloring $\mathcal{C}'$, implying that $\chi_d^t(T) \leq \chi_d^t(T') + 1$, contradicting our earlier observation that $\chi_d^t(T) = \chi_d^t(T') + 2$. Therefore, $d_{H'}(v') = 2$, and so $N_T(v') = \{u', z\}$. Recall that by assumption, $d_{H'}(u') \leq d_{H'}(v')$.

Suppose that $d_{H'}(u') = 2$. Let a be a vertex in H', different from v', that is adjacent to the vertex u' in H'. Let abc be the path of length 2 attached to a when constructing the tree T'. Let $\mathcal{C}$ be the coloring obtained from $\mathcal{C}'$ by recoloring the vertex v' with the same color used to color y, coloring the vertex u with the color 1, and coloring both v and w with a unique color. Then, $\mathcal{C}$ is a TD-coloring of T with one more color class than the coloring $\mathcal{C}'$, a contradiction. Therefore, u' is a leaf in H' (and hence in T).

Since u' is a leaf in T and $N_T(v') = \{u', z\}$, we see that $T \in \mathcal{F}$, where the underlying tree, H^*, of T has vertex set $(V(H') \setminus \{u', v'\}) \cup \{w, x, y\}$ and where x and y are the two adjacent vertices in H^* whose degrees are unchanged when constructing the tree T. This completes the proof of Claim 5. □

We now return to the proof of Theorem 18 one final time. By Claim 4 and Claim 5, we may assume that $x \in V(H')$. But then $T \in \mathcal{F}$, where the underlying tree H of T is obtained from H' by adding to it the vertex w and the edge xw. This completes the proof of Theorem 18. □

3.4 Paths

Recall from Observation 3, that for $n \geq 3$ if $G \in \{P_n, C_n\}$, then $\gamma_t(G) = n/2$ if $n \equiv 0 \,(\mathrm{mod}\, 4)$, $\gamma_t(G) = (n+1)/2$ if $n \equiv 1, 3 \,(\mathrm{mod}\, 4)$, and $\gamma_t(G) = n/2 + 1$ if $n \equiv 2 \,(\mathrm{mod}\, 4)$. For small values of $n \geq 2$, the total dominator chromatic number of a path P_n on n vertices is easy to compute (or can be checked by computer).

Observation 19 For $2 \leq n \leq 15$, we have

$$\chi_d^t(P_n) = \begin{cases} \gamma_t(P_n) & \text{for } n \in \{2, 3, 6\} \\ \gamma_t(P_n) + 1 & \text{for } n \in \{4, 5, 7, 9, 10, 11, 14\} \\ \gamma_t(P_n) + 2 & \text{for } n \in \{8, 12, 13, 15\} \end{cases}$$

For example, coloring the vertices of P_{11} with the sequence of colors 1, 2, 3, 1, 4, 5, 4, 1, 6, 7, 1 produces a $\chi_d^t(P_{11})$-coloring, while coloring the vertices of P_{14} with the sequence of colors 1, 2, 3, 1, 4, 5, 4, 6, 7, 6, 1, 8, 9, 1 produces a $\chi_d^t(P_{14})$-coloring. We determine next the total dominator chromatic number of a path P_n for $n \geq 16$.

Proposition 20 For $n \geq 16$, $\chi_d^t(P_n) = \gamma_t(P_n) + 2$.

Proof We proceed by induction on $n \geq 16$. The base cases when $n \in \{16, 17, 18, 19\}$ are easy to verify (or can be checked by computer) and a $\chi_d^t(P_n)$-coloring is shown in Fig. 4. Suppose, then, that $n \geq 20$ and that for all n' where $16 \leq n' < n$, we have $\chi_d^t(P_{n'}) = \gamma_t(P_{n'}) + 2$. Let G be the path $v_1 v_2 \ldots v_n$ and let $\mathcal{C}$ be a TD-coloring of G. Let $G' = G - \{v_1, v_2, v_3, v_4\}$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices in G'. We note that the vertex v_2 is assigned a unique color in $\mathcal{C}$. Further, the neighborhood $N(v_2) = \{v_1, v_3\}$ of v_2 contains a color class of $\mathcal{C}$. Therefore, the coloring $\mathcal{C}$ contains at least two more color classes than does the coloring $\mathcal{C}'$.

Suppose that $\mathcal{C}'$ is a TD-coloring of G'. Applying the inductive hypothesis to G' we see that $\mathcal{C}'$ has at least $\chi_d^t(G') = \gamma_t(G') + 2 = \gamma_t(P_{n-4}) + 2 = \gamma_t(P_n) = \gamma_t(G)$ color classes, implying that the TD-coloring $\mathcal{C}$ has at least $\gamma_t(G) + 2$ color classes.

Suppose that $\mathcal{C}'$ is not a TD-coloring of G'. Then since $\mathcal{C}$ is a TD-coloring of G, the only vertex in G' that is not adjacent to every vertex of some color class in $\mathcal{C}'$ is the vertex v_5. However, v_6 is the only neighbor of v_5 in G', implying that in the coloring $\mathcal{C}$ the vertex v_4 has a unique color while the color class containing the vertex v_6 contains at least two vertices. In this case, we let $\mathcal{C}''$ be obtained from the coloring $\mathcal{C}'$ by recoloring the vertex v_6 with the color assigned to the vertex v_4 in the coloring $\mathcal{C}$. Then, $\mathcal{C}''$ is a TD-coloring of G'. Since at least two color classes in the TD-coloring $\mathcal{C}$ are contained in the set $\{v_1, v_2, v_3\}$, the coloring $\mathcal{C}$ contains at least two more color classes than does the coloring $\mathcal{C}''$. Applying the inductive hypothesis to G' we see that the TD-coloring $\mathcal{C}''$ has at least $\chi_d^t(G') = \gamma_t(G') + 2 = \gamma_t(P_{n-4}) + 2 = \gamma_t(P_n) = \gamma_t(G)$ color classes, implying that the TD-coloring $\mathcal{C}$ has at least $\gamma_t(G) + 2$ color classes.

In both cases, the TD-coloring $\mathcal{C}$ has at least $\gamma_t(G) + 2$ color classes. Since $\mathcal{C}$ is an arbitrary TD-coloring of G, this implies that $\chi_d^t(G) \geq \gamma_t(G) + 2$. By Corollary 10, $\chi_d^t(G) \leq \gamma_t(G) + 2$. Consequently, $\chi_d^t(G) = \gamma_t(G) + 2$.   □

For $n \geq 16$, we define next a $\chi_d^t(P_n)$-coloring, $\mathcal{C}_n^*$, of a path P_n as follows. Let G be the path $v_1 v_2 \ldots v_n$, where $n \geq 16$. For each vertex v_i where $i \equiv 2, 3 \,(\mathrm{mod}\, 4)$,
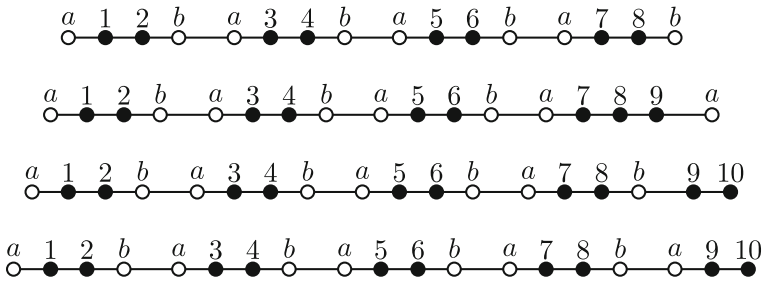
Fig. 4 A coloring $\mathcal{C}_n^t$ of a path P_n for $n \in \{16, 17, 18, 19\}$

assign a unique color. For each vertex v_i where $i \equiv 0 \pmod 4$, assign a new additional color, say b. For each vertex v_i where $i \equiv 1 \pmod 4$, assign a further additional color, say a. Let $\mathcal{C}_n$ denote the resulting coloring. We now define a coloring $\mathcal{C}_n^*$ as follows. If $n \equiv 0, 3 \pmod 4$, let $\mathcal{C}_n^* = \mathcal{C}_n$. If $n \equiv 1 \pmod 4$, then recolor the vertex v_{n-1} (currently colored with color b) with a new distinct color and let $\mathcal{C}_n^*$ denote the resulting modified coloring. If $n \equiv 2 \pmod 4$, then recolor the vertex v_{n-1} (currently colored with color a) with a new distinct color and let $\mathcal{C}_n^*$ denote the resulting modified coloring. The coloring $\mathcal{C}_n^*$ when $n \in \{16, 17, 18, 19\}$, for example, is illustrated in Fig. 4. The darkened vertices in this coloring of $\mathcal{C}_n^*$ in Fig. 4 form a $\gamma_t(P_n)$-set.

We note that $\mathcal{C}_n^*$ is a proper coloring of the vertices of G. Let S be the set of vertices in G that are not colored with the color a or b. By the way in which colors in $\mathcal{C}_n^*$ are assigned, each vertex in S is assigned a unique color. Thus since S is a TD-set in G, every vertex in G is adjacent to every vertex of some color class. Thus, $\mathcal{C}_n^*$ is a TD-coloring of G. Further, the set S is a (minimum) TD-set in G and $|S| = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor = \gamma_t(P_n)$. Thus, $\mathcal{C}_n^*$ has $|S| + 2 = \gamma_t(P_n) + 2$ color classes. Thus by Proposition 20, we see that the coloring $\mathcal{C}_n^*$ is a $\chi_d^t(P_n)$-coloring for $n \geq 16$.

We remark that analogously to the argument for a path, one can readily determine the total dominator chromatic number of a cycle C_n, $n \geq 3$. We omit the routine details.

Observation 21 $\chi_d^t(C_3) = 3$, $\chi_d^t(C_4) = 2$, and $\chi_d^t(C_{11}) = 8$. For all other values of $n \geq 5$, we have $\chi_d^t(C_n) = \chi_d^t(P_n)$.

4 Open Problems

We close with three open problems that we have yet to settle. The first open problem is to establish whether the converse of Proposition 11 is true. We state this formally as follows.

Problem 1 Prove or disprove: if G is an isolate-free graph satisfying $\chi_d^t(G) = \gamma_t(G)$, then $G \in \mathcal{G}_0$.

We remark that Problem 1 is true in the case when G is a tree as proven by Theorem 15. Recall that by Corollary 10, if T is a nontrivial tree, then $\gamma_t(T) \leq \chi_d^t(T) \leq \gamma_t(T) + 2$. The infinite family of trees T satisfying $\gamma_t(T) = \chi_d^t(T)$ are characterized in Theorem 15. We close with the following two open problems.

Problem 2 Characterize the trees T satisfying $\gamma_t(T) = \chi_d^t(T) + 1$.

Problem 3 Characterize the trees T satisfying $\gamma_t(T) = \chi_d^t(T) + 2$.

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