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ORIGINAL PAPER

Counting Colorings of a Regular Graph

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Abstract At most how many (proper) q-colorings does a regular graph admit? Galvin and Tetali conjectured that among all n-vertex, d-regular graphs with $2d|n$, none admits more q-colorings than the disjoint union of $n/2d$ copies of the complete bipartite graph $K_{d,d}$. In this note we give asymptotic evidence for this conjecture, showing that for each $q \geq 3$ the number of proper q-colorings admitted by an n-vertex, d-regular graph is at most

$$\begin{cases} (q^2/4)^{\frac{n}{2}} \binom{q}{q/2}^{\frac{n(1+o(1))}{2d}} & \text{if } q \text{ is even} \\ ((q^2-1)/4)^{\frac{n}{2}} \binom{q+1}{(q+1)/2}^{\frac{n(1+o(1))}{2d}} & \text{if } q \text{ is odd,} \end{cases}$$

where $o(1) \to 0$ as $d \to \infty$; these bounds agree up to the $o(1)$ terms with the counts of q-colorings of $n/2d$ copies of $K_{d,d}$. Along the way we obtain an upper bound on the number of colorings of a regular graph in terms of its independence number. For example, we show that for all even $q \geq 4$ and fixed $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, q)$ such that the number of proper q-colorings admitted by an n-vertex, d-regular graph with no independent set of size $n(1-\varepsilon)/2$ is at most

$$(q^2/4 - \delta)^{\frac{n}{2}},$$

with an analogous result for odd q.

Keywords Proper coloring · Enumeration · Regular graph

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1 Introduction

Throughout, G is a simple, finite loopless graph, and q is a positive integer. A *proper q-coloring* (or just *q-coloring*) of G is a function from the vertices of G to $\{1, \ldots, q\}$ with the property that adjacent vertices have different images. We write $c_q(G)$ for the number of q-colorings of G.

The following is a natural extremal enumerative question: for a family $\mathscr{G}$ of graphs, which $G \in \mathscr{G}$ maximizes $c_q(G)$? For example, for the family of n-vertex, m-edge graphs this question was raised independently by Wilf [2,12] (who encountered it in his study of the running time of a backtracking coloring algorithm) and Linial [9] (who encountered the *minimization* question in his study of the complexity of determining whether a given function on the vertices of a graph is in fact a proper coloring). Although it has only been answered completely in some very special cases many partial results have been obtained (see [10] for a good history of the problem).

The focus of this note is the family of n-vertex d-regular graphs with $d \geq 2$ (the case $d = 1$ being trivial). Galvin and Tetali [5] used an entropy argument to show that for $2d|n$ no *bipartite* n-vertex d-regular G admits more q-colorings, for each $q \geq 2$, than $\frac{n}{2d} K_{d,d}$, the disjoint union of $n/2d$ copies of the complete bipartite graph $K_{d,d}$ with d vertices in each partite set. More generally they found $c_q(G) \leq c_q(K_{d,d})^{n/2d}$ for all n, d and bipartite n-vertex d-regular G (this is [5, Prop. 1.2] in the special case $H = K_q$), and they conjectured that this bound should still hold when the biparticity assumption is dropped.

Conjecture 1.1 *For $d \geq 2, n \geq d + 1$ and $q \geq 2$, if G is any n-vertex d-regular graph then*

$$c_q(G) \leq c_q(K_{d,d})^{\frac{n}{2d}}.$$

For $q = 2$ this follows immediately from the bipartite case established in [5], so for the rest of this note we focus on $q \geq 3$. Zhao [14] established the conjecture for all $q \geq (2n)^{2n+2}$, and in the case $2d|n$ Galvin [3], using ideas introduced by Lazebnik [8] on a related problem, reduced this to $q > 2\binom{nd/2}{4}$, but neither the approach of [3] nor that of [14] seems adaptable to the case of constant $q \geq 3$.

Conjecture 1.1 is a special case of a more general conjecture concerning graph homomorphisms. A *homomorphism* from G to a graph H (which may have loops) is a map from vertices of G to vertices of H with adjacent vertices in G being mapped to adjacent vertices in H. Homomorphisms generalize q-colorings (if $H = K_q$ then the set of homomorphisms to H is in bijection with the set of q-colorings of G) as well as other graph theory notions, such as independent sets. An *independent set* in a graph is a set of pairwise non-adjacent vertices; notice that if $H = H_{\text{ind}}$ is the graph on two adjacent vertices with a loop at exactly one of the vertices, then a homomorphism from G to H may be identified, via the preimage of the unlooped vertex, with an independent set in G. Amending a false conjecture from [5], the following conjecture is made in [3]. Here we write $\hom(G, H)$ for the number of homomorphisms from G to H.

Conjecture 1.2 *For $d \geq 2$, $n \geq d+1$ and any finite graph H (perhaps with loops, but without multiple edges), if G is any n-vertex d-regular graph then*

$$\hom(G, H) \leq \max\{\hom(K_{d,d}, H)^{\frac{n}{2d}}, \hom(K_{d+1}, H)^{\frac{n}{d+1}}\},$$

where K_{d+1} is the complete graph on $d+1$ vertices.

When $d \geq q$ we have $\hom(K_{d+1}, K_q) = 0$ and so in this range Conjecture 1.2 implies Conjecture 1.1.

The inspiration for Conjecture 1.2, and the partial result of [5] that the conjecture is true for all *bipartite* G, was the special case of enumerating independent sets ($H = H_{\text{ind}}$). In what follows we use $i(G)$ to denote the number of independent sets in G. Alon [1] conjectured that for all n-vertex d-regular G we have

$$i(G) \leq i(K_{d,d})^{n/2d} = (2^{d+1} - 1)^{n/2d} = 2^{n/2 + n(1+o(1))/2d}$$

(where here and in the rest of this paragraph $o(1) \to 0$ as $d \to \infty$), and proved the weaker bound $i(G) \leq 2^{n/2 + Cn/d^{1/10}}$ for some absolute constant $C > 0$. The sharp bound was proved for *bipartite* G by Kahn [7], but it was a while before a bound for general G was obtained that came close to $i(K_{d,d})^{n/2d}$ in the second term of the exponent; this was Kahn's (unpublished) bound $i(G) \leq 2^{n/2+n(1+o(1))/d}$. This was improved to $i(G) \leq 2^{n/2+n(1+o(1))/2d}$ by Galvin [4]. Finally Zhao [13] deduced the exact bound for general G from the bipartite case.

The aim of this note is to obtain an asymptotic version of Conjecture 1.1, along the lines of Galvin's upper bound on the count of independent sets in n-vertex, d-regular graphs. Before stating the main result, we need to do some preliminary calculations. Define

$$\eta(q) = \lfloor q/2 \rfloor \lceil q/2 \rceil = \begin{cases} \frac{q^2}{4} & \text{if } q \text{ is even} \\ \frac{q^2-1}{4} & \text{if } q \text{ is odd,} \end{cases}$$

and

$$m(q) = \binom{2\lceil q/2 \rceil}{\lceil q/2 \rceil} = \begin{cases} \binom{q}{q/2} & \text{if } q \text{ is even} \\ \binom{q+1}{(q+1)/2} = \binom{q}{\lfloor q/2 \rfloor} + \binom{q}{\lceil q/2 \rceil} & \text{if } q \text{ is odd.} \end{cases}$$

Fix a bipartition $\mathcal{E} \cup \mathcal{O}$ of $K_{d,d}$. Say that an ordered pair (A, B) with $A, B \subseteq \{1, \ldots, q\}$ is *allowable* if A and B are disjoint and both are non-empty. The set of q-colorings of $K_{d,d}$ may be written as $\cup_{(A,B)} \mathcal{C}(A, B)$ where the union is over all allowable pairs, and where $\mathcal{C}(A, B)$ consists of colorings in which the set of colors appearing on $\mathcal{E}$ (resp. $\mathcal{O}$) is exactly A (resp. B). By inclusion-exclusion

$$|\mathscr{C}(A,B)| = \sum_{s=0}^{|A|}\sum_{t=0}^{|B|}(-1)^{s+t}\binom{|A|}{s}\binom{|B|}{t}((|A|-s)(|B|-t))^d$$

and so

$$|\mathscr{C}(A,B) - (|A||B|)^d| \leq O_q((|A||B|-1)^d).$$

Notation 1.3 Here, and throughout this note, we will use the notation $f = O_q(g)$ for arbitrary function f and positive function g to indicate that there is a positive constant $C(q)$, depending only on q (and so *not* on n, d or any other parameter), such that $|f| \leq C(q)g$.

The maximum possible value of $|A||B|$ is $\eta(q)$ (achieved when $A \cup B = \{1, \dots q\}$ and $|A|-|B| \in \{-1, 0, 1\}$), and there are $m(q)$ pairs that achieve this value. It follows that

$$|c_q(K_{d,d}) - m(q)\eta(q)^d| = O_q((\eta(q)-1)^d).$$

This leads to the following estimate for $c_q(K_{d,d})^{n/2d}$. For all $q \geq 3, d \geq 2$ and $n \geq d+1$ we have

$$c_q(K_{d,d})^{\frac{n}{2d}} = \eta(q)^{\frac{n}{2}} m(q)^{\frac{n}{2d}(1+O_q(1-\frac{1}{\eta(q)})^d)}.$$

In particular, for each fixed $q \geq 3$ we have

$$c_q(K_{d,d})^{\frac{n}{2d}} = \eta(q)^{\frac{n}{2}} m(q)^{\frac{n(1+o(1))}{2d}} \tag{1}$$

where $o(1) \to 0$ as d (and so n) $\to \infty$.

Our main theorem is an upper bound on $c_q(G)$ for n-vertex, d-regular G that matches (1) up to the $o(1)$ term.

Theorem 1.4 *Fix $q \geq 3$. For $d \geq 2$ and $n \geq d+1$, if G is any n-vertex, d-regular graph then*

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} m(q)^{\frac{n(1+o(1))}{2d}},$$

where $o(1) \to 0$ as $d \to \infty$, where recall that $\eta(q) = \lfloor q/2 \rfloor \lceil q/2 \rceil$ and $m(q) = \binom{2\lceil q/2 \rceil}{\lceil q/2 \rceil}$. In fact, for all $q \geq 3$ we have

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} m(q)^{\frac{n}{2d}(1+O_q(\sqrt{\frac{\log d}{d}}))}.$$

The best previous result in the direction of Theorem 1.4 was from [3], where it was shown that for fixed $q \geq 3$ we have

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} m(q)^{\frac{n(1-q)(1+o_d(1))}{dq}}.$$

(This only appears explicitly in [3] for $q = 3$, but follows immediately for general q from Proposition 1.5 below by taking $\alpha = n/q$; note that for all smaller α, $c_q(G) = 0$.)

To prove Theorem 1.4 we consider the independence number $\alpha(G)$ of G, the number of vertices in a largest independent set, and deal separately with large $\alpha(G)$ and small $\alpha(G)$. The case of large $\alpha(G)$ has already been dealt with in [3, Section 5], where an entropy approach was used to obtain the following result.

Proposition 1.5 *For $q \geq 3$, $d \geq 2$, $n \geq d + 1$ and $\varepsilon > 0$, if G is any n-vertex d-regular graph with $\alpha(G) \geq n(1-\varepsilon)/2$ then*

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} m(q)^{\frac{n(1+\varepsilon)}{2d}} O_q(1)^{\frac{n}{d^2}}.$$

(This is [3, eq. (11)] specialized to $H = K_q$.)

To bound $c_q(G)$ when G has no large independent sets we adopt an argument of Sapozhenko to obtain the following, which we prove in Sect. 2.

Lemma 1 *For $q \geq 3$ there are positive constants $c_1(q)$, $c_2(q)$ with the following property. For $d \geq 2$, $n \geq d + 1$ and $\varepsilon > 0$, if G is any n-vertex d-regular graph with $\alpha(G) \leq n(1-\varepsilon)/2$ then*

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} \exp_2\left\{c_1(q)n\sqrt{\frac{\log d}{d}} - c_2(q)\varepsilon n\right\}.$$

(For concreteness, here and throughout $\log = \log_2$.) Taking $\varepsilon = C(q)\sqrt{\log d/d}$ for suitably large $C(q)$, Proposition 1.5 and Lemma 1 combine to give Theorem 1.4.

In the process of proving Lemma 1, we will describe a very simple argument that gives the weaker bound

$$c_q(G) \leq \eta(q)^{\frac{n}{2}} 2^{O_q(n\sqrt{\frac{\log d}{d}})} \tag{2}$$

valid for *all* n-vertex, d-regular G. Note also that Lemma 1 together with [5, Prop. 1.2] (the bipartite case of Conjecture 1.1) shows that for each $q \geq 3$ there is a positive constant $C(q)$ such that the only n-vertex, d-regular G which remain as potential counterexamples to Conjecture 1.1 are those which are non-bipartite and have an independent set of size at least $(n/2)(1 - C(q)\sqrt{(\log d)/d})$. While we do not expect the approximation scheme described in this note to fully resolve Conjecture 1.1, it may be that it contributes to the complete resolution (at least for all large d) by allowing attention to be focussed on these "almost bipartite" graphs.

A simple corollary of Lemma 1 is that for each fixed $\varepsilon > 0$ and $q \geq 3$ there is $\delta = \delta(\varepsilon, q) > 0$ such that for all $d \geq 2$, $n \geq d + 1$ and n-vertex, d-regular G with $\alpha(G) \leq n(1-\varepsilon)/2$, we have

$$c_q(G) \leq (\eta - \delta)^{\frac{n}{2}}.$$

A natural question to ask is how δ (more precisely, the supremum over all δ for which the preceding statement is true) varies with ε in the range $0 \leq \varepsilon \leq 1-(2/q)$. At $\varepsilon = 0$

we have $\delta = 0$ (by Theorem 1.4 and the example of the disjoint union of $K_{d,d}$'s), and from the fact that $c_q(G) = 0$ whenever $\alpha(G) < n/q$ we conclude that $\delta = \eta(q)$ for all $\varepsilon > 1 - (2/q)$.

Question 1 For $d \geq 2$, $n \geq d+1$, $q \geq 3$ and $0 \leq \varepsilon \leq 1-(2/q)$, what is the maximum of $c_q(G)$ over all n-vertex, d-regular G with $\alpha(G) \leq n(1-\varepsilon)/2$?

2 Proof of Lemma 1: Small Independent Sets

To obtain (2) we modify an argument due to Sapozenko [11], originally used to enumerate independent sets in a regular graph; a further modification of this argument will give Lemma 1.

Let $\varphi = \sqrt{d \log d}/q$ (note that $\varphi < d$). For an independent set I in G, recursively construct sets $T = T(I)$ and $D = D(T)$ as follows. Pick $u_1 \in I$ arbitrarily and set $T_1 = \{u_1\}$. Given $T_m = \{u_1, \ldots, u_m\}$, if there is $u_{m+1} \in I$ with $N(u_{m+1}) \setminus N(T_m) \geq \varphi$, then set $T_{m+1} = \{u_1, \ldots, u_{m+1}\}$ (here $N(\cdot)$ indicates open neighborhood). If there is no such u_{m+1}, then set $T = T_m$ and

$$D = \{v \in V(G) \setminus N(T) : N(v) \setminus N(T) < \varphi\}.$$

Note that

$$|T| \leq \frac{n}{\varphi}, \tag{3}$$

since by construction $n \geq N(T) \geq (|T|-1)\varphi + d \geq |T|\varphi$; that

$$I \subseteq D$$

since if $I \setminus D \neq \emptyset$, the construction of T would not have stopped (note that $N(T) \cap I = \emptyset$); and that

$$|D| \leq \frac{nd}{2d-\varphi} \leq \frac{n}{2}\left(1+\frac{\varphi}{d}\right). \tag{4}$$

The second inequality here follows from $\varphi < d$. To see the first, consider the bipartite graph with partition classes D and $N(T)$ and edges induced from G. This graph has at most $d|N(T)| \leq d(N-|D|)$ edges (since each vertex in $N(T)$ has at most d edges to D, and there are at most $N-|D|$ such vertices), and at least $(d-\varphi)|D|$ edges (since each vertex in D has at least $d-\varphi$ edges to $N(T)$). Putting these two inequalities together gives (4).

Now a q-coloring of G is an ordered partition of $V(G)$ into q independent sets, $(I_1, \ldots, I_q)$, with I_k being the set of vertices colored k. Following Sapozhenko's argument, we associate with this partition an ordered list $(D(T(I_1)), \ldots, D(T(I_q)))$. We recover all q-colorings of G (and perhaps more) by finding all such lists, and then for each list $(D_1, \ldots, D_q)$ finding all ordered partitions of the $V(G)$ into q sets

$(I_1, \ldots, I_q)$ (not necessarily independent sets), with $I_k \subseteq D_k$ for each k. We say that such a partition is *compatible* with the D_k's.

By (3) each possible D_k is determined by a set of size at most n/φ, so the number of choices for each D_k is $\sum_{i \le n/\varphi} \binom{n}{i}$. For d satisfying $\varphi \ge 2$, we bound this sum using the inequality

$$\sum_{i \le pn} \binom{n}{i} \le 2^{H(p)n}$$

valid for $p \le 1/2$, where $H(p) = -p \log p - (1-p)\log(1-p)$ is the binary entropy function (see, e.g. [6, Corollary 22.9]). Using $H(p) \le -(1 + 1/\ln 2)p \log p$ (valid for $p \le 1/2$), we conclude that for $\varphi \ge 2$ we have $\sum_{i \le n/\varphi} \binom{n}{i} \le 2^{n(1+1/\ln 2)(\log \varphi)/\varphi}$. To extend the range of validity of this inequality to all $d \ge 2$ we only need to replace the $(1 + 1/\ln 2)$ in the exponent by a constant depending on q. We conclude that the number of choices for $(D_1, \ldots, D_q)$ is at most

$$\left(\sum_{i \le n/\varphi} \binom{n}{i}\right)^q = 2^{O_q(n\sqrt{\frac{\log d}{d}})}. \tag{5}$$

We now bound the number of partitions compatible with a particular $(D_1, \ldots, D_q)$. For each $v \in V(G)$ let a_v denote the number of D_k's with $v \in D_k$. A simple upper bound for the number of partitions is $\prod_{v \in V(G)} a_v$. Using (4) we have

$$\sum_{v \in V(G)} a_v = \sum_{k=1}^{q} |D_k| \le \frac{qn}{2}\left(1 + \frac{\varphi}{d}\right).$$

By the arithmetic mean-geometric mean inequality we get

$$\begin{aligned} \prod_{v \in V(G)} a_v &\le \left(\frac{1}{n} \sum_{v \in V(G)} a_n\right)^n \\ &\le \left(\frac{q}{2}\right)^n \left(1 + \frac{\varphi}{d}\right)^n \\ &= \left(\frac{q^2}{4}\right)^{\frac{n}{2}} 2^{O_q\left(n\sqrt{\frac{\log d}{d}}\right)}. \end{aligned} \tag{6}$$

Combining (5) and (6) we get (2) for even q.

We now work towards a better bound that incorporates the independence number of G. It will be convenient from here on to know that $|D_i| \ge n/2$ for each i. This may not initially be the case; but if it is not, then we may add vertices of G to D_i in some deterministic way until it has size $\lceil n/2 \rceil$. For example, we may totally order $V(G)$, and augment D_i by including the first $\lceil n/2 \rceil - |D_i|$ vertices in the order that are not already in D_i. Although D_i may now no longer have the form $D(T)$ for some

T, our estimates for the number of choices for D_i remain unchanged. This is because for each $D(T)$ with $|D(T)| < \lceil n/2 \rceil$, there is a *unique* extension to a set of size $\lceil n/2 \rceil$ following the process described. Also, since increasing the size of D_i can only increase the number of compatible partitions, any upper bound we can obtain in this case is also a valid upper bound in the original case.

Now we look at the subgraph induced by D_1. It inherits from G the property that no independent set has size greater than $(n/2)(1-\varepsilon)$. This means that D_1 has a matching of size at least $n\varepsilon/4$ (which may be found greedily).

Fix such a matching $M = \{x_1y_1, \ldots, x_{|M|}y_{|M|}\}$. In our naive count of colorings (i.e., compatible partitions), we had a factor $a_{x_1}a_{x_2}$ to account for the possible colors assigned to x_1 and y_1 in a compatible partition. But since x_1 and y_1 are adjacent, we cannot assign color 1 to both vertices, and so we have at most

$$a_{x_1}a_{x_2} - 1 = a_{x_1}a_{x_2}\left(1 - \frac{1}{a_{x_1}a_{x_2}}\right) \le a_{x_1}a_{x_2}\left(1 - \frac{1}{q^2}\right)$$

choices for this pair. Applying this argument to each of the pairs (x_i, y_i), we get an upper bound on the number of colorings compatible with $(D_1, \ldots, D_q)$ of

$$\left(\prod_{v \in V(G)} a_v\right)\left(1 - \frac{1}{q^2}\right)^{|M|} \le \left(\frac{q^2}{4}\right)^{\frac{n}{2}} 2^{O_q\left(n\sqrt{\frac{\log d}{d}}\right)}\left(1 - \frac{1}{q^2}\right)^{n\varepsilon/4} \tag{7}$$

$$\le \left(\frac{q^2}{4}\right)^{\frac{n}{2}} 2^{O_q\left(n\sqrt{\frac{\log d}{d}}\right) - \frac{(\log_2 e)\varepsilon n}{4q^2}}. \tag{8}$$

In (7) we use exactly the same steps that led to (6), together with our lower bound on $|M|$, and in (8) we use $1 - x \le e^{-x}$, valid for all real x. Combining with (5) we obtain Lemma 1 for even q.

Now we turn to odd q. Preceding exactly as before, we have

$$c_q(G) \le \left(\prod_{v \in V(G)} a_v\right) 2^{O_q\left(n\sqrt{\frac{\log d}{d}}\right) - \frac{(\log_2 e)\varepsilon n}{4q^2}},$$

so we are done (both with Lemma 1 and with (2) in the case of odd q) if we can bound

$$\prod_{v \in V(G)} a_v \le (\lfloor q/2 \rfloor \lceil q/2 \rceil)^{\frac{n}{2}} 2^{O_q\left(n\sqrt{\frac{\log d}{d}}\right)}. \tag{9}$$

For this we need the following simple optimization lemma.

Lemma 2 *Let $a_1, \ldots, a_m$ be positive real numbers with average a. If there is a $\delta \ge 0$ such that no a_i is in the interval $(a - \delta, a + \delta)$, then*

$$\prod_{i=1}^{m} a_i \le (a^2 - \delta^2)^{\frac{m}{2}} = (a - \delta)^{\frac{m}{2}}(a + \delta)^{\frac{m}{2}}.$$

Proof Let $f(x)$ be a continuous function that agrees with $\log x$ on $(0, a-\delta] \cup [a+\delta, \infty)$ and is linear on $(a-\delta, a+\delta)$. Since f is concave we may apply Jensen's inequality to conclude

$$\begin{aligned}\sum_{i=1}^{m} \log a_i &= \sum_{i=1}^{m} f(a_i) \\ &\leq mf(a) \\ &= \frac{m}{2}(\log(a-\delta) + \log(a+\delta)),\end{aligned}$$

from which the lemma follows immediately by exponentiation. □

To apply Lemma 2 and obtain (9) we recall our assumption that each D_i satisfies $|D_i| \geq n/2$. This, together with (4) and our specific choice of φ gives that the average of the a_v's satisfies $a \in [q/2, q/2 + (1/2)\sqrt{\log d/d}]$. Since the a_v's must be integers, and $\sqrt{\log d/d} < 1$, we may take $\delta = (1/2)(1 - \sqrt{\log d/d})$ in Lemma 2 to get

$$\begin{aligned}\prod_{v \in V(G)} a_v &\leq \left(\lfloor q/2 \rfloor + \sqrt{\frac{\log d}{d}} \right)^{\frac{n}{2}} (\lceil q/2 \rceil)^{\frac{n}{2}} \\ &= (\lfloor q/2 \rfloor \lceil q/2 \rceil)^{\frac{n}{2}} 2^{O_q(n\sqrt{\frac{\log d}{d}})},\end{aligned}$$

as required.

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