Graphs and Combinatorics (2015) 31:281–298
DOI 10.1007/s00373-013-1385-x

ORIGINAL PAPER

Convex Pentagons for Edge-to-Edge Tiling, II

Teruhisa Sugimoto

Received: 31 August 2012 / Revised: 6 October 2013 / Published online: 24 December 2013

Abstract Based on Bagina's Proposition, it has previously been demonstrated that there remain 34 cases where it is uncertain whether a convex pentagon can generate an edge-to-edge tiling. In this paper, these cases are further refined by imposing extra edge conditions. To investigate the resulting 42 cases, the properties of convex pentagonal tiles that can generate an edge-to-edge tiling are identified. These properties are the key to generating a perfect list of the types of convex pentagonal tiles that can generate an edge-to-edge tiling.

Keywords Convex pentagon · Tiling · Tile · Monohedral tiling · Edge-to-edge tiling

1 Introduction

A collection of sets ('tiles') is a *tiling* of the plane if their union is the whole plane, but the interiors of different tiles are disjoint. A tiling is *monohedral* if all tiles in the tiling are of the same size and shape [3]. In our study, a polygon in a monohedral tiling is called a *polygonal tile* [7,8]. In the classification problem of convex polygonal tiles, only the pentagonal case remains uncertain. At present, 14 essentially different types of convex pentagonal tiles are known (see Appendix A.1), but it is not known whether this list is perfect [2–7,9,11]. (Note that, as mentioned in [7], the classification problem of types of convex pentagonal tiles and the classification problem of pentagonal tilings are quite different.)

For a tiling by convex polygons, if any two polygons are either disjoint or share one vertex or an entire edge in common, we say the tiling by convex polygons is

T. Sugimoto (✉)
The Interdisciplinary Institute of Science, Technology and Art, Suzukidaini-building 211, 2-5-28 Kitahara, Asaka-shi, Saitama, 351-0036, Japan
e-mail: ismsugi@gmail.com

edge-to-edge. In our study, a vertex of an edge-to-edge tiling is called a *node*, and the number of polygons meeting at a node is called the *valence* of the node [7–10]. In addition, a convex pentagonal tile that can generate an edge-to-edge tiling is called an *EE convex pentagonal tile* [7,8].

We have attempted to obtain a perfect (or complete) list of all types of EE convex pentagonal tiles. In our previous paper [7], we used Bagina's Proposition [1] (which implies that "in any edge-to-edge tiling of the plane by convex pentagonal tiles, there is a pentagon around which at least three nodes have valence 3, in other words, there exists a tile with at least three 3-valent nodes") to produce various patterns of pentagons.[1] Examining these pentagons one-by-one, we classified them into (i) geometrically impossible cases, (ii) cases that cannot generate an edge-to-edge tiling, (iii) known types, and (iv) 34 cases where it is uncertain whether the convex pentagons can generate an edge-to-edge tiling (see Table 5 in [7]). In the present paper, these remaining 34 cases are further refined into 42 cases by imposing extra edge conditions (see Table 1). We thoroughly study the resulting 42 uncertain cases, and succeed in obtaining a perfect list of types of EE convex pentagonal tiles [8]. This investigation method requires many patterns to be considered. In this paper, we introduce some preliminary ideas for the investigation of each pattern. In a following paper, we will explain the consideration methods for the 42 uncertain cases and present the perfect list. Therefore, in this paper, we prove the following important properties of EE convex pentagonal tiles.

Lemma 1 *If the density of k-valent nodes for $k \geq 5$ in an edge-to-edge monohedral tiling by a convex pentagon is greater than zero, then there exists a tile with four or more* 3*-valent nodes.*

Lemma 2 *If the densities of k-valent nodes for $k = 4,\ 5,$ and* 6 *in an edge-to-edge monohedral tiling by a convex pentagon are equal to zero (i.e., only the densities of k-valent nodes for $k = 3$ and ≥ 7 are greater than zero), then there exists a tile with five* 3*-valent nodes.*

Note that the density of k-valent nodes is the ratio of the number of k-valent nodes to the number of pentagons in an edge-to-edge monohedral tiling by a convex pentagon. See Sect. 2 for the exact definition of density.

In Sect. 2, we show other properties of EE convex pentagonal tiles, and present the proof of Lemmas 1 and 2. In Sect. 3, we introduce the method used to refine the 34 uncertain cases of [7], and obtain the 42 uncertain cases that will be investigated in the future. Section 4 explains a property of EE convex pentagonal tiles relevant to Lemma 2.

As mentioned in [7], if it is clear that a convex pentagon belongs to one of the known types, then it is excluded from the investigation. For example, if a convex pentagon has the condition that the sum of three consecutive angles is equal to 360°,

[1] The summary of our method is as follows: Let G be an EE convex pentagonal tile candidate. Then, by Bagina's Proposition, it has at least three vertices that will become 3-valent nodes in the tiling. We choose (any) two of them as V_1, V_2, and let v_1, v_2 be conditions on the angles at V_1, V_2, respectively. By substituting all possible angle conditions in v_1, v_2 and considering the lengths of matching edges, we can produce 465 patterns of candidate pentagons G. Then, convex pentagons satisfying the conditions produced from v_1 and v_2 are referred to as patterns or cases. Refer to [7] for details.

then it is excluded from the investigation because it belongs to type 1 (see Appendix A.1 for types of convex pentagonal tiles). As mentioned in Section 2 of [7], a convex pentagonal tile belonging to type 1 or type 2 is not always able to generate an edge-to-edge tiling. In [7], it was found that, of the convex pentagons judged to belong to type 1 or type 2, there were tiles that cannot generate an edge-to-edge tiling (e.g., if a convex pentagon whose edges are all of different lengths has the relation $A + B + C = 360°$, then we judge that it belongs to type 1, and see that it can generate a non-edge-to-edge tiling, but cannot generate an edge-to-edge tiling). In this study, we have not considered in detail what kind of edge-to-edge tiling an EE convex pentagonal tile can generate. Although a convex pentagon judged to be type 1 or type 2 may generate an edge-to-edge tiling other than the representative tilings of each type (see Appendix A.1), it is not a new type (i.e., according to the present classification rule, it is only a convex pentagon belonging to type 1 or type 2).

2 Relations between EE Convex Pentagonal Tiles and Nodes

The terms 'vertices' and 'edges' are applicable to both polygons and tilings. However, in this paper, because we consider only edge-to-edge tilings, the vertices of the tilings are called nodes, as mentioned in Sect. 1. In this section, to avoid confusion, we use the terms *corners* and *sides* to refer to the vertices and edges, respectively, of a polygon.

2.1 Preparation

Let $D(r, M)$ be a closed circular disk of radius r, centered at any point M of the plane. Let us place $D(r, M)$ on a tiling, and let F_1 and F_2 denote the set of tiles contained in $D(r, M)$ and the set of tiles meeting the boundary of $D(r, M)$ but not contained in $D(r, M)$, respectively. In addition, let F_3 denote the set of tiles surrounded by those in F_2 but not belonging to F_2. The set $F_1 \cup F_2 \cup F_3$ of tiles is called the *patch* $A(r, M)$ of tiles generated by $D(r, M)$.

Let $\Im_p$ be an edge-to-edge tiling by a convex pentagonal tile. For a given tiling $\Im_p$, we denote by $p(r, M)$, $e(r, M)$, $n(r, M)$, and $n_j(r, M)$ the number of convex pentagonal tiles, edges, nodes, and j-valent nodes, respectively, in patch $A(r, M)$. In addition, $\liminf_{r\to\infty} \frac{n_j(r,M)}{p(r,M)}$, or $\liminf_{r\to\infty} \frac{n_j(r,M)}{n(r,M)}$, is called the lower *density* of j-valent nodes in $\Im_p$.

A tiling $\Im_p$ is normal in the sense of Grünbaum and Shephard [3]. Consider a disk of radius U that can contain at most one convex pentagonal tile. Let us now perform the same construction starting with the disk $D(r + U, M)$ instead of $D(r, M)$ on $\Im_p$. That is, the number of convex pentagonal tiles in $A(r + U, M)$ is $p(r + U, M)$. Form Normality Lemma (Statement 3.2.2) in [3], as $r \to \infty$,

$$\frac{p(r + U, M) - p(r, M)}{p(r, M)} \to 0 \quad \text{and} \quad \frac{p(r + U, M)}{p(r, M)} \to 1. \tag{1}$$

Two tiles are called *adjacent* if they have an edge in common, and then each is called an *adjacent* of the other. From the definition of balanced tiling (see Section 3.3 in [3]) and Statement 3.3.5 in [3] ("a normal tiling in which every tile has the same number of adjacents is balanced"), $\Im_p$ is balanced, and $\lim_{r\to\infty}\frac{n(r,M)}{p(r,M)}$ and $\lim_{r\to\infty}\frac{e(r,M)}{p(r,M)}$ exist. For the patch $A(r, M)$, $n(r, M) = \sum_{j\geq 2} n_j(r, M)$ holds. We note that $n_2(r, M)$ is the number of 2-valent nodes (i.e., nodes with exactly two tile corners in $A(r, M)$) with a defective construction (parts that are not nodes from the viewpoint of tiling in $A(r, M)$ but are nodes of $\Im_p$) on the boundary of $A(r, M)$. From similar considerations about Normality Lemma in [3], we deduce, as $r \to \infty$,

$$\frac{n_2(r, M)}{p(r, M)} \to 0. \tag{2}$$

Then, from Euler's Theorem for Tilings (Statement 3.3.3) in [3], as $r \to \infty$,

$$\frac{n(r, M)}{p(r, M)} = \frac{\sum_{j\geq 2} n_j(r, M)}{p(r, M)} = \frac{n_2(r, M)}{p(r, M)} + \frac{n_3(r, M)}{p(r, M)} + \frac{\sum_{k\geq 4} n_k(r, M)}{p(r, M)} \to \frac{3}{2}. \tag{3}$$

On the other hand, the total number of corners of $p(r, M)$ pentagons in $A(r, M)$ and of $p(r + U, M)$ pentagons in $A(r + U, M)$ on $\Im_p$ are $5p(r, M)$ and $5p(r + U, M)$, respectively. As the total number of corners belonging to nodes of each valence in $A(r, M)$ on $\Im_p$ is $\sum_{j\geq 2} j \cdot n_j(r, M)$, we have

$$5p(r, M) \leq \sum_{j\geq 2} j \cdot n_j(r, M) < 5p(r + U, M)$$

or

$$5 \leq \frac{\sum_{j\geq 2} j \cdot n_j(r, M)}{p(r, M)} < \frac{5p(r + U, M)}{p(r, M)}.$$

Therefore, from (1), as $r \to \infty$,

$$\frac{\sum_{j\geq 2} j \cdot n_j(r, M)}{p(r, M)} = \frac{2n_2(r, M)}{p(r, M)} + \frac{3n_3(r, M)}{p(r, M)} + \frac{\sum_{k\geq 4} k \cdot n_k(r, M)}{p(r, M)} \to 5. \tag{4}$$

From (2), (3), and (4), as $r \to \infty$, we obtain

$$\frac{\frac{\sum_{j\geq 2} j\cdot n_j(r,M)}{p(r,M)}}{\frac{n(r,M)}{p(r,M)}} = \frac{\sum_{j\geq 2} j \cdot n_j(r, M)}{n(r, M)} \to \frac{10}{3}. \tag{5}$$

The second term in (5) can be interpreted as the ratio of the total number of corners belonging to nodes of each valence in $A(r, M)$ to the total number of nodes in $A(r, M)$ as $r \to \infty$, i.e., it implies the *average* valence taken over all the nodes. Thus, the average valence of nodes in $\Im_p$ is $\frac{10}{3} = 3.\dot{3}$. From (2) and this property, we see that $\Im_p$ must have nodes of both valence 3 and of valence $k \geq 4$.

Then, multiplying equation (3) by 3,

$$3\frac{n_2(r, M)}{p(r, M)} + 3\frac{n_3(r, M)}{p(r, M)} + 3\frac{\sum_{k\geq 4} n_k(r, M)}{p(r, M)} \to \frac{9}{2}. \tag{6}$$

From (4) and (6), as $r \to \infty$, we obtain

$$\frac{\sum_{k\geq 4}(k-3)\cdot n_k(r, M)}{p(r, M)} - \frac{n_2(r, M)}{p(r, M)} \to \frac{1}{2}. \tag{7}$$

2.2 Proof of Lemmas 1 and 2

Let us consider nodes of valence 3 and 4 separately from nodes of other valences. Equations (3) and (7) can be expressed as follows:

$$\frac{n_2(r, M)}{p(r, M)} + \frac{n_3(r, M)}{p(r, M)} + \frac{n_4(r, M)}{p(r, M)} + \frac{\sum_{k\geq 5} n_k(r, M)}{p(r, M)} \to \frac{3}{2} \tag{8}$$

and

$$\frac{n_4(r, M)}{p(r, M)} + \frac{\sum_{k\geq 5}(k-3)\cdot n_k(r, M)}{p(r, M)} - \frac{n_2(r, M)}{p(r, M)} \to \frac{1}{2}. \tag{9}$$

From (8) and (9), as $r \to \infty$, we obtain

$$\frac{n_3(r, M)}{p(r, M)} - \frac{\sum_{k\geq 5}(k-4)\cdot n_k(r, M)}{p(r, M)} + \frac{2n_2(r, M)}{p(r, M)} \to 1. \tag{10}$$

If $\Im_p$ satisfies $\frac{n_k(r,M)}{p(r,M)} \to 0$ for $k \geq 5$ as $r \to \infty$, then we deduce from (2), (9), and (10) that

$$\frac{n_3(r, M)}{p(r, M)} \to 1 \quad \text{and} \quad \frac{n_4(r, M)}{p(r, M)} \to \frac{1}{2} \tag{11}$$

as $r \to \infty$. On the other hand, if $\Im_p$ satisfies $\liminf_{r\to\infty} \frac{n_k(r,M)}{p(r,M)} > 0$ for some $k \geq 5$, then we see that $\liminf_{r\to\infty} \frac{n_3(r,M)}{p(r,M)}$ is greater than the value of 1 implied by (2) and (10). Thus, we find the property that $\liminf_{r\to\infty} \frac{n_3(r,M)}{p(r,M)}$, i.e., the lower density of 3-valent nodes in $\Im_p$, is greater than or equal to 1. From this property and (3), we have

$$\liminf_{r\to\infty} \frac{\frac{n_3(r,M)}{p(r,M)}}{\frac{n(r,M)}{p(r,M)}} = \liminf_{r\to\infty} \frac{n_3(r, M)}{n(r, M)} \geq \frac{2}{3}. \tag{12}$$

We now consider the ratio $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)}$. From (5) and (12), we obtain

$$\liminf_{r\to\infty} \frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} = \liminf_{r\to\infty} \frac{3n_3(r,M)}{\frac{10}{3}n(r,M)} \geq \frac{9}{10}\cdot\frac{2}{3} = \frac{3}{5}. \tag{13}$$

Because $3n_3(r,M)$ is the number of corners belonging to 3-valent nodes in $A(r,M)$, and $\sum_{j\geq 2} j\cdot n_j(r,M)$ is the total number of corners in $A(r,M)$, provided that the radius r is sufficiently large, the ratio $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)}$ can be interpreted as the ratio of the total number of corners belonging to 3-valent nodes to the total number of corners in $\Im_p$. From (13), the total number of corners in $\Im_p$ belonging to 3-valent nodes is at least 60 % of the total number of corners. If convex pentagonal tiles in $\Im_p$ have at most two corners that can simultaneously belong to 3-valent nodes, then $\limsup_{r\to\infty} \frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} < \frac{3}{5}$, which is a contradiction. Therefore, convex pentagonal tiles in $\Im_p$ must have the property in which at least three corners can belong to 3-valent nodes simultaneously. Note that it is not necessary that all convex pentagonal tiles in $\Im_p$ are in such a state as three or more corners are belonging to 3-valent nodes simultaneously. In fact, there are periodic edge-to-edge tilings by convex pentagonal tiles with two corners belonging to 3-valent nodes (see Fig. 4 in [10]). That is, equation (13) implies Bagina's Proposition under the restriction that the convex pentagonal tiling is $\Im_p$.

From (2), (3), (11), and (13), as $r\to\infty$, $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} \to \frac{3}{5}$ (i.e., the total number of corners belonging to 3-valent nodes is equal to 60 % of the total number of corners) iff $\Im_p$ satisfies the relations $\frac{n_3(r,M)}{p(r,M)} \to 1$, $\frac{n_4(r,M)}{p(r,M)} \to \frac{1}{2}$, and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or 4. Therefore, if $\Im_p$ satisfies the relations $\liminf_{r\to\infty} \frac{n_k(r,M)}{p(r,M)} > 0$ for some $k \geq 5$, then $\liminf_{r\to\infty} \frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} > \frac{3}{5}$. Thus, we obtain Lemma 1.

If $\Im_p$ satisfies the relations $\frac{n_4(r,M)}{p(r,M)} \to 0$, $\frac{n_5(r,M)}{p(r,M)} \to 0$ as $r\to\infty$, and $\liminf_{r\to\infty} \frac{n_k(r,M)}{p(r,M)} > 0$ for some $k \geq 6$, then from (3) and (7) we obtain

$$\frac{n_3(r,M)}{p(r,M)} - \frac{1}{3}\frac{\sum_{k\geq 7}(k-6)\cdot n_k(r,M)}{p(r,M)} + \frac{4n_2(r,M)}{3p(r,M)} \to \frac{4}{3}. \tag{14}$$

From (2), (3), (4), and (14), as $r\to\infty$, $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} \to \frac{4}{5}$ (i.e., the total number of corners belonging to 3-valent nodes is equal to 80 % of the total number of corners) iff $\Im_p$ satisfies the relations $\frac{n_3(r,M)}{p(r,M)} \to \frac{4}{3}$, $\frac{n_6(r,M)}{p(r,M)} \to \frac{1}{6}$, and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or 6. On the other hand, as $r\to\infty$, if $\Im_p$ satisfies $\liminf_{r\to\infty} \frac{n_k(r,M)}{p(r,M)} > 0$ for some $k \geq 7$, then we can deduce that $\liminf_{r\to\infty} \frac{n_3(r,M)}{p(r,M)} > \frac{4}{3}$ and $\liminf_{r\to\infty} \frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} > \frac{4}{5}$ from (2) and (14). Therefore, if $\Im_p$ satisfies the relations $\frac{n_4(r,M)}{p(r,M)} \to 0$, $\frac{n_5(r,M)}{p(r,M)} \to 0$ as $r\to\infty$, and $\liminf_{r\to\infty} \frac{n_k(r,M)}{p(r,M)} > 0$ for some $k \geq 7$, then a convex pentagonal tile needs to have five corners belonging to 3-valent nodes simultaneously. Thus, we obtain Lemma 2. □

2.3 Remarks

From Bagina's Proposition, and Lemmas 1 and 2, there will be EE convex pentagonal tiles with exactly three corners that can simultaneously belong to 3-valent nodes, with exactly four corners that can simultaneously belong to 3-valent nodes, and with exactly five (i.e., all) corners that can simultaneously belong to 3-valent nodes.

The representative tilings of types 4, 6, 7, 8, and 9, and the representative edge-to-edge tilings of types 1 and 2 (see Appendix A.1) are the edge-to-edge tilings satisfying $\frac{n_3(r,M)}{p(r,M)} \to 1$, $\frac{n_4(r,M)}{p(r,M)} \to \frac{1}{2}$, and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or 4 (i.e., $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} \to \frac{3}{5}$) as $r \to \infty$. The representative tiling of type 5 (see Appendix A.1) is the edge-to-edge tiling satisfying $\frac{n_3(r,M)}{p(r,M)} \to \frac{4}{3}$, $\frac{n_6(r,M)}{p(r,M)} \to \frac{1}{6}$, and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or 6 (i.e., $\frac{3n_3(r,M)}{\sum_{j\geq 2} j\cdot n_j(r,M)} \to \frac{4}{5}$) as $r \to \infty$.

Let $\Im_p(3, k_0)$ be an edge-to-edge tiling by a convex pentagonal tile that satisfies $\frac{n_3(r,M)}{p(r,M)} \to \frac{3}{2} - \frac{1}{2(k_0-3)}$, $\frac{n_{k_0}(r,M)}{p(r,M)} \to \frac{1}{2(k_0-3)}$ $(k_0 \geq 4)$, and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or k_0 as $r \to \infty$. Note that, from (2), (3) and (7), we can obtain the densities of 3- and k_0-valent nodes for the case $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or k_0 as $r \to \infty$. Note that $\Im_p(3, k_0)$ is not necessarily $n_j(r, M) = 0$ for $j \neq 3$ or k_0. That is, for $\Im_p(3, k_0)$, the density of j-valent nodes for $j \neq 3$ or k_0 tends to zero as $r \to \infty$, but it is not guaranteed that the number of j-valent nodes for $j \neq 3$ or k_0 is zero. However, each EE convex pentagonal tile belonging to types 4, 6, 7, 8, or 9, and each EE convex pentagonal tile that can generate representative edge-to-edge tilings of types 1 or 2 (i.e., edge-to-edge tilings with 3-valent nodes with size 3 and 4-valent nodes with size 2 [7]) can generate $\Im_p(3, 4)$ of $n_j(r, M) = 0$ for $j \neq 3$ or 4, and EE convex pentagonal tile belonging to type 5 can generate $\Im_p(3, 6)$ of $n_j(r, M) = 0$ for $j \neq 3$ or 6, as they can generate the periodic tilings (see Appendix A.1).

We now state the following lemma.

Lemma 3 *In a tiling $\Im_p(3, k_0)$, there exists an arbitrarily large region of edge-to-edge tiling with only 3- and k_0-valent nodes containing a disk of any large radius.*

Proof of lemma 3. The tiling $\Im_p(3, k_0)$ is divided into squares of side-length L $(0 << L < r)$. If all the squares of side-length L contain at least one j-valent node for $j \neq 3$ or k_0, then we deduce

$$\liminf_{r\to\infty} \left(\frac{\sum_{k\geq 4} n_k(r, M)}{p(r, M)} - \frac{n_{k_0}(r, M)}{p(r, M)} \right) > C_1 > 0, \tag{15}$$

where C_1 is an arbitrary constant. However, relation (15) is a contradiction because, from $\frac{n_{k_0}(r,M)}{p(r,M)} \to \frac{1}{2(k_0-3)}$ and $\frac{n_j(r,M)}{p(r,M)} \to 0$ for $j \neq 3$ or k_0 as $r \to \infty$, $\frac{\sum_{k\geq 4} n_k(r,M)}{p(r,M)} -$

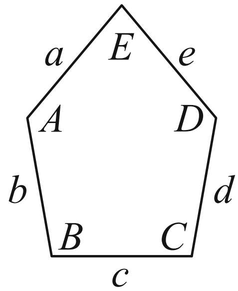

Fig. 1 Nomenclature for vertices and edges of convex pentagon

$\frac{n_{k_0}(r,M)}{p(r,M)}$ must tend to zero as $r \to \infty$. Therefore, on $\Im_p(3, k_0)$, there exists a square that does not contain a j-valent node for $j \neq 3$ or k_0. Thus, we obtain Lemma 3. □

Using Bagina's Proposition, Lemmas 1 and 3, and the properties of $\Im_p(3, 4)$, we obtain the following corollary.

Corollary 1 *If an EE convex pentagonal tile has exactly three corners that can simultaneously belong to* 3*-valent nodes, the tile can generate an arbitrarily large region of edge-to-edge tiling with only* 3*- and* 4*-valent nodes.*

3 The 42 Uncertain Cases of Whether a Convex Pentagon can Generate an Edge-to-Edge Tiling

As shown in Fig. 1, let us label the vertices (angles) of the convex pentagon A, B, C, D, and E, and its edges a, b, c, d, and e in a fixed manner.

The notation for describing the 34 cases listed in Table 5 in [7] follows the present classification rule. Because of this rule, for example, the case where v_1 is *AAB*-1 and v_2 is *EEA*-2 (conditions: $2A + B = 2E + A = 360°, a = b = c$)[2] contains convex pentagons that satisfy "$2A + B = 2E + A = 360°, e \neq a = b = c \neq d \neq e$," "$2A + B = 2E + A = 360°, a = b = c = d \neq e$," etc. For all cases, the edges with equal and unequal lengths are clarified, i.e., the 34 cases are refined by imposing extra edge conditions. Therefore, the overlaps inside each of the 11 cases among the 34 cases with three different edge lengths (i.e., with cyclic-edge-type [11223] or [11123][3]) are separated. We show two examples below.

[2] We call the multiset of vertices of polygons a *spot* if the sum of the interior angles at the vertices in the multiset is equal to 360°. If a convex pentagon has the relation $2A + B = 360°$, we say that it has the 3-valent spot $\{A, A, B\}$. Then, the spot of a convex pentagon that is supposed to become a node of an edge-to-edge tiling is called the tentative node. For example, *AAB*-1 is a tentative 3-valent node $\{A, A, B\}$. However, the tentative 3-valent node $\{A, A, B\}$ has two sub-cases, as shown in Fig. 2 and Table 2 in [7]. Therefore, *AAB*-1 expresses a sub-case of the tentative 3-valent node $\{A, A, B\}$ requesting the edge condition $a = b = c$. See Figs. 5–8 and Tables 1–4 in [7] for the sub-cases of each tentative 3-valent node. On the other hand, *EEA*-2 expresses a sub-case of the tentative 3-valent node $\{E, E, A\}$ requesting the edge condition $a = b$. Therefore, the convex pentagon of the case where v_1 is *AAB*-1 and v_2 is *EEA*-2 satisfies $2A + B = 2E + A = 360°$, $a = b = c$, and can form tentative nodes of *AAB*-1 and *EEA*-2 at least. Refer to [7] for details.

[3] Pentagons can be classified by the number of equal edges and their positions. In the following, the edges are designated symbolically as 1, 2,…in cyclic (anticlockwise) order, with the same symbol for equal edges.

Example 3.1 Case where v_1 is *AAB*-1 and v_2 is *EEA*-2 (Conditions: $2A + B = 2E + A = 360°, a = b = c$).

As *AAB*-1 implies $2A + B = 360°, a = b = c$ and *EEA*-2 implies $2E + A = 360°, a = b$, the convex pentagon satisfies $2A + B = 2E + A = 360°, a = b = c$ (see Table 5 in [7]). Then, for the edge conditions, this case contains "$e \neq a = b = c \neq d \neq e$," "$a = b = c = d \neq e$," "$a = b = c = e \neq d$," "$a = b = c \neq d = e$," and "$a = b = c = d = e$." Therefore, we consider the following cases.

If $2A + B = 2E + A = 360°, e \neq a = b = c \neq d \neq e$ is satisfied, then the convex pentagon is still an uncertain case and has two degrees of freedom (*DOFs*), besides its size (see Appendix A.2).
If $2A + B = 2E + A = 360°, a = b = c = d \neq e$ is satisfied, then it is not geometrically possible for the pentagon to be convex. This is because, with the range of values of A and C that are admitted as interior angles of a convex pentagon, the angle B is always greater than 180° in order to obtain this geometric property (this was determined using the Maple mathematical software) (see Appendix A.2).
If $2A + B = 2E + A = 360°, a = b = c = e \neq d$ is satisfied, then the convex pentagon is contained in the uncertain case where v_1 is *AAB*-1 and v_2 is *EEA*-1.
If $2A + B = 2E + A = 360°, a = b = c \neq d = e$ is satisfied, then the convex pentagon is still an uncertain case and has one *DOF*, besides its size (see Appendix A.2).
If $2A + B = 2E + A = 360°, a = b = c = d = e$ is satisfied (and generates a tiling), then the convex pentagon belongs to the set of known types (i.e., types 1, 2, or 7) from Theorem in [5].

Example 3.2 Case where v_1 is *AAB*-2 and v_2 is *CDA*-1 (Conditions: $2A + B = C + D + A = 360°, a = e, b = c$).

If $2A + B = C + D + A = 360°, d \neq a = e \neq b = c \neq d$ is satisfied, then the convex pentagon is still an uncertain case and has two *DOF*s, besides its size.
If $2A + B = C + D + A = 360°, a = e = b = c \neq d$ is satisfied, then the convex pentagon belongs to type 2.
If $2A + B = C + D + A = 360°, a = d = e \neq b = c$ is satisfied, then the convex pentagon is contained in the uncertain case where v_1 is *AAB*-2 and v_2 is *CDA*-3.
If $2A + B = C + D + A = 360°, a = e \neq b = c = d$ is satisfied, then the convex pentagon is contained in the uncertain case where v_1 is *AAB*-2 and v_2 is *CDA*-5.
If $2A + B = C + D + A = 360°, a = b = c = d = e$ is satisfied, then the convex pentagon belongs to known types from Theorem in [5].

By examining each overlapping case analogously to the above, we can classify them into four categories: (i) the convex pentagon cannot exist; (ii) the convex pentagon belongs to a known type (if it exists); (iii) the convex pentagon is contained in the known uncertain cases, except for the 11 cases mentioned previ-

Footnote 3 continued
Mirror reflections are excluded. Beginning with equilateral pentagons, followed by those with four equal edges, etc., they are classified into 12 cyclic-edge-types: [11111], [11112], [11122], [11212], [11123], [11213], [11223], [11232], [12123], [11234], [12134], [12345]. Refer to [7] for details.

ously, of Table 5 in [7]; and (iv) uncertain cases that cannot be included in the known uncertain cases. Combining the cases in category (iv) and those in Table 5 in [7] with cyclic-edge-types of [11112], [11122], or [11212], we find that 42 uncertain cases remain. These cases are listed in Table 1. The final column of Table 1 shows the *DOF*s, not including the size, of each convex pentagon. These are the cases where it is uncertain whether a convex pentagon can generate an edge-to-edge tiling, and it is these cases that we must search. The above considerations have consequently increased the total number of uncertain cases compared to those listed in [7]. However, each of the edge conditions of the convex pentagons in Table 1 is fixed.

4 Property of EE Convex Pentagonal Tiles Whose Five Vertices Belong to 3-Valent Nodes

In this section, we consider the property relevant to Lemma 2, i.e., if the five vertices of each pentagon in the 42 uncertain cases simultaneously belong to 3-valent nodes, what are the properties of the convex pentagon?

Each convex pentagon in the 42 uncertain cases can have three or more 3-valent nodes for sub-cases v_1 and v_2. Thus, these pentagons have the properties of Bagina's Proposition. On the other hand, because each of the 42 convex pentagons has 1, 2, or 3 *DOF*s, they will be able to form other sub-cases of 3-valent nodes besides v_1 and v_2. As each of the edge conditions of the 42 convex pentagons is fixed, the sub-cases of 3-valent nodes that can be formed can be obtained from the edge conditions and Tables 1–4 in [7]. (However, from the geometrical properties of each pentagon, not all sub-cases can actually be formed.) Note that, as mentioned in [7] and Sect. 1, it is not necessary to add the 3-valent nodes $\{A, B, C\}$, $\{B, C, D\}$, $\{C, D, E\}$, $\{D, E, A\}$, and $\{E, A, B\}$ to this investigation method. Using the above method, we consider the properties of the 42 types of convex pentagons, each of which has five vertices simultaneously belonging to 3-valent nodes. We give three examples below.

Example 4.1 Case where v_1 is *AAB*-1, v_2 is *DDA*-2, and the cyclic-edge-type is [11112] (Conditions: $2A + B = 2D + A = 360°$, $a = b = c = e \neq d$).

Because vertices C and E are not contained in v_1 and v_2, it is necessary to assign them to 3-valent nodes. From the edge conditions (and Tables 1–4 in [7]), the sub-cases of the 3-valent nodes that can admit C are *CDA*-1, *CDA*-2, *CCB*-1, *CCA*-1, and *CCE*-1.

If C forms *CDA*-1 or *CDA*-2, then the convex pentagon belongs to type 2, as $C + D + A = 360°, a = c, b = e$ (see Appendix A.1 for the types of convex pentagonal tiles).

If C forms *CCB*-1, then the convex pentagon belongs to type 1, as $A+B+C = 360°$ because $2A + B = 2C + B = 360°$, i.e., $A = C$.

If C forms *CCA*-1, then the convex pentagon belongs to type 2, as $C + D + A = 360°, a = c, b = e$ because $2C + A = 2D + A = 360°$, i.e., $C = D$.

Table 1 The 42 uncertain cases of whether a convex pentagon can generate an edge-to-edge tiling

Type of set	Sub-case of v_1	Sub-case of v_2	Conditions of convex pentagon		Cyclic-edge-type	*DOF*s
			Angle conditions	Edge conditions		
G_1	*ABD*-1	*ABD*-1	$A+B+D=360^{\circ}$	$b \neq a=e \neq c=d \neq b$	[11223]	3
	ABD-1	*ABD*-3	$A+B+D=360^{\circ}$	$a=e \neq b=c=d$	[11122]	2
G_2	*AAB*-1	*BBC*-1	$2A+B=2B+C=360^{\circ}$	$a=b=c=d \neq e$	[11112]	1
	AAB-1	*BBE*-1	$2A+B=2B+E=360^{\circ}$	$a=b=c=e \neq d$	[11112]	1
	AAB-1	*CCD*-2	$2A+B=2C+D=360^{\circ}$	$a=b=c \neq d=e$	[11122]	1
	AAB-1	*DDA*-1	$2A+B=2D+A=360^{\circ}$	$a=b=c=d \neq e$	[11112]	1
	AAB-1	*DDA*-2	$2A+B=2D+A=360^{\circ}$	$a=b=c=e \neq d$	[11112]	1
	AAB-1	*DDE*-2	$2A+B=2D+E=360^{\circ}$	$a=b=c=e \neq d$	[11112]	1
	AAB-1	*EEA*-1	$2A+B=2E+A=360^{\circ}$	$a=b=c=e \neq d$	[11112]	1
	AAB-1	*EEA*-2	$2A+B=2E+A=360^{\circ}$	$e \neq a=b=c \neq d \neq e$	[11123]	2
	AAB-1	*EEA*-2	$2A+B=2E+A=360^{\circ}$	$a=b=c \neq d=e$	[11122]	1
	AAB-2	*AAD*-1	$2A+B=360^{\circ}, B=D$	$a=d=e \neq b=c$	[11122]	1
	AAB-2	*BBD*-1	$2A+B=2B+D=360^{\circ}$	$b=c=d=e \neq a$	[11112]	1
	AAB-2	*CCD*-2	$2A+B=2C+D=360^{\circ}$	$a \neq b=c \neq d=e \neq a$	[11223]	2
	AAB-2	*CCD*-2	$2A+B=2C+D=360^{\circ}$	$a=d=e \neq b=c$	[11122]	1
	AAB-2	*DDB*-1	$A+B+D=360^{\circ}, A=D$	$a \neq b=c=d \neq e \neq a$	[11123]	2
	AAB-2	*DDC*-1	$2A+B=2D+C=360^{\circ}$	$a \neq b=c=d \neq e \neq a$	[11123]	2
	AAB-2	*DDC*-1	$2A+B=2D+C=360^{\circ}$	$a=e \neq b=c=d$	[11122]	1
	AAC-1	*BBA*-1	$2A+C=2B+A=360^{\circ}$	$a=b=c=d \neq e$	[11112]	1
	AAC-1	*BBD*-2	$2A+C=2B+D=360^{\circ}$	$a=c=d=e \neq b$	[11112]	1
	AAC-1	*DDB*-1	$2A+C=2D+B=360^{\circ}$	$a=b=c=d \neq e$	[11112]	1

Table 1 continued

Type of set	Sub-case of v_1	Sub-case of v_2	Conditions of convex pentagon		Cyclic-edge-type	*DOF*s
			Angle conditions	Edge conditions		
	AAC-2	*DDB*-1	$2A + C = 2D + B = 360^\circ$	$a \neq b = c = d \neq e \neq a$	[11123]	2
	AAC-2	*DDB*-1	$2A + C = 2D + B = 360^\circ$	$a = e \neq b = c = d$	[11122]	1
G_3	*AAB*-1	*EAC*-1	$2A + B = E + A + C = 360^\circ$	$a = b = c \neq d = e$	[11122]	1
	AAB-2	*ABD*-1	$A + B + D = 360^\circ, A = D$	$a = e \neq b = c = d$	[11122]	1
	AAB-2	*CDA*-1	$2A + B = C + D + A = 360^\circ$	$d \neq a = e \neq b = c \neq d$	[11223]	2
	AAB-2	*CDA*-3	$2A + B = C + D + A = 360^\circ$	$a = d = e \neq b = c$	[11122]	1
	AAB-2	*CDA*-5	$2A + B = C + D + A = 360^\circ$	$a = e \neq b = c = d$	[11122]	1
	AAB-2	*DEB*-7	$2A + B = D + E + B = 360^\circ$	$a = d \neq b = c = e$	[11212]	2
	AAB-2	*EAC*-1	$2A + B = E + A + C = 360^\circ$	$a \neq b = c \neq d = e \neq a$	[11223]	2
G_4	*ABD*-1	*AAA*	$A = 120^\circ, A + B + D = 360^\circ$	$a = b = e \neq c = d$	[11122]	1
G_5	*AAB*-1	*DDD*	$D = 120^\circ, 2A + B = 360^\circ$	$a = b = c \neq d = e$	[11122]	1
	AAB-1	*EEE*	$E = 120^\circ, 2A + B = 360^\circ$	$a = b = c = e \neq d$	[11112]	1
	AAB-2	*DDD*	$D = 120^\circ, 2A + B = 360^\circ$	$a \neq b = c \neq d = e \neq a$	[11223]	2
	AAB-2	*DDD*	$D = 120^\circ, 2A + B = 360^\circ$	$b = c = d = e \neq a$	[11112]	1
	AAB-2	*DDD*	$D = 120^\circ, 2A + B = 360^\circ$	$a = d = e \neq b = c$	[11122]	1
	AAB-2	*EEE*	$E = 120^\circ, 2A + B = 360^\circ$	$d \neq a = e \neq b = c \neq d$	[11223]	2
	AAB-2	*EEE*	$E = 120^\circ, 2A + B = 360^\circ$	$a = e \neq b = c = d$	[11122]	1
	AAC-1	*BBB*	$B = 120^\circ, 2A + C = 360^\circ$	$a = b = c = d \neq e$	[11112]	1
	AAC-2	*BBB*	$B = 120^\circ, 2A + C = 360^\circ$	$a \neq b = c = d \neq e \neq a$	[11123]	2
	AAC-2	*BBB*	$B = 120^\circ, 2A + C = 360^\circ$	$a = e \neq b = c = d$	[11122]	1
	AAC-2	*EEE*	$E = 120^\circ, 2A + C = 360^\circ$	$a = e \neq b = c = d$	[11122]	1

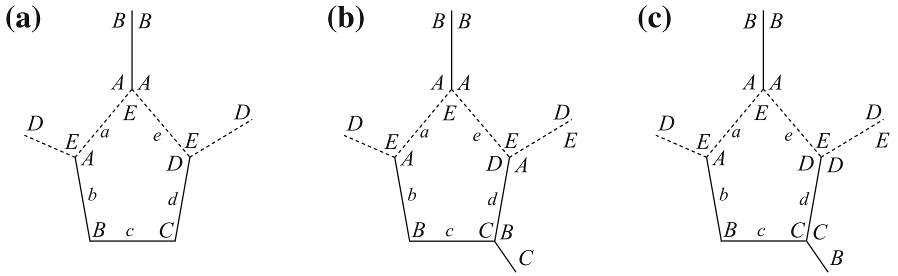

Fig. 2 Vertex E forms AAE-2 and vertex D forms the 3-valent node $\{D, E, A\}$ or $\{D, D, E\}$ under the case where v_1 is AAB-2, v_2 is ABD-1, and the cyclic-edge-type is [11122]. These figures are a simplification and do not aim to reproduce the pentagonal forms accurately

If C forms CCE-1, then the convex pentagon belongs to type 7, as $2A + B = 2C + E = 360°, a = b = c = e$. (If we change the labels from $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$, $D \rightarrow E$, and $E \rightarrow A$, then the expression of these conditions is equivalent to that in Fig. 4 in Appendix A.1).

Example 4.2 Case where v_1 is AAB-2, v_2 is ABD-2, and the cyclic-edge-type is [11122] (Conditions: $A + B + D = 360°$, $A = D, a = e \neq b = c = d$).

Because vertices C and E are not contained in v_1 and v_2, it is necessary to assign them to 3-valent nodes. From the edge conditions, the sub-cases of the 3-valent nodes that can admit E are AAE-1, DDE-2, and EEE.

If E forms AAE-1 (see Fig. 2a) and five vertices belong to 3-valent nodes, then D forms the 3-valent node $\{D, E, A\}$ or $\{D, D, E\}$. (i) If E forms AAE-1 and$D$ forms the 3-valent node $\{D, E, A\}$ (see Fig. 2b), then the convex pentagon belongs to type 1, as $D + E + A = 360°$. (ii) If E forms AAE-1 and D forms the 3-valent node $\{D, D, E\}$ (see Fig. 2c), then the convex pentagon belongs to type 1, as $D + E + A = 360°$ because $2A + E = 2D + E = 360°$, i.e., $A = D$.

If E forms DDE-2 (see Fig. 3a) and five vertices belong to 3-valent nodes, then D forms the 3-valent node $\{D, E, A\}$ or $\{D, D, E\}$. (i) If E forms DDE-2 and D forms the 3-valent node $\{D, E, A\}$ (see Fig. 3b), then the convex pentagon belongs to type 1, as $D + E + A = 360°$. (ii) If E forms DDE-2 and D forms the 3-valent node $\{D, D, E\}$ (see Fig. 3c), then A forms either $\{D, E, A\}$ or $\{A, A, E\}$. (ii-1) If E forms DDE-2,D forms $\{D, D, E\}$, and A forms the 3-valent node $\{D, E, A\}$ (see Fig. 3d), then the convex pentagon belongs to type 1, as $D + E + A = 360°$. (ii-2) If E forms DDE-2, D forms $\{D, D, E\}$, and A forms the 3-valent node $\{A, A, E\}$ (see Fig. 3e), then the convex pentagon belongs to type 1, as $D + E + A = 360°$ because $2A + E = 2D + E = 360°$, i.e., $A = D$.

If E forms EEE, then the convex pentagon belongs to type 5, as $E = 120°, C = 60°, a = e, c = d$ because $A + B + D = 3E = 360°$, i.e., $E = 120°$ and $C + E = 180°$.

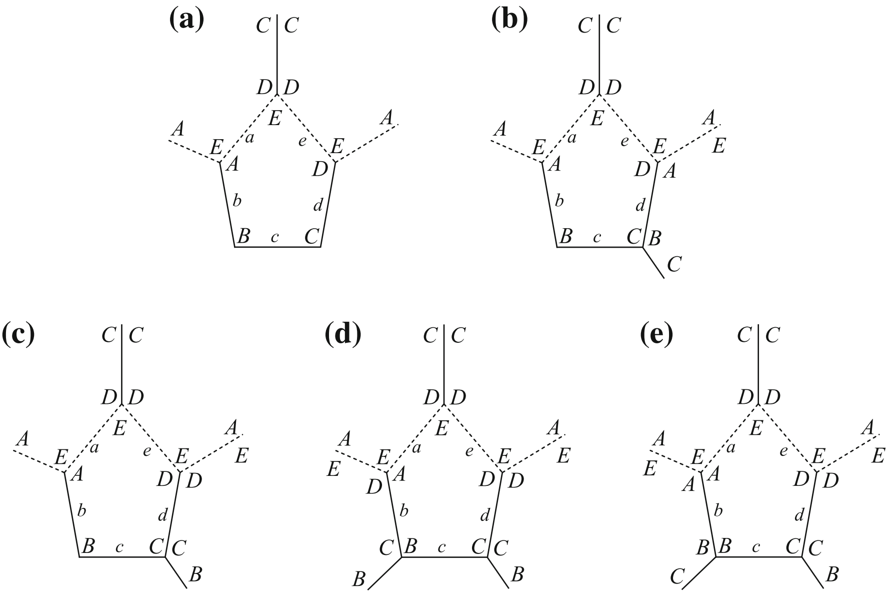

Fig. 3 Vertex E forms DDE-2 and vertex D forms the 3-valent node $\{D, E, A\}$ or $\{D, D, E\}$ under the case where v_1 is AAB-2, v_2 is ABD-1, and the cyclic-edge-type is [11122]. These figures are a simplification and do not aim to reproduce the pentagonal forms accurately

Example 4.3 Case where v_1 is AAC-2, v_2 is BBB, and the cyclic-edge-type is [11123] (Conditions: $B = 120°$, $2A + C = 360°$, $a \neq b = c = d \neq e \neq a$).

Because vertices D and E are not contained in v_1 and v_2, it is necessary to assign them to 3-valent nodes. However, from the edge conditions, there is no 3-valent node sub-case that can admit E. Thus, in this case, there is no convex pentagon with five 3-valent nodes.

By examining each of the 42 uncertain cases analogously to the above, we find that, if the convex pentagon with five 3-valent nodes at the same time is geometrically possible, it belongs to one of the known types of convex pentagon. On the other hand, from the result of [7], if convex pentagons that are not included in the 42 cases in Table 1 can generate an edge-to-edge tiling, they belong to one (or more) of types 1, 2, 4, 5, 6, 7, 8, or 9 shown in Fig. 4. Therefore, from these results and Lemma 2, we obtain the following corollary.

Corollary 2 *If a convex pentagon generates an edge-to-edge monohedral tiling with no nodes of valence* 4, 5, *or* 6 *(i.e., the valence of every node is either* 3 *or* $\geq$ 7*), then the convex pentagon belongs to one (or more) of types* 1, 2, 4, 5, 6, 7, 8, *or* 9 *shown in Fig.* 4.

5 Conclusions

We have identified the 42 cases whose edge conditions are fixed and where it is uncertain whether a convex pentagon can generate an edge-to-edge tiling (see Table 1).

This extension from the previously known 34 cases was achieved by the imposition of extra edge conditions. In our next work, we consider what types of EE convex pentagonal tiles exist in these uncertain cases. We first identify the convex pentagons with exactly three corners that can simultaneously belong to tentative 3-valent nodes in the 42 uncertain cases. Using Corollary 1, we then search the cases in which the convex pentagons can generate an edge-to-edge tiling using only 3- and 4-valent nodes. The properties shown in Lemmas 1 and 2 and Corollary 2 will be important in this investigation, because we need not consider the 7- or more valent nodes that are formed by each of the convex pentagons of the 42 uncertain cases. In other words, from the information of 3-, 4-, 5-, and 6-valent nodes that are formed by each of the convex pentagons of the 42 uncertain cases, we will be able to determine the types of EE convex pentagonal tiles that exist.

Acknowledgments The authors would like to thank Professor Shigeki AKIYAMA, University of Tsukuba, for his mathematical helpful comments and guidance, and Emeritus Professor Hiroshi MAEHARA, University of the Ryukyus, for his helpful comments. The comments of the referees also helped to clarify the article.

Appendix A.1

At present, known convex pentagonal tiles are classified into 14 essentially different types, as shown in Fig. 4. Note that the tilings in Fig. 4 are the representative tilings of each type, and the known convex pentagonal tiles must be able to generate a periodic tiling. In the present classification of convex pentagonal tiles, for example, a convex pentagonal tile of type 1 has a condition that the sum of three consecutive angles is equal to 360°, which is expressed as $A + B + C = 360°$ in Fig. 4. According to the present classification rule, a convex pentagonal tile exists that belongs to two or more types simultaneously. In addition, a convex pentagonal tile exists that can generate a tiling other than the representative tilings of each type (see Fig. 3 in [7]).

For the 14 types of convex pentagonal tiles in Fig. 4, the convex pentagons belonging to types 4, 5, 6, 7, 8, or 9 can generate an edge-to-edge tiling. On the other hand, a convex pentagonal tile belonging only to types 3, 10, 11, 12, 13, or 14 cannot generate an edge-to-edge tiling. Tilings of type 1 or type 2 are generally non-edge-to-edge, such as those shown in Fig. 4. However, in special cases, the convex pentagonal tiles that belong to type 1 or type 2 permit edge-to-edge tilings. For example, type 1 and type 2 should impose an additional condition “$a = d$” and “$c = d$,” respectively (see Fig. 5).

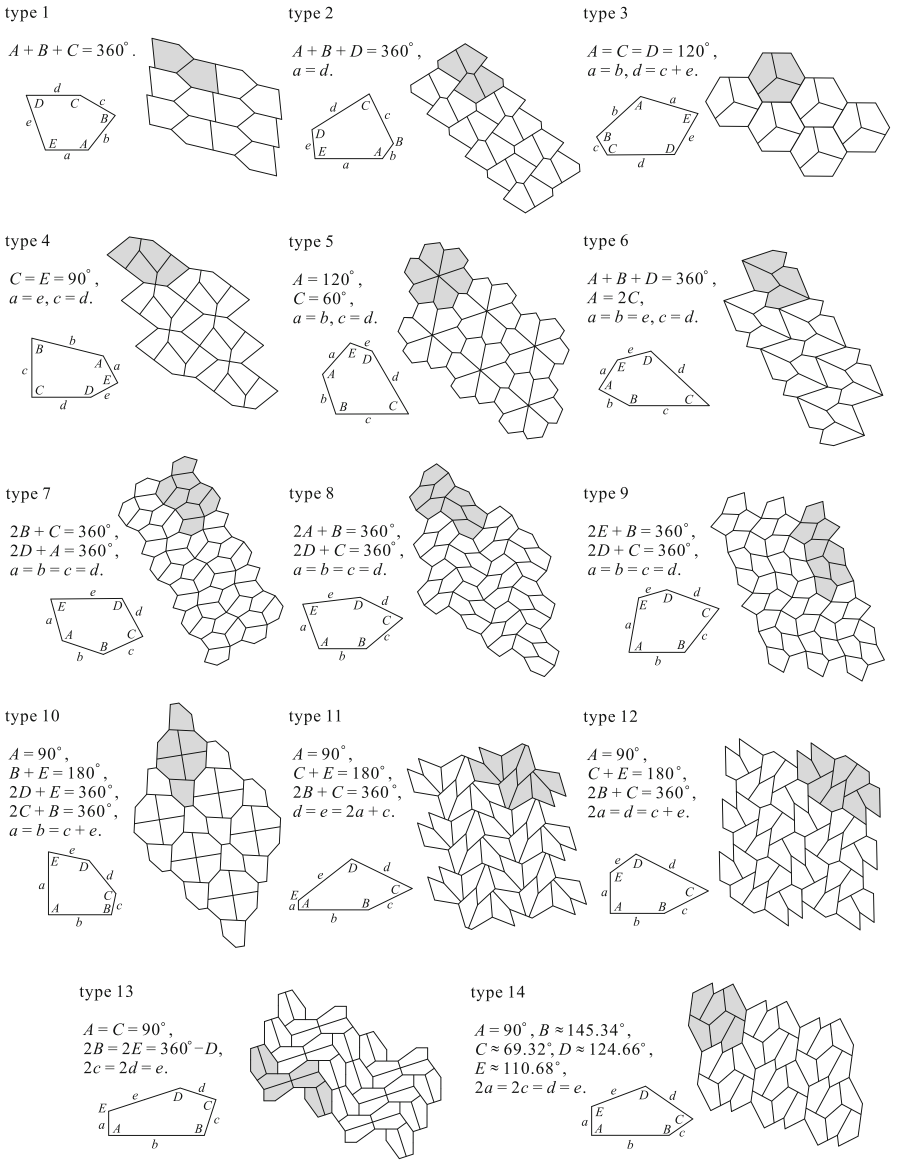

Fig. 4 Convex pentagonal tiles of 14 types. The tilings are the representative tilings of each type. The *pale gray* convex pentagons in each tiling indicate the fundamental region, which is the unit that can generate a periodic tiling by translation only

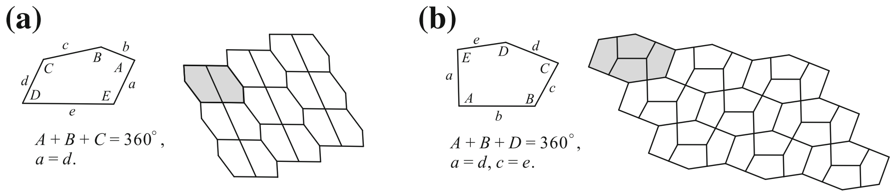

Fig. 5 Examples of edge-to-edge tilings by convex pentagonal tiles that belong to type 1 or type 2. The *pale gray* pentagons in each tiling indicate the fundamental region. **a** Convex pentagonal tiles that belong to type 1. **b** Convex pentagonal tiles that belong to type 2

Appendix A.2

Let us consider some of the cases in Example 3.1.

In the convex pentagon that satisfies $2A + B = 2E + A = 360°$, $e \neq a = b = c \neq d \neq e$, consider an isosceles triangle ABE (see Fig. 6a). When the base angles of ABE are denoted by α, the interior angles of the pentagon can be expressed as follows: $A = 180° - 2\alpha$, $B = 4\alpha$, $C = 270° - 3\alpha - D$, $E = 90° + \alpha$ where $0° < \alpha < 45°$ and $0° < D < 180°$. Therefore, this pentagon has two *DOF*s besides its size.

The convex pentagon that satisfies $2A + B = 2E + A = 360°$, $a = b = c = d \neq e$ is assumed to exist geometrically, and two isosceles triangles, ABE with base angles α and BCD with base angles β, are considered inside the convex pentagon (see Fig. 6b). We find $0° < \alpha < 45°$ and $0° < \beta < 90°$, because $0° < B = 4\alpha < 180°$ and $0° < C = 180° - 2\beta < 180°$. Considering the triangle BDE and the sine formula., β can be expressed as a function of α (the equation of β is omitted for brevity). Therefore, we have a pentagon with one *DOF* besides its size, and find that the value of β is outside the range $(0°, 180°)$ for $0° < \alpha < 45°$. Thus, this convex pentagon does not exist geometrically.

In the convex pentagon that satisfies $2A + B = 2E + A = 360°$, $a = b = c \neq d = e$, two isosceles triangles ABE and CDE are considered (see Fig. 6c). When the base angles of isosceles triangle ABE are denoted by α, the base angles of CDE can be written as $\gamma = \tan^{-1}\left((2\cos\alpha - \cos 3\alpha)/\sin 3\alpha\right)$ by considering triangle BDE and the sine formula. Then, the interior angles of the pentagon can be expressed as follows: $A = 180° - 2\alpha$, $B = 4\alpha$, $C = 90° + 2\gamma - 3\alpha$, $D = 180° - 2\gamma$, $E = 90° + \alpha$. Therefore, this pentagon has one *DOF* besides its size.

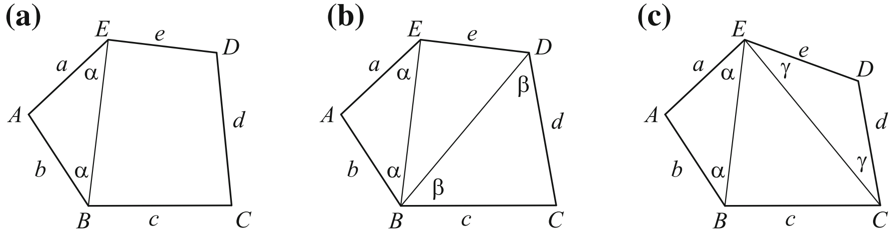

Fig. 6 *Triangles* in convex pentagons. Note that this figure does not aim to reproduce the pentagonal forms accurately

References

1. Bagina, O.: Tiling the plane with congruent equilateral convex pentagons. J. Comb. Theory Ser. A **105**, 221–232 (2004)
2. Gardner, M.: On Tessellating the Plane with Convex Polygon Tiles. Scientific American, pp. 112–117 (1975)
3. Grünbaum, B.; Shephard, G.C.: Tilings and Patterns. W.H. Freeman and Company, New York, pp.15–35 ((Chapter 1), pp. 113–134 (Chapter 3), pp. 471–487, pp. 492–497 and pp. 517–518 (Chapter 9) (1987))
4. Hallard, T.C.; Kenneth, J.F.; Richard, K.G.: Unsolved Problems in Geometry, Springer, New York, pp. 95–96 (C14) (1994)
5. Hirschhorn, M.D., Hunt, D.C.: Equilateral convex pentagons which tile the plane. J. Comb. Theory. Ser. A **39**, 1–18 (1985)
6. Kershner, R.B.: On paving the plane. Am. Math. Mon. **75**, 839–844 (1968)
7. Sugimoto, T.: Convex pentagons for edge-to-edge tiling, I. Forma, **27**, 93–103
8. Sugimoto, T.: Classification of convex pentagons that can generate edge-to-edge Monohedral tilings of the plane. Forma (accepted)
9. Sugimoto, T., Ogawa, T.: Properties of tilings by convex pentagons. Forma **21**, 113–128 (2006)
10. Sugimoto, T., Ogawa, T.: Properties of nodes in pentagonal tilings. Forma **24**, 117–121 (2009)
11. Wells, D.: The Penguin Dictionary of Curious and Interesting Geometry. Penguin Books, London, pp. 177–179 (1991)