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Note on Enomoto and Ota's Conjecture for Short Paths in Large Graphs

Martin Hall · Colton Magnant · Hua Wang

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Abstract With sufficient minimum degree sum, Enomoto and Ota conjectured that for any selected set of vertices, there exists a spanning collection of disjoint paths, each starting at one of the selected vertices and each having a prescribed length. Using the Regularity Lemma, we prove that this claim holds without the spanning assumption if the vertex set of the host graph is sufficiently large.

Keywords Disjoint paths · Degree sum · Regularity lemma

1 Introduction

Given a graph G, the *degree* of a vertex $d(v)$ is the number of edges in G that are incident to v. For a set of vertices $A \subseteq V(G)$, the degree of a vertex into A, $d_A(v)$ is the number of edges between v and vertices of A. Let $\sigma_2(G)$ denote the minimum sum of degrees of two nonadjacent vertices in G, a natural extension of the minimum degree of a graph, which is denoted as $\delta(G)$. Conditions on $\sigma_2(G)$ have been used to produce many extremal results in graph theory, the most well known of which is certainly Ore's Theorem [7] which states that a graph with $\sigma_2(G) \geq n$ is hamiltonian, meaning it contains a spanning cycle. All other standard terminology can be found in [1].

In 2000, Enomoto and Ota conjectured the following and proved the result for some special cases.

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M. Hall · C. Magnant (✉) · H. Wang
Department of Mathematical Sciences, Georgia Southern University,
Statesboro, GA 30460, USA
e-mail: dr.colton.magnant@gmail.com

Conjecture 1 (Enomoto, Ota [2]) Let G be a graph of order n, t be an integer with $t \leq n$, $\{u_1, u_2, \ldots, u_t\}$ be a set of vertices in $V(G)$ and $\{n_1, n_2, \ldots, n_t\}$ be a set of integers, each at least 1 with $\sum n_i = n$. If $\sigma_2(G) \geq n + t - 1$, then G contains a set of vertex disjoint paths $P_1, P_2, \ldots, P_t$ where P_i has u_i at one end and $|P_i| = n_i$.

The following asymptotic version of Conjecture 1 was proven in [5].

Theorem 1 (Magnant, Martin [5]) *Let $t \geq 2$ be an integer and let $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be fractions with $\sum \gamma_i = 1$ and let $\epsilon > 0$. Then if n is sufficiently large, G is a graph of order n satisfying $\sigma_2(G) \geq n + t - 1$, and $\{u_1, u_2, \ldots, u_t\}$ is any set of t vertices in G, then G contains a set of vertex disjoint paths $P_1, P_2, \ldots, P_t$ where P_i has u_i at one end, $|P_i - \gamma_i n| \leq \epsilon n$ and $\sum |P_i| = n$.*

In this work, we prove the following result, which produces a structure similar to that provided by Conjecture 1 but not spanning.

Theorem 2 *Let $t \geq 1$ be an integer, $\{n_1, n_2, \ldots, n_t\}$ be positive integers. Then there exists n_0 such that if G is a graph of order $n \geq n_0$ with $\sigma_2(G) \geq n + t - 2$, then for any choice of t vertices $\{u_1, u_2, \ldots, u_t\}$ in $V(G)$, there exists a set of vertex disjoint paths $P_1, P_2, \ldots, P_t$ in G where P_i starts at the vertex u_i and has order n_i for all $1 \leq i \leq t$. Furthermore, this degree condition is the best possible.*

Note that here n is much larger than $\sum n_i$. In fact, this result is almost a corollary of Theorem 1. The difference is that the degree sum assumption in Theorem 2 is slightly lower than that of Conjecture 1 or Theorem 1. We present the proof of Theorem 2 to outline a possible proof strategy for Conjecture 1 in the case when n is large using the regularity lemma.

Although Theorem 2 is likely not the best possible in terms of the restriction on the lengths of the paths (since these lengths are not allowed to even be a function of n), the degree condition is actually sharp by the following example. Let G be a graph containing a vertex v with degree $t - 1$ and let $G \setminus \{v\}$ be complete. Then $\sigma_2(G) = d(v) + (n - 2) = n + t - 3$ but if we select $N(v) \cup \{v\}$ to be the set $\{u_1, u_2, \ldots, u_t\}$, there is no way to construct a path starting at v that avoids $N(v)$.

Our proof makes use of the following extremely powerful tools. For the next statement, we need some definitions. The *density* of edges between two disjoint vertex sets A and B is defined to be

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For two sets of vertices A and B and a real number $\epsilon > 0$, the pair (A, B) is called *ϵ-regular* if, for any subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$, we have $|d(A', B') - d(A, B)| < \epsilon$.

Lemma 1 (Regularity Lemma: Szemerédi [8]) *For every $\epsilon > 0$ and each integer ℓ_0 there is an $M = M(\epsilon, \ell_0)$ such that if G is any graph on at least M vertices and $d \in (0, 1)$ then there exists a partition of $V(G)$ into $\ell + 1$ classes $V_0, V_1, \ldots, V_\ell$, and a spanning subgraph $G' \subseteq G$ with the following properties:*

– $\ell_0 \le \ell \le M$, $|V_0| \le \epsilon|G|$, $|V_1| = \cdots = |V_\ell| = L$,
– $d_{G'}(v) > d_G(v) - (d+\epsilon)|G|$ *for all* $v \in V(G)$,
– $e(G'[V_i]) = 0$ *for all* $i \ge 1$,
– *for all* $1 \le i < j \le \ell$ *the graph* $(V_i, V_j)_{G'}$ *is* ϵ*-regular and has density either* 0 *or greater than* d.

The sets V_i in Lemma 2 are called *clusters* and we call the cluster V_0 the *garbage cluster*. From these clusters, we create the reduced graph R as follows. The graph R contains a vertex v_i for each cluster V_i for all $i \ge 1$ and has an edge between v_i and v_j if and only if the pair of clusters has density greater than d.

Lemma 2 (Degree-Sum Regularity: Kühn, Osthus, Treglown [4]) *Given a constant* c*, let* G *be a graph such that* $\sigma_2(G) \ge c|G|$*. Suppose we have applied Lemma 1 with parameters* ϵ *and* d *on* G *and let* R *be the corresponding reduced graph. Then* $\sigma_2(R) > (c - 2d - 4\epsilon)|R|$.

Similar to the definition of ϵ-regular, we now define super-regularity. A pair of sets A and B are called (ϵ, δ)*-super-regular* if, for all subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge \epsilon|A|$ and $|B'| \ge \epsilon|B|$, we have $d(A', B') > \delta$ and furthermore $d_B(a) > \delta|B|$ for all $a \in A$ and $d_A(b) > \delta|A|$ for all $b \in B$. In order to obtain large super-regular pairs, we apply the following lemma noted in [1] among others.

Lemma 3 [1] *Let* (A, B) *be an* ϵ*-regular pair with density greater than* d *and* B' *be a subset of* B *of size at least* $\epsilon|B|$*. Then there are at most* $\epsilon|A|$ *vertices* v *in the set* A *with* $|N(v) \cap B'| < (d - \epsilon)|B'|$.

Lemma 4 (Blow-Up Lemma: Komlós, Sárközy, Szemerédi [3]) *Given a graph* R *of order* r *and positive parameters* d, Δ*, there exists a positive* $\epsilon = \epsilon(d, \Delta, r)$ *such that the following holds. Let* $\{n_1, n_2, \ldots, n_r\}$ *be an arbitrary set of positive integers and replace the set of vertices* $\{v_1, v_2, \ldots, v_r\}$ *of* R *with pairwise disjoint sets* $V_1, V_2, \ldots, V_{r-1}$ *and* V_r *of sizes* $n_1, n_2, \ldots, n_{r-1}$ *and* n_r *respectively (blowing up). We construct two graphs on the same vertex-set* $V = \cup V_i$*. The first graph* $\mathbf{R}$ *is obtained by replacing each edge* v_i, v_j *of* R *with the complete bipartite graph between the corresponding vertex-sets* V_i *and* V_j*. A sparser graph* G *is constructed by replacing each edge* v_i, v_j *arbitrarily with an* (ϵ, d)*-super-regular pair between* V_i *and* V_j*. If a graph* H *with* $\Delta(H) \le \Delta$ *is embeddable into* $\mathbf{R}$ *then it is also embeddable into* G.

The proof of Theorem 2 is presented in Sect. 2. Then, in Sect. 3, we conclude with an easy (but sharp) result providing a minimum degree condition which implies the same structure as Theorem 2.

2 Proof of Theorem 2

Proof Let $t \ge 1$ be an integer and let $\{n_1, n_2, \ldots, n_t\}$ be a set of integers, each at least 1. Choose $\ell_0 = 2$ and other constants as follows:

$$0 < \epsilon \ll d \ll \frac{1}{3}.$$

Suppose further that n is sufficiently large as a function of t and ϵ.

Let G be a graph of order n with $\sigma_2(G) \geq n+t-2$. Let $U = \{u_1, u_2, \ldots, u_t\} \subseteq G$ be any set of t vertices. By Lemmas 1 and 2, there exists a partition of $V(G)$ into a bounded number $\ell + 1 \geq 3$ of clusters $V_0, V_1, \ldots, V_\ell$ such that the reduced graph R has $\sigma_2(R) \geq (1-\gamma)|R|$ where $\gamma = 2d + 4\epsilon \ll \frac{1}{3}$. Thus, R contains at least one edge.

Between a pair of clusters V_i and V_j corresponding to an edge of R, we apply Lemma 3 to obtain a super-regular pair (V_i', V_j') where $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$. By Lemma 4, for the purpose of constructing disjoint paths, we may assume the pair (V_i', V_j') induces a complete bipartite graph in G. Let $B = V_i' \cup V_j'$ be the vertex set of this complete bipartite graph. By Lemmas 1 and 3, we know $|B| \geq 2(1-\epsilon)L$ so n can be chosen to be sufficiently large such that $|B| > t + \sum n_i$. This means that if all the chosen vertices are in B, the conclusion follows immediately from Lemma 4.

With $\sigma_2(G) \geq n + t - 2$, we know $\kappa(G) \geq t$. By Menger's Theorem [6], there exists a set of $|U \setminus B| \leq t$ disjoint paths from $U \setminus B$ to $B \setminus U$. Choose such a set of paths that is as short as possible so each path contains exactly one vertex of $B \setminus U$. For each vertex u_i, let $w_i \in B \setminus U$ be the other end of the provided path. If the path from u_i to w_i already has order at least n_i, a subpath of this path is the desired path for u_i and we simply ignore the remaining vertices. If the path from u_i to w_i has $m_i < n_i$ vertices, then we set w_i as a proxy for u_i using $n_i' = n_i - m_i + 1$ as the new desired order for each path. All desired paths may then be constructed within B to complete the proof. □

3 Minimum Degree Result

Given a set of orders of paths, the following provides a sharp minimum degree condition for the existence of paths of the desired orders, beginning at any chosen set of vertices.

Theorem 3 *Let $t \geq 1$ be an integer, $\{n_1, n_2, \ldots, n_t\}$ be a set of positive integers. If G is a graph of order $n \geq \sum n_i$ with $\delta(G) \geq (\sum n_i) - 1$, then for any choice of t vertices $\{u_1, u_2, \ldots, u_t\} \subseteq V(G)$, there exists a set of vertex disjoint paths $P_1, P_2, \ldots, P_t$ in G where P_i starts at the vertex u_i and has order n_i for all $1 \leq i \leq t$. Furthermore, this degree condition is sharp.*

Proof The proof is by induction on $\sum n_i$. If $\sum n_i = t$, the result is trivial so suppose the result holds for $\sum n_i = n'$ for some integer n' and suppose $\sum n_i = n' + 1$. Since $n' + 1 > t$, there must be at least one integer in the set, say n_t, with $n_t \geq 2$. If $\delta(G) \geq n'$, then by induction, for any set of vertices $\{u_1, u_2, \ldots, u_t\}$, there exists a set of paths $P_1, P_2, \ldots, P_t$ with P_i starting at u_i for all i, with $|P_i| = n_i$ for all $i \neq t$, and with $|P_t| = n_t - 1$. Let w_t be the ending vertex of P_t (the opposite end of the path from u_t). Then since $d(w_t) \geq n'$, there exists an edge from w_t to a vertex $v \in G \setminus (P_1 \cup P_2 \cup \cdots \cup P_t)$. Extending P_t to include the edge $w_t v$ produces the desired set of paths.

The sharpness of this result when $\sum n_i = n'$ is given by any graph G containing $K_{n'-1}$ as a component and $\delta(G) \geq n' - 2$ otherwise. Then $\delta(G) \geq n' - 2$ but if all of the selected vertices are in a component isomorphic to $K_{n'-1}$, there is no way to construct the desired paths. □

References

1. Diestel, R.: Graph Theory. Graduate Texts in Mathematics, 4th edn, vol. 173. Springer, Heidelberg (2010)
2. Enomoto, H., Ota, K.: Partitions of a graph into paths with prescribed endvertices and lengths. J. Graph Theory **34**(2), 163–169 (2000)
3. Komlós, J., Sárközy, G.N., Szemerédi, E.: Blow-up lemma. Combinatorica **17**(1), 109–123 (1997)
4. Kühn, D., Osthus, D., Treglown, A.: An Ore-type theorem for perfect packings in graphs. SIAM J. Discrete Math. **23**(3), 1335–1355 (2009)
5. Magnant, C., Martin, D.M.: An asymptotic version of a conjecture by Enomoto and Ota. J. Graph Theory **64**(1), 37–51 (2010)
6. Menger, K.: Über reguläre Baumkurven. Math. Ann. **96**(1), 572–582 (1927)
7. Ore, O.: Note on Hamilton circuits. Am. Math. Monthly **67**, 55 (1960)
8. Szemerédi, E.: Regular partitions of graphs. In: Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976). Colloq. Internat. CNRS, vol. 260, pp. 399–401. CNRS, Paris (1978)