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The Chromatic Index of a Graph Whose Core is a Cycle of Order at Most 13

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Abstract Let G be a graph. The core of G, denoted by G_Δ, is the subgraph of G induced by the vertices of degree $\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. A k *-edge coloring* of G is a function $f : E(G) \to L$ such that $|L| = k$ and $f(e_1) \neq f(e_2)$ for all two adjacent edges e_1 and e_2 of G. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number k for which G has a k-edge coloring. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G) + 1$. In this paper it is shown that every connected graph G of even order whose core is a cycle of order at most 13 is Class 1.

Keywords Edge coloring · Core · Class 1

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1 Introduction

All graphs considered in this paper are finite, undirected, with no loops or multiple edges. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The number of vertices of G is called the *order* of G and denoted by $|G|$. Also, $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. The *core* of G, denoted by G_Δ, is the subgraph of G induced by all vertices of degree $\Delta(G)$. We denote the cycle of order n by C_n. Let H be a subgraph of G. For a vertex v of G, $d_G(v)$ and $N_G(v)$ denote the degree and the neighborhood of v in G, respectively. A *star graph* is a graph containing a vertex adjacent to all other vertices and with no extra edges.

A *matching* in a graph G is a set of pairwise non-adjacent edges and a 1 *-factor* is a matching which covers $V(G)$. A component of a graph is called *odd* if its order is odd. The number of odd components of G is denoted by $o(G)$. For a subset $X \subseteq V(G)$ ($Y \subseteq E(G)$), $G \setminus X$ ($G \setminus Y$) denotes the graph obtained from G by deleting all vertices (edges) of X (Y), respectively. Moreover, by $G \setminus H$ we mean the induced subgraph of $V(G) \setminus V(H)$.

A *k-edge coloring* of a graph G is a function $f : E(G) \longrightarrow L$ such that $|L| = k$ and $f(e_1) \neq f(e_2)$ for all two adjacent edges e_1 and e_2 of G. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number k for which G has a k-edge coloring. For a general introduction to the edge coloring, the interested reader is referred to [8]. If α is a color and a vertex v is incident with an edge colored α, we say that v sees α and otherwise, we say that color α is missed at v.

A celebrated result due to Vizing [17] states that for every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G) + 1$. Moreover, a connected graph G is called *critical* if it is Class 2 and $G \setminus \{e\}$ is Class 1 for every edge $e \in E(G)$. A graph G is called *overfull* if $|E(G)| > \lfloor \frac{|V(G)|}{2} \rfloor \Delta(G)$. It is easy to see that, if G is overfull, then G is Class 2. For more information about overfull graphs see [10]. In [16] it was proved that there is no critical connected graph G of even order with $|G_\Delta| \leq 5$.

Let H, Q and R be subgraphs of G. We denote the number of edges of H with one end point in Q and another end point in R by $e_H(Q, R)$. For a subset $S \subseteq V(G)$, we denote the induced subgraph of G on S by $\langle S \rangle$.

Classifying a graph into Class 1 and Class 2 is a difficult problem in general (indeed, NP hard), even when restricted to the class of graphs with maximum degree 3 (see [15]). As a consequence, this problem is usually considered on classes of graphs with particular classes of cores. One possibility is to consider a graph whose core has a simple structure, see [5–7,9,11–14,18]. Vizing [18] proved that, if G_Δ has no edge, then G is Class 1. Fournier [9] generalized Vizing's result by showing that, if G_Δ contains no cycle, then G is Class 1. Thus a necessary condition for a graph to be Class 2 is to have a core containing cycles. Hilton and Zhao [12,13] considered the problem of classifying graphs whose cores are a disjoint union of cycles. Only a few such graphs are known to be Class 2. These include the overfull graphs and the graph P^*, which is obtained from the Petersen graph by removing one vertex and has order 9. Furthermore, they posed the following conjecture.

Conjecture 1 *Let G be a connected graph such that* $\Delta(G_\Delta) \leq 2$. *Then G is Class* 2 *if and only if G is overfull or* $G = P^*$.

In [4], the following theorem was proved:

Theorem 1 *Let G be a connected graph such that* $\Delta(G_\Delta) \leq 2$, $\Delta(G) = 3$ *and* $G \neq P^*$. *Then G is Class* 1.

Theorem 2 [9] *If* G_Δ *is a forest, then G is Class* 1.

Theorem 3 [13] *Let G be a connected graph of Class* 2 *and* $\Delta(G_\Delta) \leq 2$. *Then the following statements hold:*

(i) *G is critical;*
(ii) $\delta(G_\Delta) = 2$;
(iii) $\delta(G) = \Delta(G) - 1$, *unless G is an odd cycle.*

Theorem 4 [13] *Let G be a critical connected graph. Then every vertex of G is adjacent to at least two vertices of* G_Δ.

A connected graph is called *unicyclic* if it contains precisely one cycle. In [1], the following results are given.

Theorem 5 *Let G be a connected graph. If every component of* G_Δ *is a unicyclic graph or a tree and* G_Δ *is not a disjoint union of cycles, then G is Class* 1.

Theorem 6 *Let G be a connected graph with* $\Delta(G_\Delta) \leq 2$. *Suppose that G has an edge cut of size at most* $\Delta(G) - 2$ *which is a matching or a star. Then G is Class* 1.

Theorem 7 *Let G be a connected graph of even order. If* $\Delta(G_\Delta) \leq 2$ *and* $|G_\Delta|$ *is odd, then G is Class* 1.

The following theorem provides a condition on the core of a graph under which the graph is Class 1.

Theorem 8 [2] *Let G be a connected graph of even order and* $\Delta(G_\Delta) \leq 2$. *If* $|G_\Delta| \leq 9$ *or* $G_\Delta = C_{10}$, *then G is Class* 1.

Now, we propose the following theorem which will help to prove the main theorem of the paper.

Theorem 9 *Let G be a connected graph with* $G_\Delta = C_k$. *If* $\Delta(G) \geq 4$ *and G has an edge cut of size at most* 3 *which is not a star, then G is Class* 1. *Moreover, if* $\Delta(G) \geq 5$ *and G has an edge cut of size at most* 3, *then G is Class* 1.

Proof For simplicity, let $\Delta = \Delta(G)$. To the contrary assume that G is Class 2. Now, by Theorem 3, G is critical and $\delta(G) = \Delta - 1$. Now, by Theorem 4

$$|N(x) \cap V(G_\Delta)| \geq 2, \quad \text{for every} \quad x \in V(G). \tag{1}$$

Let F be an edge cut of G. Note that if $|F| \leq 2$ or F is a star, then by Theorem 6, G is Class 1 and we get a contradiction. So, we can assume that $|F| = 3$ and F is not a star.

Thus G is one of the graphs shown in Fig. 1, where $G \setminus F = G_1 \cup G_2$.

First note that since $d_{G_1}(u_1), d_{G_2}(v_2) \geq \Delta - 2$, $|G_i| \geq \Delta - 1 \geq 3$, for $i = 1, 2$. So, by (1) and noting that G_Δ is a cycle, it is not hard to see that $|V(G_i) \cap V(G_\Delta)| \geq 2$, for $i = 1, 2$. Now, with no loss of generality, let $u_1, u_2, v_1, v_2 \in V(G_\Delta)$. Note that since G_Δ is a cycle, G is one of the graphs in Parts (a) and (c). Now, two cases may appear:

First assume that G is the graph shown in Fig. 1, Part (a). Add two new vertices x_1 and x_2 to $G \setminus F$, join x_1 to u_i and join x_2 to v_i, for $i = 1, 2, 3$ and let $H_1 = \langle V(G_1) \cup \{x_1\} \rangle$ and $H_2 = \langle V(G_2) \cup \{x_2\} \rangle$. Note that H_1 and H_2 are connected, $\Delta(H_i) = \Delta(G)$ and the core of H_i is a path, for $i = 1, 2$. Then by Theorem 2, H_i has a $\Delta(G)$-edge coloring ϕ_i, for $i = 1, 2$. Now, by a suitable permutation of colors, one may assume that $\phi_1(x_1u_i) = \phi_2(x_2v_i)$, for $i = 1, 2, 3$. Then define an edge coloring $c : E(G) \longrightarrow \{1, \ldots, \Delta(G)\}$ as follows:
Let $c(e) = \phi_1(e)$ and $c(e') = \phi_2(e')$, for every $e \in E(G_1)$, $e' \in E(G_2)$ and $c(u_iv_i) = \phi_1(x_1u_i)$, for $i = 1, 2, 3$ and so G is Class 1, a contradiction.

Next, suppose that G is the graph shown in Fig. 1, Part (c). Since $u_1, u_2, v_1, v_2 \in V(G_\Delta)$, $w \notin V(G_\Delta)$. Now, three cases may be considered:

(i) $u_1u_2, v_1v_2 \notin E(G)$.

Consider G_1, join u_1 to u_2 and call the resultant graph by H. Clearly, $\Delta(H) = \Delta$ and $\Delta(H_\Delta) \leq 2$. Note that since $w \notin V(G_\Delta)$, $d_G(w) = \Delta - 1$ and so $d_H(w) = d_{G_1}(w) =$

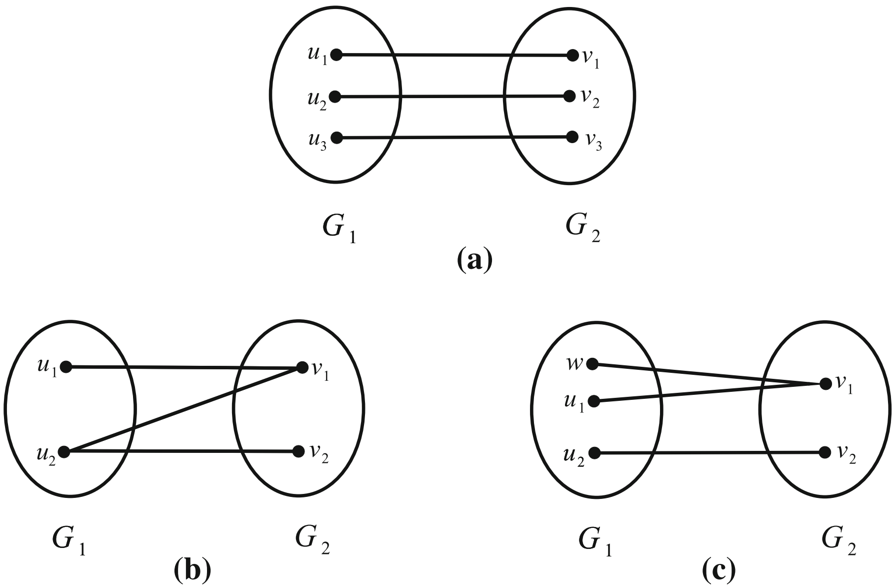

Fig. 1 Three possibilities of an edge cut of size 3 for G

$\Delta - 2$. Now, by Theorem 3, since $\delta(H) < \Delta(H) - 1$, H has a Δ-edge coloring ϕ by the colors $\{1, \dots, \Delta\}$ such that $\phi(u_1u_2) = 1$. Moreover, since $d_H(w) = \Delta - 2$, with no loss of generality we can assume that color 2 is missed at w. Now, consider G_2, join v_1 to v_2 and call the resultant graph by K. Clearly, $\Delta(K) = \Delta$. Moreover, since $d_G(v_1) = \Delta, d_K(v_1) = \Delta - 1$ and so $v_1 \notin V(K_\Delta)$. Thus the core of K is a path and by Theorem 2, K has a Δ-edge coloring θ by the colors $\{1, \dots, \Delta\}$ such that $\theta(v_1v_2) = 1$ and color 2 is missed at v_1. Now, define a Δ-edge coloring $c : E(G) \longrightarrow \{1, \dots, \Delta\}$ as follows:
Let $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(G_1)$, $e' \in E(G_2)$, $c(u_1v_1) = c(u_2v_2) = 1$ and $c(wv_1) = 2$. Hence G is Class 1, a contradiction.

(ii) $u_1u_2 \in E(G)$.

Clearly, since $d_{G_1}(u_1) = \Delta - 1, |G_1| \geq \Delta$. Moreover, since $u_1u_2 \in E(G_\Delta)$, we have $V(G_1) \cap V(G_\Delta) = \{u_1, u_2\}$. Note that if $|G_1| \geq \Delta + 1$, then by (1), for every $v \in V(G_1) \setminus \{w, u_1, u_2\}$, $\{u_1, u_2\} \subseteq N(v) \cap V(G_1)$ and so $e_{G_1}(u_i, \langle V(G_1) \setminus \{w, u_1, u_2\}\rangle) \geq \Delta - 2$, for $i = 1, 2$. Moreover, by (1) and with no loss of generality, $u_1 \in N(w) \cap V(G_1)$. This implies that $e_{G_1}(u_1, \langle V(G_1) \setminus \{u_1, u_2\}\rangle) \geq \Delta - 1$ and since $u_1u_2 \in E(G)$, $d_G(u_1) > \Delta$, a contradiction. Thus assume that $|G_1| = \Delta$. Clearly, for every $v \in V(G_1) \setminus \{w\}$, $d_{G_1}(v) = \Delta - 1$. Thus $d_{G_1}(w) = \Delta - 1$ which contradicts $w_1 \notin V(G_\Delta)$.

(iii) $v_1v_2 \in E(G)$.

Clearly, since $d_{G_2}(v_2) = \Delta - 1$, $|G_2| \geq \Delta$. Moreover, since $v_1v_2 \in E(G_\Delta)$, $V(G_2) \cap V(G_\Delta) = \{v_1, v_2\}$. Now, by (1), for every $v \in V(G_2) \setminus \{v_1, v_2\}$, $\{v_1, v_2\} \subseteq N(v) \cap V(G_2)$ and since $v_1v_2 \in E(G)$, $d_G(v_1) > \Delta$, a contradiction and the proof is complete.

Now, to prove the main result of this paper, we need a lemma. Before proving the lemma, we state a result without proof.

Theorem 10 [19] *If G is critical and* $\Delta(G) \geq 4$, *then*

$$n_\Delta \geq 2 \sum_{j=2}^{\Delta(G)-1} \frac{n_j}{j-1} + \frac{1}{2} n_3,$$

where n_j *is the number of vertices having degree* j *in G.*

Lemma 1 *Let G be a connected graph of even order,* $\Delta = 4$ *and* $G_\Delta = C_{12}$. *Then G is Class* 1.

Proof Let $n = |G|$. To the contrary assume that G is Class 2. Now, by Theorem 3, G is critical and $\delta(G) = 3$. Moreover, since G is critical by Theorem 10, $n \leq 20$ and since $\delta(G) = 3$,

$$2 \times 12 = e_G(G_\Delta, G \setminus G_\Delta) \leq 3(n - 12)$$

which implies that $n \geq 20$. Thus $n = 20$ and $V(G) \setminus V(G_\Delta)$ is an independent set. We show that G has a 1-factor. To see this assume the contrary. Then by Tutte's 1-factor Theorem [3, p. 44], there exists a subset $T \subseteq V(G)$ such that $o(G \setminus T) > |T|$. Let $m = o(G \setminus T)$. Since n is even, we have $m \equiv |T| \pmod 2$, which implies that $m \geq |T| + 2$. Let $B_1, \ldots, B_b$, $S_1, \ldots, S_s$, $P_1, \ldots, P_p$ and $Q_1, \ldots, Q_q$ be the odd components of $G \setminus T$ such that

$$\begin{cases} |B_i| \geq 5 & \text{for } i = 1, \ldots, b, \\ S_i = \{u_i\}, u_i \in V(G_\Delta) & \text{for } i = 1, \ldots, s, \\ P_i = \{v_i\}, v_i \notin V(G_\Delta) & \text{for } i = 1, \ldots, p, \\ |Q_i| = 3 & \text{for } i = 1, \ldots, q. \end{cases}$$

Now, since $m \geq |T| + 2$, we have

$$b + s + p + q \geq |T| + 2. \tag{2}$$

By the definition, $e_G(T, S_i) = 4$, for $i = 1, \ldots, s$ and $e_G(T, P_i) = 3$, for $i = 1, \ldots, p$. Moreover, since $V(G) \setminus V(G_\Delta)$ is an independent set, every component D of $G \setminus T$ with $|D| > 1$ has at least a vertex of G_Δ. Thus we have $e_G(T, Q_i) \geq 4$, for $i = 1, \ldots, q$. Now, we claim that $e_G(T, B_i) \geq 4$, for $i = 1, \ldots, b$. We prove it for B_1. First note that if $V(B_1) \cap V(G_\Delta) = \{w\}$, then since B_1 is a connected component and $V(G) \setminus V(G_\Delta)$ is independent, for every $v \in V(B_1) \setminus \{w\}$, $vw \in E(B_1)$. Also, $e_{G_\Delta}(w, T) = 2$ which implies that $d_G(w) \geq 6$, a contradiction. So, assume that $|V(B_1) \cap V(G_\Delta)| \geq 2$. It is not hard to see that there are two vertices w_1 and w_2 in $V(B_1) \cap V(G_\Delta)$ such that $|N(w_i) \cap V(G_\Delta) \cap T| \geq 1$. Let $x_i \in N(w_i) \cap V(G_\Delta) \cap T$, for $i = 1, 2$. First, assume that $x_1 \neq x_2$. Now, if the set of edges between B_1 and T is at most 3, then this edge cut is not a star and by Theorem 9, G is Class 1, a contradiction. Now, suppose that $x_1 = x_2$. Then since G_Δ is a cycle, it is not hard to see that $e_{G_\Delta}(T, B_1) \geq 4$ and we are done. So, the claim is proved and we have $e_G(T, B_i) \geq 4$, for $i = 1, \ldots, b$. Now, by counting the number of edges between T and $G \setminus T$, we obtain,

$$4|T| \geq e_G(T, G \setminus T) \geq 4b + 4s + 3p + 4q.$$

Now, by (2), $p \geq 8$. Since $|V(G) \setminus V(G_\Delta)| = 8$, we have $p = 8$ which implies that $T \subseteq V(G_\Delta)$. If $G \setminus T$ has 8 components, then by (2), $|T| \leq 6$ and so $n \leq 14$, a contradiction. Hence, $G \setminus T$ has at least a component $D \neq P_i$, for $i = 1, \ldots, 8$. Now, since $p = 8$, $V(D) \subseteq V(G_\Delta)$. So, for every vertex $v \in V(D)$, $N(v) \subseteq V(G_\Delta)$ which is a contradiction. Therefore G has a 1-factor. Let M be a 1-factor of G and $H = G \setminus M$. Clearly, H_Δ is a forest and so by Theorem 2, H is Class 1 and G is Class 1, too. This completes the proof.

Now, we are in a position to prove the main theorem of the paper.

Theorem 11 *Let G be a connected graph of even order and G_Δ is a cycle of order at most* 13. *Then G is Class* 1.

Proof For simplicity, let $\Delta = \Delta(G)$ and $n = |G|$. The proof is by induction on Δ. First note that if G_Δ has odd order or $|G_\Delta| \leq 10$, then by Theorems 7 and 8, we are done. Moreover, if G is not critical or $\delta(G) < \Delta - 1$, then by Theorem 3, G is Class 1 and the theorem is proved. So, we can assume that

$$G_\Delta = C_{12},$$
$$G \text{ is critical},$$
$$\delta(G) = \Delta - 1.$$

Now, since G is critical by Theorem 4,

$$|N(x) \cap V(G_\Delta)| \geq 2, \quad \text{for every} \quad x \in V(G). \tag{3}$$

By (3), we find that $12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 2(n - 12)$, and so

$$n \leq 6\Delta. \tag{4}$$

Note that since G_Δ is a cycle, $\Delta \geq 2$. Now, since G has even order, if $\Delta \leq 4$, then by Theorem 1 and Lemma 1, G is Class 1 and we are done. So we may assume that

$$\Delta \geq 5.$$

Now, if G has an edge cut of size at most 3, then by Theorem 9, G is Class 1 and we are done. Thus we can suppose that G is 4-edge connected. We show that G has a 1-factor or G is Class 1. To see this suppose to the contrary that G has a 1-factor. Then by Tutte's 1-factor Theorem [3, p. 44], we can assume that there exists a subset $T \subseteq V(G)$ such that $o(G \setminus T) > |T|$. Let $m = o(G \setminus T)$. Since n is even, we have $m \equiv |T| \pmod 2$, which implies that $m \geq |T| + 2$.

Let $B_1, \ldots, B_b$ and $S_1, \ldots, S_s$ be the odd components of $G \setminus T$ such that $|B_i| \geq \Delta$ for $i = 1, \ldots, b$ and $|S_j| \leq \Delta - 1$ for $j = 1, \ldots, s$, where $m = b + s$. Since $|T| \leq m - 2$, we find that

$$|T| \leq b + s - 2. \tag{5}$$

Also, since G is 4-edge connected,

$$e_G(T, B_i) \geq 4, \quad \text{for} \quad i = 1, \ldots, b. \tag{6}$$

For $j = 1, \ldots, s$, since $1 \le |S_j| \le \Delta - 1 = \delta(G)$, the following hold:

$$\begin{aligned} e_G(T, S_j) &= \sum_{x \in V(S_j)} e_G(T, x) \\ &\ge (\delta(G) - (|S_j| - 1))|S_j| \\ &= (\Delta - |S_j|)|S_j| \qquad (7) \\ &\ge \Delta - 1. \qquad (8) \end{aligned}$$

Let $q = |T \cap V(G_\Delta)|, r = |E(\langle T \rangle) \cap E(G_\Delta)|$. Since $G_\Delta = C_{12}$, the number of edges of G_Δ joining T to $V(G) \setminus T$ satisfies

$$2q - 2r = e_{G_\Delta}(T, G \setminus T) \le 2(12 - q).$$

Hence

$$q \le 6 + \frac{r}{2}. \qquad (9)$$

Now, by (3) and noting that $|B_j| \ge \Delta$, for $j = 1, \ldots, b$ and $G_\Delta = C_{12}$, we obtain that

$$e_G(T, B_j) \ge \begin{cases} \Delta + 1 & \text{if} \quad |V(B_j) \cap V(G_\Delta)| = 1, \\ 2\Delta & \text{if} \quad V(B_j) \cap V(G_\Delta) = \emptyset. \end{cases} \qquad (10)$$

Let b_0, b_1 and b_2 be the number of B_j such that $V(B_j) \cap V(G_\Delta) = \emptyset$, $|V(B_j) \cap V(G_\Delta)| = 1$ and $|V(B_j) \cap V(G_\Delta)| \ge 2$, respectively. We have $b = b_0 + b_1 + b_2$. Now, by (6) and (10), we find that

$$\begin{aligned} e_G(T, B_1 \cup \cdots \cup B_b) &\ge 4b_2 + (\Delta + 1)b_1 + 2\Delta b_0 \\ &= (\Delta - 1)b - (\Delta - 5)b_2 + 2b_1 + (\Delta + 1)b_0. \qquad (11) \end{aligned}$$

Obviously, using (8) and (11), we have

$$\begin{aligned} q\Delta - 2r + (|T| - q)(\Delta - 1) &\ge e_G(T, B_1 \cup \cdots \cup B_b \cup S_1 \cup \cdots \cup S_s) \\ &\ge (\Delta - 1)b - (\Delta - 5)b_2 + 2b_1 + (\Delta + 1)b_0 + (\Delta - 1)s. \end{aligned}$$

This implies that

$$q - 2r + (|T| - b - s)(\Delta - 1) + (\Delta - 5)b_2 - 2b_1 - (\Delta + 1)b_0 \ge 0. \qquad (12)$$

Now, if $b_2 \le 2$, then by (5) and (9), we have

$$\begin{aligned} &q - 2r + (|T| - b - s)(\Delta - 1) + (\Delta - 5)b_2 - 2b_1 - (\Delta + 1)b_0 \\ &\le 6 - \frac{3r}{2} - 2(\Delta - 1) + 2(\Delta - 5) - 2b_1 - (\Delta + 1)b_0 \\ &< 0. \end{aligned}$$

This contradicts (12) and we conclude that $b_2 \geq 3$ and so $q \leq 12 - 2b_2 \leq 6$. Now, we need the following claim:

Claim 1 The number of components of $G \setminus T$ containing at least one vertex of G_Δ is at most 4.

Proof of Claim 1. Let l be the number of components of $G \setminus T$ containing at least one vertex of G_Δ. Since $b_2 \geq 3, l \geq 3$. It is not hard to see that $2l \leq e_{G_\Delta}(T, G \setminus T) \leq 2q$ and so

$$l \leq q \leq |T|. \tag{13}$$

Now, suppose to the contrary that $l \geq 5$. Thus $q \geq 5$ and since $b_2 \geq 3$ we find that $|G_\Delta| \geq 13$ which is a contradiction and the claim is proved.

Therefore we have the following:

$$3 \leq b_2 \leq 4, \tag{14}$$

and by (13),

$$q \geq 3. \tag{15}$$

Note that if $|T| \leq b + s - 4$, then by (9) and (14)

$$\begin{aligned} & q - 2r + (|T| - b - s)(\Delta - 1) + (\Delta - 5)b_2 - 2b_1 - (\Delta + 1)b_0 \\ & \leq 6 - \frac{3r}{2} - 4(\Delta - 1) + 4(\Delta - 5) - 2b_1 - (\Delta + 1)b_0 \\ & < 0. \end{aligned}$$

This contradicts (12) and we conclude that $|T| \geq b + s - 3$. Now, by (5) and noting that $m = b + s \equiv |T| \pmod 2$, we conclude that

$$|T| = b + s - 2. \tag{16}$$

Moreover, since $n \leq 6\Delta$, we find that $b \leq 5$. Now, three cases may be considered:

Case 1 Assume that $b = 5$. Then by (16), $s = |T| - 3$. Since for every $v \in V(G)$, $d_G(v) \geq \Delta - 1$, $|S_j| \geq \Delta - 1 - |T| + 1 = \Delta - |T|$. Now, by (4),

$$6\Delta \geq n \geq 5\Delta + |T| + (|T| - 3)(\Delta - |T|).$$

Since $\Delta \geq 5$, we conclude that $|T| \leq 4$ or $|T| \geq \Delta$. Now, if $|T| \geq \Delta$, then $s > 0$ and clearly $n > 6\Delta$, a contradiction. Thus, by (15) we can suppose that $3 \leq |T| \leq 4$. This implies that $s \leq 1$ and $|S_1| \geq \Delta - 4$. Moreover, by (13), $G \setminus T$ contains at most $|T|$ components, each containing at least one vertex of G_Δ. Now, by (16), $G \setminus T$ contains $|T| + 2$ odd components and so at least two odd components, say D_1 and D_2, have no vertex of G_Δ. With no loss of generality we can assume that $|D_1| \geq \Delta$ and $|D_2| \geq \Delta - 4$. Now, by (3), $e_G(T, D_1 \cup D_2) \geq 2(2\Delta - 4)$. Thus by (6),

$$|T|\Delta \geq e_G(T, G \setminus T) \geq 4|T| + 2(2\Delta - 4).$$

So, $|T| \geq 5$, a contradiction and the proof of this case is complete.

Case 2 Suppose that $b = 4$ and $|V(B_i) \cap V(G_\Delta)| \geq 1$, for $i = 1, \ldots, 4$.
By (16),

$$s = |T| - 2 \geq q - 2 \geq 3 - 2 = 1. \tag{17}$$

First note that since by (14) $b_2 \geq 3$, $|(\cup_{i=1}^4 V(B_i)) \cap V(G_\Delta)| \geq 7$ and so $q \leq 5$. Moreover, by (13) we have $4 \leq q \leq 5$. Also, by the Claim 1,

$$V(G_\Delta) \subseteq T \cup \left(\cup_{i=1}^4 V(B_i)\right). \tag{18}$$

Moreover, by (6) and (7), we have

$$q\Delta + (|T| - q)(\Delta - 1) \geq e_G(T, G \setminus T) \geq 4 \times 4 + \sum_{j=1}^{s}(\Delta - |S_j|)|S_j|.$$

Thus, we find that

$$q - 16 + (|T| - s)(\Delta - 1) - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \geq 0. \tag{19}$$

On the other hand by (17) and $q \leq 5$, we find that

$$\begin{aligned} & q - 16 + (|T| - s)(\Delta - 1) - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ & \leq 5 - 16 + 2\Delta - 2 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ & = 2\Delta - 13 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)). \end{aligned}$$

Note that by (7) and (8), $(\Delta - |S_j|)|S_j| - (\Delta - 1) \geq 0$. Now, if $\Delta \leq 6$, then

$$q - 16 + (|T| - s)(\Delta - 1) - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) < 0$$

which contradicts (19). Thus suppose that

$$\Delta \geq 7. \tag{20}$$

Now, if $3 \leq |S_k| \leq \Delta - 3$ for some k, then $-\big((\Delta - |S_k|)|S_k| - (\Delta - 1)\big) \leq -2\Delta + 8$. Thus

$$\begin{aligned} &2\Delta - 13 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ &\leq 2\Delta - 13 - 2\Delta + 8 \\ &< 0. \end{aligned}$$

This contradicts (19). So, since $|S_j|$ is odd, $|S_j| \in \{1, \Delta - 2, \Delta - 1\}$, for $j = 1, \ldots, s$. Now, we prove the following claim:

Claim 2 $|S_j| = 1$, for $j = 1, \ldots, s$.

Proof of Claim 2. First we show that $|S_j| = 1$, for some j, $1 \leq j \leq s$. To see this, suppose to the contrary that $|S_j| \geq \Delta - 2$, for $j = 1, \ldots, s$. Thus, by (3) and (18), $e_G(T, S_j) \geq 2(\Delta - 2)$, for $j = 1, \ldots, s$. Now, since $b = 4$, by (6) and (17), we have

$$q\Delta + (|T| - q)(\Delta - 1) \geq e_G(T, G \setminus T) \geq 4 \times 4 + (|T| - 2)(2\Delta - 4),$$

which is a contradiction with $4 \leq q \leq 5$ and $|T| \geq 4$. So, with no loss of generality, assume that $S_1 = \{u_1\}$. Now, since $d_G(u_1) \geq \Delta - 1$, $|T| \geq \Delta - 1$ and so $s \geq \Delta - 3$. To complete the proof of the claim, suppose the contrary and with no loss of generality assume that $|S_s| \geq \Delta - 2$. Thus by (14) we find that

$$6\Delta \geq n \geq 4\Delta + (\Delta - 1) + (\Delta - 4) + (\Delta - 2).$$

So, $\Delta \leq 7$. On the other hand, by (3) and (18), $e_G(T, S_s) \geq 2\Delta - 4$. Thus by (6), (8), (17) and since $q \leq 5$,

$$5\Delta + (|T| - 5)(\Delta - 1) \geq e_G(T, G \setminus T) \geq 4 \times 4 + (|T| - 3)(\Delta - 1) + 2\Delta - 4.$$

This implies that $\Delta \geq 10$ which contradicts $\Delta \leq 7$. So, the claim is proved and we can assume that $S_j = \{u_j\}$, for $j = 1, \ldots, s$.

Now, since $6\Delta \geq n \geq 4\Delta + |T| + |T| - 2$, we find that $|T| \leq \Delta + 1$. Moreover, since $N(u_j) \subseteq T$, we conclude that $\Delta - 1 \leq |T| \leq \Delta + 1$. Now, three subcases may be considered:

Subcase 2.1 $|T| = \Delta + 1$.
By (17), $s = \Delta - 1$ and so $n \geq 4\Delta + \Delta + 1 + \Delta - 1 = 6\Delta$, so $n = 6\Delta$. Note that if there exists a vertex $x \in V(G)$ such that $|N(x) \cap V(G_\Delta)| \geq 3$, then by (3) we have

$$12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 3 + 2(n - 13).$$

This implies that $n \leq \frac{12\Delta - 1}{2}$ which contradicts $n \geq 6\Delta$. Thus we can suppose that for every $x \in V(G)$, $|N(x) \cap V(G_\Delta)| = 2$. Let $T = \{v_1, \ldots, v_{\Delta+1}\}$. Note that if

$x \in \cup_{j=1}^{s} S_j$, then x has degree $\Delta - 1$ and so is joined to 2 vertices of T of degree Δ and to $\Delta - 3$ vertices of T of degree $\Delta - 1$. Therefore $q \leq |T| - (\Delta - 3) = 4$, so $q = 4$. Let $T \cap V(G_\Delta) = \{v_1, \dots, v_4\}$ and so $\{v_5, \dots, v_{\Delta+1}\} \subseteq N(u_i) \cap V(T)$, for $i = 1, \dots, \Delta - 1$. This implies that $e_G(v_i, \cup_{j=1}^{s} S_j) = \Delta - 1$, for $i = 5, \dots, \Delta + 1$. So, for $i = 5, \dots, \Delta+1$, $\Delta - 1$ edges join v_i to $\cup_{j=1}^{s} S_j$, so by (18) there are no edges joining v_i to any other vertex of G_Δ, contradicting (3).

Subcase 2.2 $|T| = \Delta$.
By (17), $s = \Delta - 2$ and so $n \geq 4\Delta + \Delta + \Delta - 2 = 6\Delta - 2$. Since $q \geq 4$, $|N(u_i) \cap V(G_\Delta)| \geq 3$, for $i = 1, \dots, \Delta - 2$. So, by (3) we have

$$12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 3(\Delta - 2) + 2(n - (\Delta - 2 + 12)).$$

This implies that $n \leq \frac{11\Delta+2}{2}$ and by $n \geq 6\Delta - 2$, we conclude that $\Delta \leq 6$ which contradicts (20).

Subcase 2.3 $|T| = \Delta - 1$.
By (17), $s = \Delta - 3$ and since $q \geq 4$, $|N(u_i) \cap V(G_\Delta)| \geq 4$, for $i = 1, \dots, \Delta - 3$. Thus, by (3) we have

$$12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 4(\Delta - 3) + 2(n - (\Delta - 3 + 12)).$$

This implies that $n \leq 5\Delta + 3$. Moreover, we have $n \geq 4\Delta + (\Delta - 1) + (\Delta - 3) = 6\Delta - 4$. Now, by (20), we conclude that $\Delta = 7$. Thus, $|T| = 6$, $s = 4$ and $n = 38$. Now, since $b = 4$ and $| \cup_{i=1}^{4} V(B_i)| = 28$ and $|B_i| \geq 7$, we conclude that $|B_i| = 7$, for $i = 1, \dots, 4$. Note that if $|V(B_i) \cap V(G_\Delta)| = 1$ for some i, $1 \leq i \leq 4$, then by (10), $e_G(T, B_i) \geq \Delta + 1 = 8$ and we conclude that

$$6 \times 7 \geq e_G(T, G \setminus T) \geq 8 + 3 \times 4 + 4 \times 6,$$

a contradiction. Thus $q = 4$ and so $|V(B_i) \cap V(G_\Delta)| = 2$, for $i = 1, \dots, 4$. Moreover,

$$4 \times 7 + 2 \times 6 \geq e_G(T, G \setminus T) \geq 4 \times 4 + 4 \times 6.$$

This implies that T is an independent set and $e_G(T, B_i) = 4$, for $i = 1, \dots, 4$. Now, let $T = \{x_1, \dots, x_4, y_1, y_2\}$, where $T \cap V(G_\Delta) = \{x_1, \dots, x_4\}$. Also, suppose that $V(B_i) = \{v_{i1}, \dots, v_{i5}, w_{i1}, w_{i2}\}$, where $V(B_i) \cap V(G_\Delta) = \{w_{i1}, w_{i2}\}$, for $i = 1, \dots, 4$. Now, we prove the following claim:

Claim 3 $e_G(T, w_{i1}) + e_G(T, w_{i2}) = 3$, for $i = 1, \dots, 4$.

Proof of Claim 3. First note that $w_{i1}w_{i2} \in E(B_i)$, for $i = 1, \dots, 4$. Because otherwise, we deduce that

$$4 \times 2 \geq e_{G_\Delta}(T, G \setminus T) \geq 4 + 3 \times 2,$$

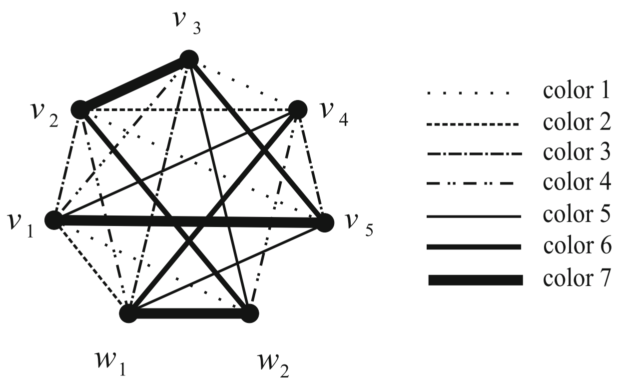

Fig. 2 A 7-edge coloring of B_1 with the desired properties

a contradiction. Now, if $e_G(T, w_{i1}) + e_G(T, w_{i2}) \geq 4$, for some i, $i = 1, \ldots, 4$, then since $e_G(T, B_i) = 4$, $e_G(T, v_{ij}) = 0$, for $j = 1, \ldots, 5$. Thus, since $d_G(v_{ij}) = 6$, for $j = 1, \ldots, 5$, $d_G(w_{i1}) > 7$ or $d_G(w_{i2}) > 7$ which contradicts $\Delta = 7$. Thus we have

$$e_G(T, w_{i1}) + e_G(T, w_{i2}) \leq 3, \text{ for } i = 1, \ldots, 4.$$

By (3), (18) and since T is an independent set, we conclude that $e_G(y_i, \cup_{i=1}^4 B_i) \geq 2$, for $i = 1, 2$. Now, since $e_G(T, w_{i1}) + e_G(T, w_{i2}) \leq 3$, for $i = 1, \ldots, 4$, we find that $e_G(y_1 \cup y_2, B_i) = 1$, for $i = 1, \ldots, 4$. This implies that $e_G(T, w_{i1}) + e_G(T, w_{i2}) = 3$, for $i = 1, \ldots, 4$ and the claim is proved.

For simplicity, let $V(B_1) = \{v_1, \ldots, v_5, w_1, w_2\}$, where $V(B_1) \cap V(G_\Delta) = \{w_1, w_2\}$. Now, by the Claim 3 and noting that $G_\Delta = C_{12}$, with no loss of generality, let $N(w_1) \cap T = \{x_1\}$ and $N(w_2) \cap T = \{x_2, y_1\}$. Now, since $e_G(T, w_2) = 2$, we have $d_{B_1}(w_2) = 5$. Then noting that $w_1 w_2 \in E(G)$, with no loss of generality we can suppose that $v_5 w_2 \notin E(G)$. Now, by (3), $N(v_5) \cap V(T) = \{x_i\}$, for some i, $i = 1, \ldots, 4$. Now, two cases may occur:

- $i \notin \{1, 2\}$.

Let $H = \langle V(G) \setminus V(B_1) \rangle$. Add a new vertex z to H and join z to x_1, x_2, x_i, y_1 and call the resultant graph by H'. It is easy to see that H'_Δ is a path and so by Theorem 2, H' has a 7-edge coloring $\phi : V(H') \longrightarrow \{1, \ldots, 7\}$. Let $\phi(zx_1) = 1$, $\phi(zx_2) = 2$, $\phi(zy_1) = 3$ and $\phi(zx_i) = 4$.

Now, we introduce a 7-edge coloring of B_1, called θ, in which color 1 is missed at w_1, colors 2, 3 are missed at w_2 and color 4 is missed at v_5, see Fig. 2.

Now, define an edge coloring $c : E(G) \longrightarrow \{1, \ldots, 7\}$ as follows:

Let $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(H)$, $e' \in E(B_1)$ and $c(w_1 x_1) = 1$, $c(w_2 x_2) = 2$, $c(w_2 y_1) = 3$ and $c(v_5 x_i) = 4$. Hence in this case we are done.

- $i \in \{1, 2\}$.

Let $H = \langle V(G) \setminus V(B_1) \rangle$. Add a new vertex z to H and join z to x_1, x_2 and y_1. Moreover, add a new edge $x_i y_1$ and call the resultant graph by H'. We show that

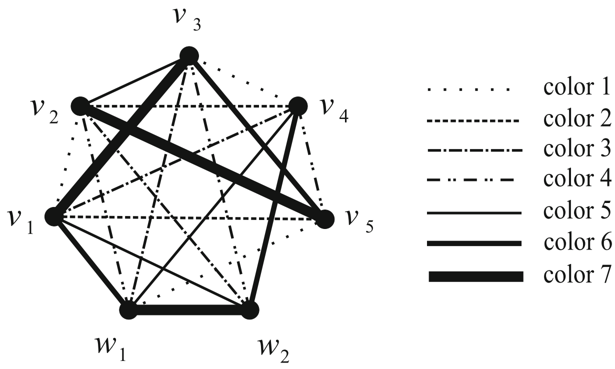

Fig. 3 A 7-edge coloring of B_1 with the desired properties

H' is Class 1. Clearly, $x_1, x_2, y_1 \in V(H'_\Delta)$. Note that since T is an independent set, $d_G(y_1) = 6$ and $y_1u_j \in E(G)$, for $j = 1, \ldots, 4$, then $e_G(y_1, \cup_{i=1}^4 B_i) = 2$ and so $|N(y_1) \cap V(G_\Delta)| = 2$. This implies that $|N(y_1) \cap V(H'_\Delta)| = 2$. It is not hard to see that H'_Δ is a unicyclic graph and $d_{H'_\Delta}(x_j) = 1$, for $j \in \{1, 2\} \setminus \{i\}$. Now, by Theorem 5, H' has a 7-edge coloring $\phi : V(H') \longrightarrow \{1, \ldots, 7\}$. Clearly, $|\{\phi(zx_1), \phi(zx_2), \phi(zy_1)\}| = 3$. Now, if $\phi(x_iy_1) \neq \phi(zx_j)$, where $j = \{1, 2\} \setminus \{i\}$, then with no loss of generality, we can assume that $\phi(zx_1) = 1, \phi(zx_2) = 2, \phi(zy_1) = 3$ and $\phi(x_iy_1) = 4$. Now, using Fig. 2, there exists a 7-edge coloring of B_1 such that color 1 is missed at w_1, colors 2, 3 are missed at w_2 and color 4 is missed at v_5 and so we obtain a 7-edge coloring of G. So, assume that $\phi(x_iy_1) = \phi(zx_j)$. Then, let $\phi(zy_1) = 1, \phi(zx_j) = \phi(x_iy_1) = 2$, where $j \neq i$, and $\phi(zx_i) = 3$.

Now, we introduce a 7-edge coloring of B_1, called θ, in which color 2 is missed at w_1, colors 1, 2 are missed at w_2 and color 3 is missed at v_5, see Fig. 3.

Now, define an edge coloring $c : E(G) \longrightarrow \{1, \ldots, 7\}$ as follows:
Let $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(H)$, $e' \in E(B_1)$ and $c(w_1x_1) = c(w_2x_2) = 2$, $c(w_2y_1) = 1$ and $c(v_5x_i) = 3$. This implies that G is Class 1 and in this case we are done.

Case 3 Suppose that $b_2 = 3$. Thus since $|G_\Delta| = 12$, $q \leq 6$. First note that if $b_1 \neq 0$, then $b \geq 4$ and by Cases 1 and 2 we are done. So, assume that $b_1 = 0$. Moreover, if $b_0 \neq 0$, then by (16),

$$
\begin{aligned}
&q - 2r + (|T| - b - s)(\Delta - 1) + (\Delta - 5)b_2 - 2b_1 - (\Delta + 1)b_0 \\
&\leq 6 - 2(\Delta - 1) + 3(\Delta - 5) - (\Delta + 1) \\
&< 0.
\end{aligned}
$$

This contradicts (12) and we are done. Thus one can assume that $b_0 = b_1 = 0$ and so $b = b_2 = 3$. Moreover, by (16) and noting that $b = 3$, we have

$$s = |T| - 1. \tag{21}$$

Also, by (6) and (7), we have

$$q\Delta + (|T| - q)(\Delta - 1) \geq e_G(T, G \setminus T) \geq 3 \times 4 + \sum_{j=1}^{s}(\Delta - |S_j|)|S_j|.$$

This implies that

$$q - 12 + (|T| - s)(\Delta - 1) - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \geq 0. \tag{22}$$

On the other hand by (21), we find that

$$\begin{aligned} &q - 12 + (|T| - s)(\Delta - 1) - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ &\leq 6 - 12 + \Delta - 1 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ &= \Delta - 7 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)). \end{aligned}$$

Note that if $\Delta \leq 6$, then we get a contradiction with (22). Thus, we can assume that

$$\Delta \geq 7. \tag{23}$$

Now, if $3 \leq |S_k| \leq \Delta - 2$, for some k, then $-\big((\Delta - |S_k|)|S_k| - (\Delta - 1)\big) \leq -\Delta + 3$ from which we conclude that

$$\begin{aligned} &\Delta - 7 - \sum_{j=1}^{s}((\Delta - |S_j|)|S_j| - (\Delta - 1)) \\ &\leq \Delta - 7 - \Delta + 3 \\ &< 0. \end{aligned}$$

This contradicts (22). So, we find that $|S_j| \in \{1, \Delta - 1\}$, for $j = 1, \ldots, s$. Now, we prove the following claim:

Claim 4 $|S_j| = 1$, for $j = 1, \ldots, s$.

Proof of Claim 4. First we show that $|S_j| = 1$, for some j, $1 \leq j \leq s$. To see this, by the contrary assume that $|S_j| = \Delta - 1$, for $j = 1, \ldots, s$. Now, since $|S_j|$ is odd, Δ is even and so $|B_i| \geq \Delta + 1$, for $i = 1, 2, 3$. Now, by (21),

$$6\Delta \geq n \geq 3(\Delta + 1) + |T| + (|T| - 1)(\Delta - 1).$$

This implies that $|T| \leq 3$. Now, since $b = 3$, by (15), $q \geq 3$ and so by (13), $|T| = 3$. Also, by (13) and noting that $b_2 = 3$, $(\cup_{j=1}^{s} S_j) \cap V(G_\Delta) = \emptyset$ and so by (3), $e_G(T, S_j) \geq 2\Delta - 2$, for $j = 1, \ldots, s$. Now, since $s = |T| - 1 = 2$, we have

$$3\Delta \geq e_G(T, G \setminus T) \geq 3 \times 4 + 2(2\Delta - 2),$$

a contradiction. Thus, with no loss of generality assume that $S_1 = \{u_1\}$. Then $|T| \geq \Delta - 1$. Now, to complete the proof of the claim, suppose the contrary. Then with no loss of generality we may assume that $|S_s| = \Delta - 1$. Then Δ is even and so $|B_i| \geq \Delta + 1$. Thus

$$6\Delta \geq n \geq 3(\Delta + 1) + |T| + |T| - 2 + \Delta - 1 \tag{24}$$

which implies that $|T| \leq \Delta$. Two cases may occur:

First assume that $|T| = \Delta$ and so by (24), $n = 6\Delta$. Now, if $q \geq 4$, then since $N(u_1) \subseteq T$, we have $u_1 \notin V(G_\Delta)$. Moreover, $|N(u_1) \cap V(G_\Delta)| \geq 3$ and by (3) we conclude that

$$12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 3 + 2(n - 13)$$

which implies that $n < 6\Delta$, a contradiction. Thus, suppose that $q \leq 3$. Now, by $b_2 = 3$ and (13), we have $q = 3$. Now, by (13), $(\cup_{j=1}^{s} S_j) \cap V(G_\Delta) = \emptyset$. Thus, by (3), $e_G(T, S_s) \geq 2(\Delta - 1)$ and so the following holds:

$$3\Delta + (\Delta - 3)(\Delta - 1) \geq e_G(T, G \setminus T) \geq 3 \times 4 + (\Delta - 2)(\Delta - 1) + 2(\Delta - 1),$$

a contradiction.

Now, suppose that $|T| = \Delta - 1$. Thus, by (21), $s = \Delta - 2$. First note that if there exists $k \neq s$ such that $|S_k| = \Delta - 1$, then

$$6\Delta \geq n \geq 3(\Delta + 1) + (\Delta - 1) + 2(\Delta - 1) + (\Delta - 4).$$

This implies that $\Delta \leq 4$, a contradiction with (23). Thus we can assume that $S_i = \{u_i\}$, for $i = 1, \ldots, s - 1$ and $|S_s| = \Delta - 1$. Hence

$$n \geq 3(\Delta + 1) + \Delta - 1 + \Delta - 3 + \Delta - 1 = 6\Delta - 2.$$

Now, since $q \geq 3$, $|N(u_i) \cap V(G_\Delta)| \geq 3$, for $i = 1, \ldots, \Delta - 3$. So, we have

$$12(\Delta - 2) = e_G(G_\Delta, G \setminus G_\Delta) \geq 3(\Delta - 3) + 2(n - (\Delta - 3 + 12)).$$

This implies that $n \leq \frac{11\Delta+3}{2}$. Now, since Δ is even, $n \leq \frac{11\Delta+2}{2}$ and $n \geq 6\Delta - 2$, we obtain a contradiction of (23). So, the proof of the claim is complete and we have $S_j = \{u_j\}$, for $j = 1, \ldots, s$.

Now, two cases may be occurred:

First assume that $u_1 \in V(G_\Delta)$. Then u_1 is joined to 2 vertices of degree Δ, and so to $\Delta-2$ vertices of degree $\Delta-1$, so that $|T| \geq q+\Delta-2$. Let $K = \langle \cup_{i=1}^{3} V(B_i) \cup \{u_1\} \cup T \rangle$. Moreover, since $b_2 = 3$, there are at least 4 components of $G \setminus T$ each of them containing at least a vertex of $V(G_\Delta)$. Now, by the Claim 1, $V(G_\Delta) \subseteq V(K)$. Thus by (3), for every vertex $v \in T$, $d_K(v) \geq 2$. So, using (21) we have

$$q(\Delta-2) + (|T|-q)(\Delta-3) \geq e_G\left(T, \cup_{j=2}^{s} S_j\right) = (|T|-2)(\Delta-1).$$

This implies that $2|T| \leq q + 2\Delta - 2$. Now, since $|T| \geq q + \Delta - 2$, we have $q \leq 2$ which is a contradiction.

Next, suppose that $(\cup_{j=1}^{s} S_j) \cap V(G_\Delta) = \emptyset$. Then

$$q(\Delta-2) + (|T|-q)(\Delta-3) \geq e_G\left(T, \cup_{j=1}^{s} S_j\right) = (|T|-1)(\Delta-1).$$

So, $2|T| \leq q + \Delta - 1$. Now, since $|T| \geq \Delta - 1$, we conclude that $q \geq \Delta - 1$. Now, since $\Delta \geq 7$ and $q \leq 6$, we conclude that $q = 6$, $\Delta = 7$ and $|T| = \Delta - 1 = 6$ and $s = 5$. Thus, we find that $|V(B_i) \cap V(G_\Delta)| = 2$, for $i = 1, 2, 3$. On the other hand, since for every $v \in T$, $e_G(v, \cup_{j=1}^{s} S_j) = 5$, $e_G(v, B_1 \cup B_2 \cup B_3) \leq 2$. Thus, by (6) and noting that $b = 3$, we have

$$6 \times 2 \geq e_G(T, B_1 \cup B_2 \cup B_3) \geq 3 \times 4.$$

This implies that T is an independent set and $e_G(T, B_i) = 4$, for $i = 1, 2, 3$. Thus, we find that for every $e = uv$, $u \in \cup_{i=1}^{3} V(B_i)$ and $v \in T$, $e \in E(G_\Delta)$. Moreover, by (3) and since for every u_i, $i = 1, \ldots, 5$, $N(u_i) = T$ and $|T| = q = 6$, we have

$$12 \times 5 = e_G(G_\Delta, G \setminus G_\Delta) \geq 5 \times 6 + 2(n - 17).$$

This implies that $n \leq 32$. On the other hand, $n \geq 3 \times 7 + 6 + 5$ and so $n = 32$. Thus, we have $|B_i| = 7$, for $i = 1, 2, 3$. Let $V(B_1) = \{v_1, \ldots, v_5, w_1, w_2\}$, where $V(B_1) \cap V(G_\Delta) = \{w_1, w_2\}$. Since $|(\cup_{i=1}^{3} V(B_i)) \cap V(G_\Delta)| = 6$ and T is an independent set, $e_G(T, w_i) = 2$, for $i = 1, 2$. Let $N(w_1) \cap T = \{x_1, x_2\}$ and $N(w_2) \cap T = \{x_3, x_4\}$. Since $G_\Delta = C_{12}$, $|\{x_1, x_2\} \cap \{x_3, x_4\}| \leq 1$. Let $H = \langle V(G) \setminus V(B_1) \rangle$. Add a new vertex z to H and join z to each vertex v contained in $\{x_1, x_2\} \cup \{x_3, x_4\}$ and call the resultant graph by H'. Clearly, H'_Δ is obtained from C_{12} by removing four edges, so H'_Δ is a forest and so by Theorem 2, H' has a 7-edge coloring $\phi : V(H') \longrightarrow \{1, \ldots, 7\}$. Now, we consider two following cases:

- $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$.

With no loss of generality, let $\phi(zx_i) = i$, for $i = 1, \ldots, 4$. Now, we introduce a 7-edge coloring of B_1, called θ, in which the colors $1, 2$ are missed at w_1 and the colors $3, 4$ are missed at w_2, see Fig. 4.

Now, define an edge coloring $c : E(G) \longrightarrow \{1, \ldots, 7\}$ as follows: Let $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(H)$, $e' \in E(B_1)$ and $c(w_1x_1) = 1$, $c(w_1x_2) = 2$, $c(w_2x_3) = 3$ and $c(w_2x_4) = 4$. Thus we obtain a 7-edge coloring of G and so G is Class 1 and in this case we are done.

- $N(w_1) \cap T = \{x_1, x_2\}$ and $N(w_2) \cap T = \{x_2, x_3\}$.

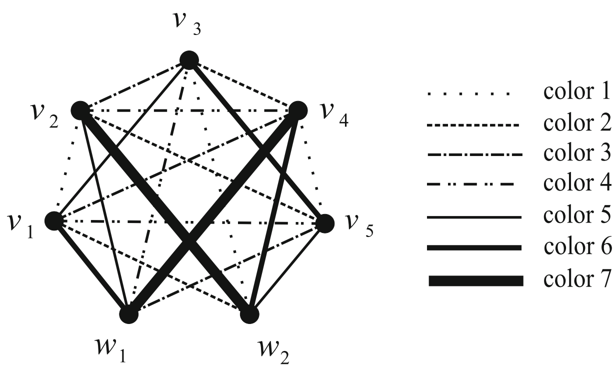

Fig. 4 A 7-edge coloring of B_1 with the desired properties

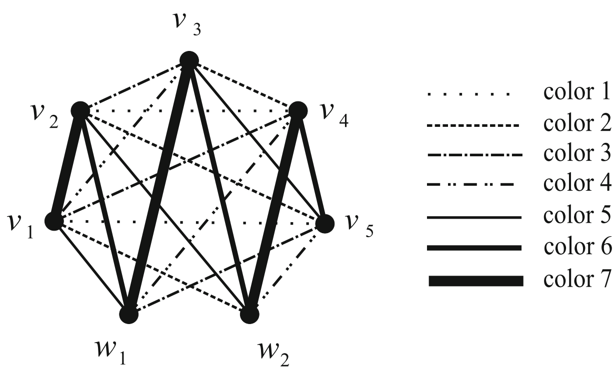

Fig. 5 A 7-edge coloring of B_1 with the desired properties

Let $\phi(zx_i) = i$, for $i = 1, 2, 3$. Since $d_{H'}(x_2) = 6$, there exists a color α which is missed at x_2. If $\alpha \notin \{1, 3\}$, then with no loss of generality let $\alpha = 4$. Then by Fig. 4, there exists a 7-edge coloring of B_1 such that colors 1, 2 are missed at w_1 and colors 3, 4 are missed at w_2. Therefore G is Class 1 and in this case we are done. So, with no loss of generality assume that $\alpha = 1$. Now, we introduce a 7-edge coloring of B_1, called θ, in which colors 1, 2 are missed at w_1 and colors 1, 3 are missed at w_2, see Fig. 5.

Now, define an edge coloring $c : E(G) \longrightarrow \{1, \ldots, 7\}$ as follows:
Let $c(e) = \phi(e)$ and $c(e') = \theta(e')$, for every $e \in E(H)$, $e' \in E(B_1)$ and $c(w_1x_1) = c(w_2x_2) = 1$, $c(w_1x_2) = 2$ and $c(w_2x_3) = 3$. Therefore G is Class 1 and in this case we are done.

Thus by the assumption of theorem we showed that G has a 1-factor or G is Class 1. Call the 1-factor of G by M. Let $H = G \setminus M$. If H_Δ is a forest, then by Theorem 2, H is Class 1 and G is Class 1, too. If $H_\Delta = C_{12}$, then by (3), $|N(v) \cap V(H_\Delta)| \geq 1$, for every $v \in V(G) \setminus V(G_\Delta)$ and so H is connected. Now, by the induction hypothesis H is Class 1. Thus G is Class 1 and we are done. This completes the proof.

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References

1. Akbari, S., Cariolaro, D., Chavoshi, M., Ghanbari, M., Zare, S.: Some Criteria for a graph to be Class 1. Discrete Math. **312**, 2593–2598 (2012)
2. Akbari, S., Ghanbari, M., Kano, M., Nikmehr, M.J.: The chromatic index of a graph whose core has maximum degree 2. Electron. J. Comb. **19**, P58 (2012)
3. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications. North Holland, New York (1976)
4. Cariolaro, D., Cariolaro, G.: Colouring the petals of a graph. Electron. J. Comb. **10**, R6 (2003)
5. Chetwynd, A.G., Hilton, A.J.W.: A Δ-subgraph condition for a graph to be Class 1. J. Combin. Theory Ser. B. **46**, 37–45 (1989)
6. Chetwynd, A.G., Hilton, A.J.W.: 1-factorizing regular graphs of high degree—an improved bound. Discret. Math. **75**, 103–112 (1989)
7. Dugdale, J.K., Hilton, A.J.W.: A sufficient condition for a graph to be the core of a Class 1 graph. Comb. Probab. Comput. **9**, 97–104 (2000)
8. Fiorini, S., Wilson R.J.: Edge-Colourings of Graphs. Research Notes in Mathematics, Pitman (1977)
9. Fournier, J.-C.: Coloration des arêtes d'un graphe. Cahiers du CERO (Bruxelles) **15**, 311–314 (1973)
10. Hilton, A.J.W.: Two conjectures on edge-colouring. Discrete Math. **74**, 61–64 (1989)
11. Hilton, A.J.W., Zhao, C.: A sufficient condition for a regular graph to be Class 1. J. Graph Theory **17**(6), 701–712 (1993)
12. Hilton, A.J.W., Zhao, C.: On the edge-colouring of graphs whose core has maximum degree two. J. Comb. Math. Comb. Comput. **21**, 97–108 (1996)
13. Hilton, A.J.W., Zhao, C.: The chromatic index of a graph whose core has maximum degree two. Discrete Math **101**, 135–147 (1992)
14. Hoffman, D.G.: Cores of Class II graphs. J. Graph Theory **20**, 397–402 (1995)
15. Holyer, I.: The NP-completeness of edge-colouring. SIAM J. Comput. **10**, 718–720 (1981)
16. Song, Z., Yap, H.P.: Chromatic index critical graphs of even order with five major vertices. Graph Comb. **21**, 239–246 (2005)
17. Vizing, V.G. : On an estimate of the chromatic class of a *p*-graph. Diskret. Analiz. **3**, 25–30 (in Russian) (1964)
18. Vizing, V.G. : Critical graphs with a given chromatic class. Diskret. Analiz. **5**, 6–17 (in Russian) (1965)
19. Zhanga, L., Shi, W., Huanga, X., Li, G.: New results on chromatic index critical graphs. Discrete Math. **309**, 3733–3737 (2009)