Graphs and Combinatorics (2014) 30:183–195
DOI 10.1007/s00373-012-1251-2

ORIGINAL PAPER

d-strong Edge Colorings of Graphs

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Received: 1 April 2011 / Revised: 18 June 2012 / Published online: 30 October 2012

Abstract For a proper edge coloring of a graph G the palette $S(v)$ of a vertex v is the set of the colors of the incident edges. If $S(u) \neq S(v)$ then the two vertices u and v of G are distinguished by the coloring. A d-strong edge coloring of G is a proper edge coloring that distinguishes all pairs of vertices u and v with distance $1 \leq d(u, v) \leq d$. The d-strong chromatic index $\chi'_d(G)$ of G is the minimum number of colors of a d-strong edge coloring of G. Such colorings generalize strong edge colorings and adjacent strong edge colorings as well. We prove some general bounds for $\chi'_d(G)$, determine $\chi'_d(G)$ completely for paths and give exact values for cycles disproving a general conjecture of Zhang et al. (Acta Math Sinica Chin Ser 49:703–708 (2006)).

Keywords Edge coloring · Strong edge coloring · Observability · Vertex distinguishing index

Mathematics Subject Classification (2010) 05C15

1 Introduction

If $c : E \to \{1, 2, \ldots, k\}$ is a proper edge coloring of a graph $G = (V, E)$ then the *palette* $S(v)$ of a vertex $v \in V$ is the set of colors of the incident edges: $S(v) = \{c(e) : e = vw \in E\}$. An edge coloring c *distinguishes* vertices u and v if $S(u) \neq S(v)$. A *d-strong edge coloring* of G is a proper edge coloring that distinguishes all pairs of vertices u and v with distance $1 \leq d(u, v) \leq d$. The minimum number of colors of a d-strong edge coloring is called *d-strong chromatic index* $\chi'_d(G)$ of G. A d-strong

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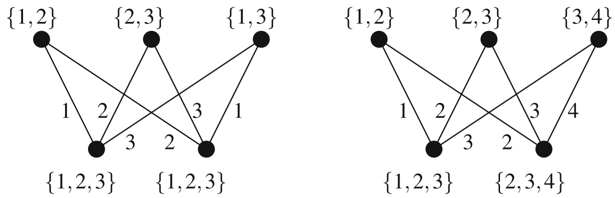


Fig. 1 $\chi'_1(K_{2,3}) = 3,\ \chi'_2(K_{2,3}) = 4$

edge coloring of G exists if and only if G does not contain a complete graph K_2 as a component.

This chromatic invariant is also called $D(d)$-vertex distinguishing edge chromatic number (see [14]).

Figure 1 shows a 1-strong edge coloring of the complete bipartite graph $K_{2,3}$ with 3 colors and a 2-strong edge coloring with 4 colors.

In [12] and [6] the *strong chromatic index* $\chi'_s(G)$ was introduced as the minimum number of colors of a proper edge coloring of a graph G without a component K_2 and without two components K_1 that distinguishes every pair of distinct vertices. Other names of this chromatic invariant are *vertex distinguishing index* vdi(G) (see [13]) or *observability* obs(G) (see [12]).

On the other hand, in [17] the *adjacent strong chromatic index* $\chi'_{as}(G)$ is defined as the minimum number of colors of a proper edge coloring of a graph G without components K_2 that distinguishes every pair of adjacent vertices. Other names which are used for this chromatic invariant are *neighbor distinguishing index* ndi(G) (see [9]) or *adjacent vertex distinguishing chromatic index* $\chi'_a(G)$ (see [15]).

Note that $\chi'_{as}(G) = \chi'_1(G)$ and that for connected graphs $\chi'_s(G) = \chi'_d(G)$ if $d \geq \text{diam}(G)$ where diam(G) is the *diameter* of the graph G. Therefore, d-strong edge colorings are generalizations of strong edge colorings and of adjacent strong edge colorings as well.

The following properties of $\chi'_d(G)$ are obvious.

Lemma 1 (Monotonicity) *If* $d \leq t$ *then* $\chi'_d(G) \leq \chi'_t(G)$.

Proof A t-strong edge coloring of G with $t \geq d$ clearly distinguishes all pairs of vertices of distance at most d. □

Lemma 2 (Additivity) *If* $G = H_1 \cup H_2$ *then* $\chi'_d(G) = \max\{\chi'_d(H_1), \chi'_d(H_2)\}$.

Proof The components of G can be colored independently since vertices in different components have not to be distinguished. □

If G is a subgraph of H, $G \subseteq H$, then note that this does not imply $\chi'_d(G) \leq \chi'_d(H)$ in general (see Fig. 2 for a counterexample), that is, the property $\chi'_d(G) \leq k$ is not a hereditary property.

Some partial results on d-strong edge colorings can be found in [1,14,16].

In this note we will present some general bounds on d-strong chromatic indexes as well as some results for specific graph classes as paths and cycles.

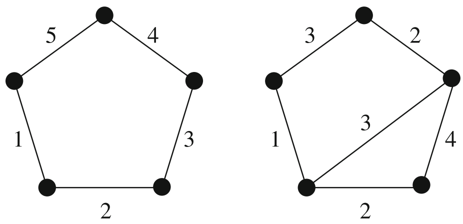


Fig. 2 $\chi'_s(C_5) = 5$, $\chi'_s(C_5 + e) = 4$

2 General Upper Bounds

Let G be a graph of order p without isolated edges and with at most one isolated vertex. It is easy to prove that $\chi'_s(G) \le \Delta(G) + p - 1$ (see [10]) where $\Delta(G)$ is the maximum degree of G. It is conjectured in [7] that $\chi'_s(G) \le p + 1$ which was proved in [3]. For graphs G with minimum degree $\delta(G) > \frac{p}{3}$ it holds $\chi'_s(G) \le \Delta(G) + 5$ (see [4]).

Soták [12] and independently Burris and Shelp [6,7] conjectured that the strong chromatic index attains as the chromatic index just one of two values. If d_i is the number of vertices of G of degree i and $\mu(G) = \max\{\min\{j : \binom{j}{i} \ge d_i\} : \delta(G) \le i \le \Delta(G)\}$ then they conjectured that $\mu(G) \le \chi'_s(G) \le \mu(G) + 1$. This conjecture is true, e.g., for paths P_p, cycles C_p, complete graphs K_p, complete bipartite graphs $K_{m,n}$, and wheels W_p.

Obviously, any upper bound for $\chi'_s(G)$ is also an upper bound for $\chi'_d(G)$.

Let now G be a graph of order p without isolated edges. It is proved in [1] that $\chi'_1(G) \le 3\Delta(G)$. In [2] it is shown that $\chi'_1(G) \le \Delta(G) + O(\log k)$ where $k = \chi(G)$ is the chromatic number of G which implies that $\chi'_1(G) \le 2\Delta(G)$.

It is conjectured (see [17]) that $\chi'_1(G) \le \Delta(G) + 2$ for connected graphs of order at least $p \ge 3$ that are not isomorphic to the cycle C_5. This conjecture is true, e.g., for paths P_p, cycles C_p, complete graphs K_p, complete bipartite graphs $K_{m,n}$, and wheels W_p.

In [14] it is proved by probabilistic methods that $\chi'_2(G) \le 32\Delta(G)^2$ if $\Delta(G) \ge 4$, $\chi'_3(G) \le 8\Delta(G)^{5/2}$ if $\Delta(G) \ge 6$ and $\chi'_d(G) \le 2\sqrt{2(d-1)}\Delta(G)^{(d+2)/2}$ for $d \ge 4$ if $\Delta(G) \ge 4$.

Let n_i denote the maximum number of vertices of degree i that are of pairwise distance at most d and let

$$\mu_d(G) = \max\{\min\{j : \binom{j}{i} \ge n_i\} : \delta(G) \le i \le \Delta(G)\}.$$

Obviously, $\chi'_d(G) \ge \mu_d(G)$. Zhang et al. [16] conjectured that the d-strong chromatic index attains one of two possible values (as the strong chromatic index, see above).

Conjecture 1 [16] $\chi'_d(G) \le \mu_d(G) + 1$ *for all graphs G and $d \ge 2$.*

We prove in Sect. 4 that Conjecture 1 is not true in general.

3 Paths

The strong chromatic index of paths is well-known.

Theorem 1 [7,8] *Let $p \geq 3$ and j be the minimum integer such that $\binom{j}{2} \geq p-2$. Then $\chi'_s(P_p) = j+1$ if j is odd and $p = (j^2 - j + 4)/2$ or if j is even and $p > (j^2 - 2j + 6)/2$, and $\chi'_s(P_p) = j$ otherwise.*

Note that $j = \mu(P_p)$ for $p \geq 3$ since P_p has $p-2 \geq 1$ vertices of degree 2 and 2 vertices of degree 1.

The result of Theorem 1 can be rewritten as follows.

Corollary 1 *Let $p \geq 3$. Then*

$$\chi'_s(P_p) = k \text{ if } \begin{cases} k \text{ odd and } \binom{k-1}{2} - \frac{k-1}{2} + 4 \leq p \leq \binom{k}{2} + 1, \\ k \text{ even and } \binom{k-1}{2} + 2 \leq p \leq \binom{k}{2} - \frac{k}{2} + 3. \end{cases}$$

Note that the sequence $(\chi'_s(P_p))$ is nondecreasing with increasing p. The proof of Corollary 1 is similar to the following proof of Theorem 2.

Theorem 2 *Let P_∞ be a one-sided or two-sided infinite path. Then*

$$\chi'_d(P_\infty) = k \text{ if } \begin{cases} k \text{ odd and } \binom{k-1}{2} - \frac{k-1}{2} \leq d \leq \binom{k}{2} - 1, \\ k \text{ even and } \binom{k-1}{2} \leq d \leq \binom{k}{2} - \frac{k}{2} - 1. \end{cases}$$

Note that the sequence $(\chi'_d(P_\infty))$ is nondecreasing with increasing d and that Theorem 2 covers all $d \geq 1$.

Proof The key idea is the following: The d-strong edge colorings of the path P_∞ with k colors correspond to the one-sided or two-sided infinite walks in the complete graph K_k where in the walks between same edges there must be at least d other edges, and vice versa. The edge colors of P_∞ correspond to the vertices of K_k, and the palettes of the vertices of P_∞ correspond to the sets of end-vertices of the edges of K_k.

Let $k \geq 3$ be odd. In a d-strong edge coloring of P_∞ each $d+1$ successive palettes must be pairwise different. This corresponds to a trail of length $d+1$ in a complete graph with the set of colors as vertex set. Since each trail in K_{k-1} is of length at most $\binom{k-1}{2} - \frac{k-1}{2} + 1$ (it corresponds to a subgraph of K_{k-1} having 0 or 2 vertices of odd degree) it holds $d+1 \leq \binom{k-1}{2} - \frac{k-1}{2} + 1$ which implies $\chi'_d(P_\infty) \geq k$ for $d \geq \binom{k-1}{2} - \frac{k-1}{2} + 1$.

The complete graph K_k is Eulerian and has an Eulerian circuit that contains all $\binom{k}{2}$ edges exactly once. A periodic coloring of the edges of P_∞ with respect to this Eulerian circuit results in an edge coloring of P_∞ that generates all $\binom{k}{2}$ palettes at the vertices. Equal palettes occur at distance $\binom{k}{2}$ which implies that this edge coloring is d-strong for $1 \leq d \leq \binom{k}{2} - 1$.

If $d = \binom{k-1}{2} - \frac{k-1}{2}$ and $k \geq 5$ ($k = 3$ implies $d = 0$) then assume that $\chi'_d(P_\infty) \leq k-1$, i.e., there is a d-strong edge coloring of P_∞ with $k-1$ colors. Since $d+1$

successive palettes must be pairwise different and the maximum length of a trail in K_{k-1} is $d+1=\binom{k-1}{2}-\frac{k-1}{2}+1$ the palettes must repeat periodically with period length $d+1$. Successive palettes have exactly one color in common which implies that also the edge colors periodically repeat with period length $d+1$. This corresponds to a circuit in K_{k-1} of length $\binom{k-1}{2}-\frac{k-1}{2}+1$ which is not possible. Therefore, $\chi'_d(P_\infty)\geq k$. This implies $\chi'_d(P_\infty)=k$ for $\binom{k-1}{2}-\frac{k-1}{2}\leq d\leq\binom{k}{2}-1$ if k is odd.

Let $k\geq 4$ be even. If $d\geq\binom{k-1}{2}$ then $\chi'_d(P_\infty)\geq k$ since $n_2=d+1\geq\binom{k-1}{2}+1$ vertices must be distinguishable which is not possible with $k-1$ colors.

Since K_k without $\frac{k}{2}$ independent edges is Eulerian there exists a circuit of length $\binom{k}{2}-\frac{k}{2}$ in K_k. As above one obtains a d-strong edge coloring of P_∞ with k colors for $1\leq d\leq\binom{k}{2}-\frac{k}{2}-1$ by a periodic edge coloring of P_∞. This implies $\chi'_d(P_\infty)=k$ for $\binom{k-1}{2}\leq d\leq\binom{k}{2}-\frac{k}{2}-1$ if k is even. □

In the following we consider d-strong edge colorings of finite paths P_p.

If $p>d+1$ then the palettes of the end-vertices of a path P_p are not forced to be different which implies that $\chi'_d(P_{d+2})\leq\chi'_d(P_{d+3})\leq\ldots\leq\chi'_d(P_\infty)$ since a d-strong edge coloring of P_{p+i}, $i\geq 1$, induces a d-strong edge coloring of P_p.

Consider a d-strong edge coloring of P_{d+2}. Then the coloring induced on at least one of the two subpaths P_{d+1} of P_{d+2} is a d-strong edge coloring which implies $\chi'_d(P_{d+1})\leq\chi'_d(P_{d+2})$.

Since $\chi'_d(P_p)=\chi'_s(P_p)$ if $p\leq d+1$ and $(\chi'_s(P_p))$, $p\geq 3$, is nondecreasing it holds that $\chi'_d(P_3)\leq\chi'_d(P_4)\leq\ldots\leq\chi'_d(P_{d+1})$.

Altogether it follows

Lemma 3 $\chi'_d(P_3)\leq\chi'_d(P_4)\leq\ldots\leq\chi'_d(P_p)\leq\ldots\leq\chi'_d(P_\infty)$.

In the following all values of the d-strong chromatic indexes of finite paths will be determined. In the proof of the general result of Theorem 3 we use the observation of the following proposition.

Proposition 1 ([17]) $\chi'_1(P_p)=\chi'_2(P_p)=\begin{cases}2 & \textit{if } p=3,\\ 3 & \textit{if } p\geq 4.\end{cases}$

Proof Coloring the edges of P_3 with two colors results in a strong edge coloring implying that $\chi'_1(P_3)=\chi'_2(P_3)=\chi'_s(P_3)=2$. If $p\geq 4$ any three successive edges of P_p must be colored with three distinct colors which gives $\chi'_2(P_p)\geq\chi'_1(P_p)\geq 3$ according to Lemma 1. A periodic coloring $(1,2,3,1,2,3,\ldots)$ of the edges of P_p is 2-strong. Therefore, $\chi'_2(P_p)=\chi'_1(P_p)=3$ if $p\geq 4$. □

In the proof of the following theorem methods of the proof of Theorem 2 are used.

Theorem 3 *Let* $d\geq 1$, $p\geq 3$, *and* $m=\min\{\chi'_d(P_\infty),\chi'_s(P_p)\}$. *Then*

$$\chi'_d(P_p)=\begin{cases}m-1 & \textit{if } m\geq 4 \textit{ even}, d=\binom{m-1}{2}, \textit{ and } p=d+2,\\ & \textit{or } m\geq 5 \textit{ odd}, d=\binom{m-1}{2}-\frac{m-1}{2}, \textit{ and } p=d+4,\\ m & \textit{otherwise}.\end{cases}$$

Proof If $m \geq 4$ is even, $d = \binom{m-1}{2}$, and $p = d + 2$ then $n_2 = \min\{p - 2, d + 1\} = d = \binom{m-1}{2}$ where n_2 is the maximum number of vertices in P_p of degree 2 and of pairwise distance at most d. Color the edges of P_{d+2} with $m - 1$ colors according to an Eulerian circuit of the complete graph K_{m-1} implying that the d inner vertices of the path contain mutually distinct palettes. This edge coloring is d-strong since the distance of the end-vertices of P_{d+2} is larger than d. On the other hand, $\chi'_d(P_{d+2}) \geq \chi'_d(P_{d+1}) = \chi'_s(P_{d+1}) = m-1$ according to Lemma 3 and Theorem 1. Therefore, $\chi'_d(P_{d+2}) = m - 1$.

If $m \geq 5$ is odd, $d = \binom{m-1}{2} - \frac{m-1}{2}$, and $p = d+4$ then $n_2 = \min\{p-2, d+1\} = d + 1 = \binom{m-1}{2} - \frac{m-1}{2} + 1$. Let G be the complete graph K_{m-1} without $\frac{m-1}{2}$ independent edges, $Z = (y, \ldots, y)$ an Eulerian circuit of G and $xy \notin E(G)$. Color the inner edges of P_{d+4} according to Z with $m - 1$ colors and the end-edges with color x. This implies that the end-vertices of P_{d+4} have the same palettes and the same is true for the two neighbors of the end-vertices of P_{d+4}. The edge coloring is d-strong since the vertices with same palettes are of distance larger than d which implies that $\chi'_d(P_{d+4}) \leq m-1$. On the other hand, $\chi'_d(P_{d+4}) \geq \chi'_d(P_{d+1}) = \chi'_s(P_{d+1}) = m-1$ which implies equality.

In the following it is shown that $\chi'_d(P_p) = m$ in all other cases.

According to Lemma 3 and Lemma 1 it holds that $\chi'_d(P_p) \leq \chi'_d(P_\infty)$ and $\chi'_d(P_p) \leq \chi'_s(P_p)$ and therefore $\chi'_d(P_p) \leq m$.

If $m = 2$ then $p = 3$ according to Theorems 1 and 2 and therefore $\chi'_d(P_p) \geq \chi'_1(P_3) = 2 = m$ by Proposition 1.

If $m = 3$ then $p = 4$ or $p \geq 5$ and $d \leq 2$ and thus $\chi'_d(P_p) \geq \chi'_1(P_4) = 3 = m$ again by Proposition 1.

If $m \geq 4$ is even then $d \geq \binom{m-1}{2}$ and $p \geq \binom{m-1}{2}+2$. If $d \geq \binom{m-1}{2}$ and $p > \binom{m-1}{2}+2$ then, by Lemmas 1 and 3, $\chi'_d(P_p) \geq \chi'_{d^*}(P_{p^*})$ with $d^* = \binom{m-1}{2}$ and $p^* = \binom{m-1}{2}+3$. In this case, $n_2 = \min\{p^* - 2, d^* + 1\} = d^* + 1 = \binom{m-1}{2} + 1$ implying that $\chi'_{d^*}(P_{p^*}) \geq m$ since $m - 1$ colors induce at most $\binom{m-1}{2}$ pairwise distinct 2-element palettes.

If $m \geq 4$ is even, $d > \binom{m-1}{2}$, and $p \geq \binom{m-1}{2} + 2$ then $\chi'_d(P_p) \geq \chi'_{d^*}(P_{p^*})$ with $d^* = \binom{m-1}{2} + 1$ and $p^* = \binom{m-1}{2} + 2$. In a d^*-strong edge coloring of P_{p^*} with $m - 1$ colors all mutually distinct $\binom{m-1}{2}$ palettes must occur at the inner vertices of P_{p^*} implying that the edges of P_{p^*} must be colored according to an Eulerian circuit of K_{m-1}. This induces that the end-vertices of P_{p^*} which are at distance $p^* - 1 = d^*$ have the same palettes which is not possible. Therefore, $\chi'_{d^*}(P_{p^*}) \geq m$.

If $m \geq 5$ is odd then $d \geq \binom{m-1}{2} - \frac{m-1}{2}$ and $p \geq \binom{m-1}{2} - \frac{m-1}{2} + 4$ according to Theorems 1 and 2. If $d \geq \binom{m-1}{2} - \frac{m-1}{2}$ and $p > \binom{m-1}{2} - \frac{m-1}{2} + 4$ then $\chi'_d(P_p) \geq \chi'_{d^*}(P_{p^*})$ with $d^* = \binom{m-1}{2} - \frac{m-1}{2}$ and $p^* = \binom{m-1}{2} - \frac{m-1}{2} + 5$. Therefore, $n_2 = \min\{p^* - 2, d^* + 1\} = d^* + 1 = \binom{m-1}{2} - \frac{m-1}{2} + 1$ which means that any $d^* + 1$ successive 2-element palettes of P_{p^*} must be distinct. Since a longest trail in K_{m-1} is of length $d^* + 1$ the palettes and therefore also the edge colors periodically repeat with period length $d^* + 1$. By this a circuit of length $d^* + 1 = \binom{m-1}{2} - \frac{m-1}{2} + 1$ would be induced in K_{m-1} which is not possible. Therefore, $\chi'_{d^*}(P_{p^*}) \geq m$.

If $m \geq 5$ is odd, $d > \binom{m-1}{2} - \frac{m-1}{2}$, and $p \geq \binom{m-1}{2} - \frac{m-1}{2} + 4$ then $\chi'_d(P_p) \geq \chi'_{d^*}(P_{p^*})$ with $d^* = \binom{m-1}{2} - \frac{m-1}{2} + 1$ and $p^* = \binom{m-1}{2} - \frac{m-1}{2} + 4$. Since $n_2 = \min\{p^* - 2, d^* + 1\} = d^* + 1 = \binom{m-1}{2} - \frac{m-1}{2} + 2$ successive 2-element palettes must be distinct in this subcase but the complete graph K_{m-1} has only trails of length at most $\binom{m-1}{2} - \frac{m-1}{2} + 1$ it follows that $\chi'_{d^*}(P_{p^*}) \geq m$. □

4 Cycles

In this section we consider the class of cycles C_p. Let $V(C_p) = \{v_0, v_1, \ldots, v_{p-1}\}$ and $E(C_p) = \{e_i : e_i = v_i v_{(i+1) \bmod p}, 0 \leq i \leq p-1\} = \{v_0v_1, v_1v_2, \ldots, v_{p-2}v_{p-1}, v_{p-1}v_0\}$.

In Theorem 4 we repeat the result for the strong chromatic index of cycles. This result can be rewritten by elementary transformations—see Corollary 2. The succeeding results provide general lower and upper bounds for the d-strong chromatic index $\chi'_d(C_p)$. In Propositions 2–6 we give exact values for $\chi'_d(C_p)$ for $d = 1, \ldots, 6$.

Theorem 4 ([7,8]) *Let $p \geq 3$ and j be the minimum integer such that $\binom{j}{2} \geq p$. Then*

$$\chi'_s(C_p) = \begin{cases} j+1 & \text{if } j \text{ is odd and } \binom{j}{2} - 2 \leq p \leq \binom{j}{2} - 1 \text{ or} \\ & \text{if } j \text{ is even and } p > (j^2 - 2j)/2, \\ j & \text{otherwise.} \end{cases}$$

Note that $j = \mu(C_p)$ since C_p has p vertices of degree 2.

The result of Theorem 4 can be rewritten as follows.

Corollary 2 *Let $p \geq 3$. Then*

$$\chi'_s(C_p) = k \text{ if } \begin{cases} k \text{ odd and } \binom{k-1}{2} - \frac{k-1}{2} + 1 \leq p \leq \binom{k}{2} - 3 \text{ or } p = \binom{k}{2}, \\ k \text{ even and } p = \binom{k-1}{2} - 2,\ p = \binom{k-1}{2} - 1,\ \text{ or } \binom{k-1}{2} + 1 \leq p \leq \binom{k}{2} - \frac{k}{2}. \end{cases}$$

Since $\operatorname{diam}(C_p) = \lfloor p/2 \rfloor$ for cycles C_p it follows that

$$\chi'_d(C_p) = \chi'_s(C_p) \text{ if } d \geq \lfloor p/2 \rfloor.$$

Therefore, we consider in the following the case that $d < \lfloor p/2 \rfloor$, i.e., $p \geq 2d + 2$. Lemma 4 provides a general lower bound for the d-strong chromatic index.

Lemma 4 *Let $p \geq 2d + 2$. Then $\chi'_d(C_p) \geq \mu_d(C_p) = \left\lceil \frac{1}{2} + \frac{1}{2}\sqrt{8d+9} \right\rceil$.*

Proof If $p \geq 2d + 2$ then $n_2 = d + 1$ implying that $\mu_d(C_p) = \min\{j : \binom{j}{2} \geq n_2 = d+1\}$. This condition for j is equivalent to $\mu_d(C_p) = j$ with $\binom{j}{2} = \frac{1}{2}j(j-1) \geq d + 1 > \frac{1}{2}(j-1)(j-2) = \binom{j-1}{2}$. This gives $j \geq \frac{1}{2} + \frac{1}{2}\sqrt{8d+9}$ and $j < \frac{3}{2} + \frac{1}{2}\sqrt{8d+9}$, i.e., $j = \left\lceil \frac{1}{2} + \frac{1}{2}\sqrt{8d+9} \right\rceil$. □

A somewhat better lower bound is provided using the result of Theorem 2.

Lemma 5 *If* $p \geq d+1$ *then*

(a) $\chi'_d\left(C_p\right) \geq \chi'_d\left(P_\infty\right)$,
(b) $\chi'_d\left(C_p\right) \geq \chi'_d\left(C_{kp}\right)$.

Proof A d-strong edge coloring of C_p can be transferred to a d-strong edge coloring of an infinite path or of a cycle C_{kp} by periodic repetition. □

In the following theorems we prove some general lower and upper bounds for the d-strong chromatic index of cycles C_p.

Theorem 5 *If* $k \geq 3,\ \ p = 2\binom{k}{2} - 1$, *and* $\frac{1}{3}(p-2) \leq d \leq \frac{1}{2}(p-2)$, *then* $\chi'_d\left(C_p\right) \geq k+1$.

Proof The bounds for d imply that $2d+2 \leq p \leq 3d+2$. Therefore, in a d-strong edge coloring of C_p with k colors each palette occurs at most twice since $p < 3(d+1) = 3n_2$. According to $p = 2\binom{k}{2} - 1$ all but one palettes must occur twice and one palette once.

A d-strong edge coloring of C_p with k colors corresponds to a closed walk in the complete graph K_k such that all but one edges occur twice and one edge once in the walk. The multigraph induced by the edges of this closed walk has two vertices of odd degree and hence does not contain an Eulerian circuit. Therefore, a d-strong edge coloring of C_p with k colors does not exist. □

Theorem 6 *If* $d \geq 5$ *and* $p \geq 2d+2$ *then* $\chi'_d\left(C_p\right) \leq d+1$.

Proof Let $d \geq 5$ and $p = m(d+1)+i,\ \ m \geq 2,\ \ 0 \leq i \leq d$. Color $m(d+1)$ edges of C_p periodically with colors $1, 2, \ldots, d+1$ and the remaining i edges for $i \geq 1$ as follows to obtain a d-strong edge coloring with $d+1$ colors:

If $i = 1$ then use color $\lceil d/2 \rceil$. If $i = 2$ then color the two remaining edges with colors 2 and $d-1$. If $i = 3$ or $i = 4$ then use colors $1, d, \lceil d/2 \rceil$ or $1, d, 2, d-1$, respectively. If $5 \leq i \leq d$, color the remaining edges with the first i colors of the sequence $(1, d, 2, d+1, 3, d-1, 4, d-2, \ldots, d/2, d/2+2)$ for d even and of the sequence $(1, d, 2, d+1, 3, d-1, 4, d-2, \ldots, (d+1)/2)$ for d odd. □

The following propositions provide exact values for $\chi'_d\left(C_p\right)$ and $d \leq 6$.

Proposition 2 ([17]) $\chi'_1\left(C_p\right) = \chi'_2\left(C_p\right) = \begin{cases} 5 & \textit{if } p = 5, \\ 3 & \textit{if } 3 \mid p, \\ 4 & \textit{otherwise.} \end{cases}$

Proposition 3 *Let* $p \geq 8$. *Then* $\chi'_3\left(C_p\right) = \begin{cases} 5 & \textit{if } p = 11, \\ 4 & \textit{otherwise.} \end{cases}$

Proof Lemma 4 implies $\chi'_3\left(C_p\right) \geq 4$.

A periodic edge coloring with colors 1, 2, 3, 4 is 3-strong if $p = 4k,\ \ k \in \mathbb{N}$. If $p = 4k+1,\ \ k \geq 2$, then a periodic coloring 1, 2, 3, 4 of $p-5$ edges and a coloring 2, 1, 3, 2, 4 of the remaining 5 edges is 3-strong. A periodic coloring 1, 2, 3, 4 of $p-6$

edges together with a coloring 2, 1, 3, 2, 4, 3 of the remaining 6 edges is 3-strong if $p = 4k+2,\ k \geq 2$. If $p = 4k+3,\ k \geq 3$, then a periodic coloring 1, 2, 3, 4 of $p-11$ edges and a coloring 1, 2, 3, 1, 4, 2, 1, 3, 2, 4, 3 of the remaining edges is 3-strong. Therefore, $\chi_3'\left(C_p\right) = 4$ for $p \geq 8$ and $p \neq 11$.

If $p = 11$ then 1, 2, 3, 4, 5, 1, 2, 3, 1, 4, 5 is a 3-strong edge coloring of C_{11} with 5 colors. On the other hand, Theorem 5 implies $\chi_3'(C_{11}) \geq 5$ (setting $k = 4$). □

Proposition 4 *Let* $p \geq 10$. *Then* $\chi_4'\left(C_p\right) = 5$.

Proof Let $p \geq 10$. Since $\chi_4'(P_\infty) = 5$ according to Theorem 2, $\chi_4'\left(C_p\right) \geq 5$ by Lemma 5.

The following colorings of C_p with five colors are 4-strong:

$p = 5k$:	1, 2, 3, 4, 5, ..., 1, 2, 3, 4, 5
$p = 5k+1$:	1, 2, 3, 4, 5, ..., 1, 2, 3, 4, 5, 3
$p = 5k+2$:	1, 2, 3, 4, 5, ..., 1, 2, 3, 4, 5, 2, 4
$p = 5k+3, k \geq 2$:	1, 2, 3, 4, 5, ..., 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 3
$p = 5k+4, k \geq 2$:	1, 2, 3, 4, 5, ..., 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 2, 4

□

In the proofs of the following results we use a well-known theorem of Frobenius.

Lemma 6 (Frobenius [11]) *Let* $a, b \in \mathbb{N}$ *with* $\gcd(a, b) = 1$. *The equation* $p = na + mb$ *has at least one solution with nonnegative integers* n, m *for all integers* $p > a \cdot b - a - b$.

Proposition 5 *Let* $p \geq 12$. *Then* $\chi_5'\left(C_p\right) = \begin{cases} 6 & \text{if } p = 15, \\ 5 & \text{otherwise.} \end{cases}$

Proof Let $p \geq 12$. By Lemma 5 and Theorem 2 it holds that $\chi_5'\left(C_p\right) \geq \chi_5'(P_\infty) = 5$.

The equation $p = 6l + 7m + 10n$ has a solution with $l, m, n \in \mathbb{N}_0$ for all integers $p > 29$ by Lemma 6 (setting $n = 0$). There exist also solutions if $p = 6k,\ p = 6k + 1 = 6(k-1)+7,\ p = 6k+2 = 6(k-2)+2\cdot 7 \geq 14,\ p = 6k+3 = 6(k-3)+3\cdot 7 \geq 21,\ p = 6k+4 = 6(k-1)+10 \geq 10$, and $p = 6k+5 = 6(k-2)+7+10 \geq 17$, i.e., for all $p \geq 12$ and $p \neq 15$.

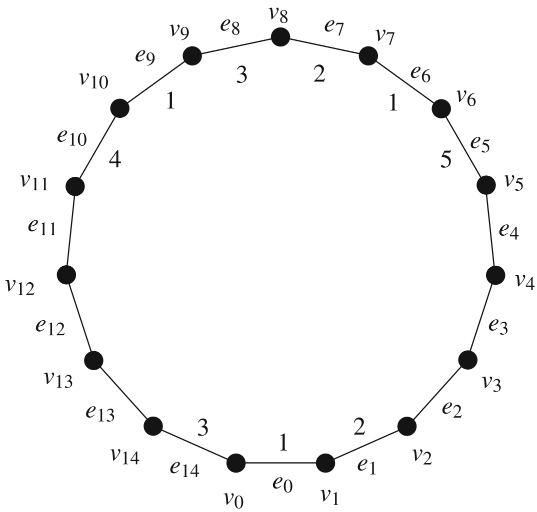
A periodic coloring 1, 2, 3, 4, 5, 3 of $6l$ edges, 1, 2, 3, 4, 5, 2, 4 of $7m$ edges, or 1, 2, 3, 4, 5, 1, 3, 5, 2, 4 of $10n$ edges, respectively, is 5-strong. This follows by the fact that these three colorings all begin with the 5 colors 1, 2, 3, 4, 5. Therefore, $\chi_5'\left(C_p\right) = 5$ for $p \geq 12,\ p \neq 15$.

The edge coloring 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 2, 4, 6 of C_{15} is 5-strong implying that $\chi_5'(C_{15}) \leq 6$. Assume that there exists a 5-strong edge coloring f of C_{15} with 5 colors. Recall that such an edge coloring f corresponds to a closed walk of the edges of K_5 of length 15.

There are $\binom{5}{2} = 10$ pairwise distinct palettes which can occur at most 2 times since $p = 15 < 3 \cdot 6 = 3(d+1)$. Therefore, at least 5 palettes must occur twice.

Consider two equal palettes, say $S(v_1) = S(v_i) = \{1, 2\}$ with $f(e_0) = 1$ and $f(e_1) = 2$. Since $S(v_j) \neq S(v_1)$ for $1 \leq d(v_j, v_1) \leq 5$ it follows that $6 \leq d(v_i, v_1) \leq 7$.

Fig. 3 Edge coloring f of C_{15}



1. Let $f(e_{i-1}) = 1,\ f(e_i) = 2$, and $d(v_i, v_1) = 7$.
   Since $S(v_1) = S(v_i)$ the walk is subdivided in this case into two closed walks of lengths 7 and 8, respectively. Since circuits of length 8 do not exist in K_5 there must occur equal palettes in the closed walk of length 8 which is impossible for $d = 5$.
2. Let $f(e_{i-1}) = 1,\ f(e_i) = 2$, and $d(v_i, v_1) = 6$.
   As above, the walk is subdivided into two closed walks of lengths 6 and 9, respectively. Since circuits of length 9 do not exist in K_5 there must be equal palettes in the closed walk of length 9. Let, without loss of generality, $v_i = v_7$ and $S(v_0) = S(v_9) = \{1, 3\}$ implying that $f(e_8) = f(e_{14}) = 3$ and $f(e_9) = 1$. Since the palettes $\{1, 2\}$ and $\{1, 3\}$ occur twice the edges e_5 and e_{10} must be colored with the remaining colors 4 and 5. Let $f(e_5) = 5$ and $f(e_{10}) = 4$, without loss of generality (see Fig. 3).
   If one of the remaining edges is colored with color 1 (e_3 and e_{12} are possible) then both adjacent edges must be colored with colors 4 and 5 implying two palettes $\{1, 4\}$ or $\{1, 5\}$ of distance 3 which is impossible.
   If $f(e_{13}) = 2$ then this forces $f(e_{12}) = 5$ and then $f(e_2) = 4,\ f(e_3) = 3$ implying that e_4 can not be colored properly. Analogously, $f(e_2) = 3$ forces a contradiction. Therefore, $f(e_2), f(e_{13}) \in \{4, 5\}$.
   If $f(e_2) = 4$ or $f(e_{13}) = 5$ then $S(v_4) = \{2, 3\}$ or $S(v_{12}) = \{2, 3\}$, a contradiction to $S(v_8) = \{2, 3\}$. Therefore, $f(e_2) = 5$ and $f(e_{13}) = 4$ which forces $S(v_4) = \{3, 4\}$, a contradiction to $S(v_{14}) = \{3, 4\}$.
   Note that from above we obtain that in the sequence of edge colors of the cycle the same pair of colors can only occur in different order: $f(e_i) = f(e_j) \implies f(e_{i+1}) \neq f(e_{j+1})$.
3. Let $f(e_{i-1}) = 2,\ f(e_i) = 1$, and $d(v_i, v_1) = 6$.
   The equal palettes $S(v_1) = S(v_i) = \{1, 2\}$ generate three closed walks of lengths $2,\ 5$, and 8. Since circuits of length 8 do not exist in K_5 there must be equal palettes in the closed walk of length 8. Let, without loss of generality, $v_i = v_7$. Since

$1 \notin S(v_9)$ we obtain $S(v_9) \neq S(v_0)$, and since $1 \notin S(v_{14})$ we get $S(v_{14}) \neq S(v_8)$. Therefore, $S(v_0) = S(v_8)$. The missing case of two equal palettes of distance 7 is covered in 4.

4. Let $f(e_{i-1}) = 2,\ f(e_i) = 1$, and $d(v_i, v_1) = 7$.
   Let, without loss of generality, $v_i = v_8$. The equal palettes $S(v_1) = S(v_8) = \{1, 2\}$ generate three closed walks of lengths 2, 6, and 7. Since $d = 5$ the closed walk $S(v_2), \dots, S(v_7)$ of length 6 is a circuit. This circuit is unique up to permutation of the colors, one color occurs twice and the other 4 colors exactly once.
   The colors 1 and 2 cannot occur twice since otherwise this coloring would generate palettes $\{1, 2\}$ of distance ≤ 5 which is not possible.
   Without loss of generality, let color 3 occur twice such that $f(e_2) = f(e_5) = 3$ and therefore $f(e_6) \in \{4, 5\}$, say $f(e_6) = 5$, implying $\{f(e_3), f(e_4)\} = \{1, 4\}$.
   1 $f(e_3) = 4,\ f(e_4) = 1$:
   Since $(f(e_0), f(e_1), \dots, f(e_8)) = (1, 2, 3, 4, 1, 3, 5, 2, 1)$ by the above construction this enforces $f(e_{14}) = f(e_9) = 5,\ f(e_{10}) = 4$, and $f(e_{13}) \in \{2, 3\}$. Furthermore, since $f(e_{10}) = f(e_3) = 4$, we obtain $f(e_{11}) \neq f(e_4) = 1$ according to the remark at the end of case 2. Therefore, $f(e_{11}) \in \{2, 3\}$ implying that $f(e_{12}) = 1$ which is impossible.
   2 $f(e_3) = 1,\ f(e_4) = 4$:
   Since $(f(e_0), f(e_1), \dots, f(e_8)) = (1, 2, 3, 1, 4, 3, 5, 2, 1)$ we obtain $f(e_{14}) = 5,\ f(e_{13}) \in \{2, 4\},\ f(e_9) \in \{3, 5\}$.
   If $f(e_9) = 5$ then $f(e_{10}) = 4,\ f(e_{13}) = 2,\ f(e_{11}) = 1$ ($f(e_{11}) = 3$ is impossible by the remark at the end of case 2), and $f(e_{12}) = 3$ which contradicts the definition of a 5-strong edge coloring.
   If $f(e_9) = 3$ this forces $f(e_{10}) = 2,\ f(e_{11}) = 4$, and $S(v_{13}) = \{1, 2\}$ or $\{2, 3\}$, a contradiction. □

Proposition 6 *Let* $p \geq 14$. *Then* $\chi'_6(C_p) = \begin{cases} 6 & \textit{if } p = 15,\ p = 19, \textit{ or } p = 22, \\ 5 & \textit{otherwise.} \end{cases}$

Proof Let $p \geq 14$. Lemma 5 and Theorem 2 imply that $\chi'_6(C_p) \geq \chi'_6(P_\infty) = 5$.

The equation $p = 7l + 10m + 16n$ has a solution with $l, m, n \in \mathbb{N}_0$ for all $p > 53$ by Lemma 6 (setting $n = 0$). There exist also solutions for $p = 7k,\ p = 7k{+}1 = 7(k-5) + 2 \cdot 10 + 16 \geq 36,\ p = 7k + 2 = 7(k-2) + 16,\ p = 7k + 3 = 7(k-1) + 10,\ p = 7k{+}4 = 7(k{-}4){+}2{\cdot}16 \geq 32,\ p = 7k{+}5 = 7(k{-}3){+}10{+}16 \geq 26,\ p = 7k + 6 = 7(k-2) + 2 \cdot 10 \geq 20$, i.e., for all $p \geq 14$ and $p \notin \{15, 18, 19, 22, 25, 29\}$.

A periodic coloring 1, 2, 3, 1, 4, 2, 5 of $7l$ edges, 1, 2, 3, 1, 4, 2, 5, 3, 4, 5 of $10m$ edges, or 1, 2, 3, 1, 4, 2, 5, 4, 3, 2, 1, 3, 5, 2, 4, 5 of $16n$ edges, respectively, is 6-strong. The edge colorings 1, 2, 3, 1, 4, 3, 5, 2, 1, 5, 4, 1, 3, 2, 5, 3, 4, 5 of C_{18}, 1, 2, 3, 1, 4, 2, 5, 1, 2, 3, 1, 4, 3, 5, 2, 1, 5, 4, 1, 3, 2, 5, 3, 4, 5 of C_{25}, and 1, 2, 3, 1, 4, 2, 5, 1, 2, 3, 4, 5, 3, 1, 5, 2, 3, 4, 1, 2, 4, 5, 1, 3, 2, 5, 3, 4, 5 of C_{29} are also 6-strong. Therefore, $\chi'_6(C_p) = 5$ in all above cases.

The edge coloring 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 2, 4, 6 of C_{15} is 6-strong, implying together with $\chi'_5(C_{15}) = 6$ that $\chi'_6(C_{15}) = 6$. The edge colorings 1, 2, 3, 4, 5, 2, 4, 6, 1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 6 of C_{19} and 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 2, 4, 1, 2, 3, 4, 5, 2, 4, 6 of C_{22} imply that $\chi'_6(C_{19}) \leq 6$ and $\chi'_6(C_{22}) \leq 6$, respectively. Theorem 5 implies that $\chi'_6(C_{19}) \geq 6$ (setting $k = 5$). With the help of a computer we proved that a 6-strong edge coloring of C_{22} with 5 colors does not exist [5]. □

The results of Proposition 2–6 are summarized in the following theorem.

Theorem 7 *If* $1 \leq d \leq 6$ *and* $p \geq 2d + 2$ *then*

$$\chi'_d(C_p) = \begin{cases} \mu_d(C_p) + 2 & \text{if } d = 1,\, p = 5 \text{ or } d = 5,\, p = 15, \\ \mu_d(C_p) & \text{if } d \in \{1, 2\},\, 3 \mid p \text{ or } d = 3,\, p \neq 11 \\ & \quad \text{or } d = 6,\, p \notin \{15, 19, 22\}, \\ \mu_d(C_p) + 1 & \text{otherwise.} \end{cases}$$

Using similar methods we proved that $\chi'_8(C_p) = 5$ if $10 \mid p$ and $\chi'_8(C_p) = 6$ otherwise for all $p \geq 18$, and $\chi'_9(C_p) = 5$ if $10 \mid p$, $\chi'_9(C_{29}) = 7$ and $\chi'_9(C_p) = 6$ otherwise for all $p \geq 20$. For $d = 7$ and $p \geq 16$ it holds $5 \leq \chi'_7(C_p) \leq 6$.

Theorem 7 proves that Conjecture 1 is not true in general since $\chi'_5(C_{15}) = 6$ (as proved in Proposition 5) whereas $\mu_5(C_{15}) = \min\{j : \binom{j}{2} \geq n_2 = 6\} = 4$. The following result provides an infinite class of counterexamples to this conjecture.

Theorem 8 *If* $k \geq 6$, $p = 2\binom{k}{2} - 1$, *and* $d = \binom{k-1}{2} - 1$ *then* $\chi'_d(C_p) \geq k + 1$ *and* $\mu_d(C_p) = k - 1$.

Proof Since $2d+2 = (k-1)(k-2) \leq p = k(k-1)-1$ if $k \geq 2$ and $p = k^2-k-1 < \frac{3}{2}(k-1)(k-2) = 3\binom{k-1}{2} = 3d + 3$ if $k \geq 6$ it holds that $\frac{1}{3}(p-2) \leq d \leq \frac{1}{2}(p-2)$ and therefore $\chi'_d(C_p) \geq k + 1$ by Theorem 5.

On the other hand, $p \geq 2d + 2$ implies that $n_2 = d + 1 = \binom{k-1}{2}$ and therefore $\mu_d(C_p) = k - 1$. □

Therefore, $\chi'_9(C_{29}) \geq 7$, $\chi'_{14}(C_{41}) \geq 8$, $\chi'_{20}(C_{55}) \geq 9, \ldots$ are counterexamples to Conjecture 1.

Note that the coloring 1, 2, 3, 1, 4, 2, 5, 3, 4, 5, 1, 2, 3, 6, 2, 4, 6, 1, 3, 5, 1, 4, 5, 2, 6, 3, 4, 6, 7 shows that $\chi'_9(C_{29}) \leq 7$, i.e., $\chi'_9(C_{29}) = 7$.

Our results give cause to state the following

Conjecture 2 $\chi'_d(G) \leq \mu_d(G) + 2$ *for cycles* C_p.

We would like to thank an anonymous referee for useful comments.

References

1. Akbari, S., Bidkhori, H., Nosrati, N.: r-Strong edge colorings of graphs. Discrete Math. **306**, 3005–3010 (2006)
2. Balister, P.N., Győri, E., Lehel, J., Schelp, R.H.: Adjacent vertex distinguishing edge-colorings. SIAM J. Discrete Math. **21**, 237–250 (2007)
3. Bazgan, C., Harkat-Benhamdine, A.H., Li, H., Woźniak, M.: On the vertex-distinguishing proper edge-colorings of graphs. J. Combin. Theory Ser. B **75**, 288–301 (1999)

4. Bazgan, C., Harkat-Benhamdine, A.H., Li, H., Woźniak, M.: A note on the vertex-distinguishing proper coloring of graphs with large minimum degree. Discrete Math. **236**, 37–42 (2001)
5. Bode, J.-P.: private communication
6. Burris, A.C.: Vertex-distinguishing edge colorings. Doctorate thesis, Memphis State University (1993)
7. Burris , A.C., Schelp, R.H.: Vertex-distinguishing proper edge-colorings. J. Graph Theory **26**, 73–82 (1997)
8. Černý, J., Horňák, M., Soták, R.: Observability of a graph. Math. Slovaca **46**, 21–31 (1996)
9. Edwards, K., Horňák, M., Woźniak, M.: On the neighbour-distinguishing index of a graph. Graphs Combin. **22**, 341–350 (2006)
10. Favaron, O., Li, H., Schelp, R.H.: Strong edge colorings of graphs. Discrete Math. **159**, 103–109 (1996)
11. Frobenius, G.: Über Matrizen aus nichtnegativen Elementen. Sitzungsber. Preuss. Akad. Wiss. Berlin, 456–477 (1912)
12. Soták, R.: Invariants of regular graphs (in Slovak). Diploma thesis, Košice (1992)
13. Taczuk, K., Woźniak, M.: A note on the vertex-distinguishing index for some cubic graphs. Opuscula Math. **23**, 223–229 (2004)
14. Tian, J.J., Liu, X., Zhang, Z., Deng, F.: Upper bounds on the $D(\beta)$-vertex-distinguishing edge-chromatic-numbers of graphs. LNCS **4489**, 453–456 (2007)
15. Wang, W., Wang, Y.: Adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree. J. Comb. Optim. **19**, 471–485 (2010)
16. Zhang, Z., Li, J., Chen, X., et al.: $D(\beta)$-vertex-distinguishing proper edge-coloring of graphs. Acta Math. Sinica Chin. Ser. **49**, 703–708 (2006)
17. Zhang, Z., Liu, L., Wang, J.: Adjacent strong edge coloring of graphs. Appl. Math. Lett. **15**, 623–626 (2002)